

BOUNDARY REPRESENTATIONS ON CO-INVARIANT SUBSPACES OF BERGMAN SPACE

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ABSTRACT. Let M be an invariant subspace of the Bergman space $L_a^2(\mathbb{D})$ and S_M be the compression of the coordinate multiplication operator M_z to the co-invariant subspace $L_a^2(\mathbb{D}) \ominus M$. The present paper determines when the identity representation of $C^*(S_M)$ is a boundary representation for the Banach subalgebra $\mathcal{B}(S_M)$. The paper also considers boundary representations on the co-invariant subspaces of $L_a^2(\mathbb{B}_n)$.

1. INTRODUCTION

In his two Acta papers [Arv1, Arv2], Arveson systematically studied relations between non-self-adjoint operator algebras on Hilbert space and C^* -algebras generated by them. One of the main purposes is to understand to what extent an algebra of operators on a Hilbert space can determine the structure of the C^* -algebra generated by it. As a typical example, Arveson investigated for an invariant subspace M of the Hardy space $H^2(\mathbb{D})$, when the structure of the C^* -algebra generated by S_M and the identity can be determined by the Banach subalgebra generated by S_M and the identity, where S_M is the compression of the coordinate multiplication operator M_z to the co-invariant subspace $H^2(\mathbb{D}) \ominus M$. The present paper considers the same problem on the Bergman space over the unit disk.

For convenience, let us first introduce some notation and recall some basic definitions. For a Hilbert space H , let $\mathcal{L}(H)$ (resp. $\mathcal{K}(H)$) denote the set of all bounded (resp. compact) operators on H . For a subset $S \subseteq \mathcal{L}(H)$, let $C^*(S)$ be the C^* -algebra generated by S and the identity, and let $\mathcal{B}(S)$ be the operator norm-closed Banach algebra generated by S and the identity. Let B_1, B_2 be two C^* -algebras, and let ϕ be a linear map of B_1 into B_2 . If $M_k, k = 1, 2, \dots$, denotes the C^* -algebra of all complex $k \times k$ matrices and id_k denotes the identity map of M_k , then $id_k \otimes \phi$ is a linear map of $M_k \otimes B_1$ into $M_k \otimes B_2$. The map ϕ is called *completely positive* (resp. *completely isometric*) provided that every map in the sequence $id_1 \otimes \phi, id_2 \otimes \phi, \dots$ is positive (resp. isometric).

One of the principal concepts in Arveson's work [Arv1, Arv2] is that of boundary representations. Let B be a C^* -algebra and let A be a linear subspace of B such that $B = C^*(A)$. An irreducible representation ω of B is called a *boundary representation* for A if $\omega|_A$ has a unique completely positive linear extension to B ,

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namely ω itself. Boundary representations of C^* -algebras have been comprehensively investigated; see [Arv1, Arv2, Arv3, Arv4, GHX] for more information. In particular, if the identity representation of B is a boundary representation for A , then the subspace A can determine the structure of the C^* -algebra B .

Arveson explored a typical example in [Arv1, Arv2]. We restate his results as follows. Let \mathbb{D} be the open unit disk, and let \mathbb{T} be the unit circle. The Hardy space $H^2(\mathbb{D})$ consists of all functions in $L^2(\mathbb{T})$ whose negative Fourier coefficients vanish. Let M be an invariant subspace of $H^2(\mathbb{D})$; i.e., M is closed and invariant under the coordinate multiplication operator M_z . By the well known Beurling theorem, $M = \eta H^2(\mathbb{D})$ for some inner function η . To exclude trivial cases, suppose η is not a Möbius transform. Let $N = H^2(\mathbb{D}) \ominus M$, then N is co-invariant; that is, N is invariant under M_z^* . Conversely, each co-invariant subspace has such a form. Set $S_M = P_N M_z|_N$, where P_N is the orthogonal projection from $H^2(\mathbb{D})$ to N . Arveson determined when the identity representation of $C^*(S_M)$ is a boundary representation for the Banach subalgebra $\mathcal{B}(S_M)$.

Theorem 1.1 ([Arv1, Arv2]). *The identity representation of $C^*(S_M)$ is a boundary representation for the subalgebra $\mathcal{B}(S_M)$ if and only if $Z_\eta \cap \mathbb{T}$ is a proper subset of \mathbb{T} , where Z_η consists of the zeroes of η inside \mathbb{D} , along with all points λ on \mathbb{T} for which η cannot be continued analytically from \mathbb{D} to λ .*

The proof of this theorem is very complicated. In [Arv1], Arveson only proved the extreme case $Z_\eta \cap \mathbb{T} = \mathbb{T}$ and $Z_\eta \cap \mathbb{T}$ has Lebesgue measure zero. The above general result was obtained in [Arv2].

The present paper deals with the same problem on the Bergman space $L_a^2(\mathbb{D})$. Recall that the Bergman space $L_a^2(\mathbb{D})$ is the closed subspace of $L^2(\mathbb{D}, dA)$ consisting of analytic functions, where dA is the normalized area measure on \mathbb{D} . For each $f \in L_a^2(\mathbb{D})$, set

$$Z_*(f) = \{\lambda \in \mathbb{D} : f(\lambda) = 0\} \cup \{\lambda \in \mathbb{T} : \liminf_{z \in \mathbb{D}, z \rightarrow \lambda} |f(z)| = 0\}.$$

Given an invariant subspace M of $L_a^2(\mathbb{D})$, set

$$Z_*(M) = \bigcap_{f \in M} Z_*(f).$$

To exclude trivial cases, we always assume that M is infinite co-dimensional; that is, the co-invariant subspace $L_a^2(\mathbb{D}) \ominus M$ is infinite dimensional. Now our main result can be stated as follows.

Theorem 1.2. *Let M be an infinite co-dimensional invariant subspace of $L_a^2(\mathbb{D})$, $N = L_a^2(\mathbb{D}) \ominus M$ and $S_M = P_N M_z|_N$. Then the identity representation of $C^*(S_M)$ is a boundary representation for the subalgebra $\mathcal{B}(S_M)$ if and only if $\dim M \ominus zM = 1$ and $Z_*(M) \cap \mathbb{T}$ is a proper subset of \mathbb{T} .*

The proof of this theorem will be presented in Section 2. One will find in the proof that the distinction between our result and that of Arveson is essentially due to the difference between the invariant subspace structure of the two spaces $L_a^2(\mathbb{D})$ and $H^2(\mathbb{D})$.

In Section 3, we obtain some results on boundary representations on the co-invariant subspaces of $L_a^2(\mathbb{B}_n)$, where \mathbb{B}_n denotes the unit ball of \mathbb{C}^n .

2. PROOF OF THE MAIN RESULT

In this section, we will give the proof of the main result Theorem 1.2.

For an irreducible set S of operators, there is a general criterion for determining whether the identity representation of $C^*(S)$ is a boundary representation for S . It is the following Arveson's boundary theorem.

Theorem 2.1 ([Arv2]). *Let S be an irreducible set of operators on a Hilbert space H , such that S contains the identity and $C^*(S)$ contains the algebra $\mathcal{K}(H)$ of all compact operators on H . Then the identity representation of $C^*(S)$ is a boundary representation for S if and only if the quotient map $q : \mathcal{L}(H) \rightarrow \mathcal{L}(H)/\mathcal{K}(H)$ is not completely isometric on the linear span of $S \cup S^*$.*

From the above theorem, Arveson obtained the following corollary which is more convenient for applications. In the following, let $\sigma(T)$ denote the spectrum of an operator T , and $\sigma_e(T)$ denote the essential spectrum of T . Let $|\sigma_e(T)| = \sup\{|\lambda| : \lambda \in \sigma_e(T)\}$, the essential spectral radius of T . An operator T is said to be *essentially normal* if the self-commutator, $[T^*, T] = T^*T - TT^*$, is compact.

Corollary 2.2 ([Arv2]). *Let S be an irreducible set of commuting essentially normal operators which contains the identity. Assume that $|\sigma_e(T)| < \|T\|$ for some element $T \in S$. Then the identity representation of $C^*(S)$ is a boundary representation for S .*

The following result is also useful in our proof.

Lemma 2.3 ([Arv1, Theorem 3.6.3]). *Let T be a contraction on a Hilbert space. Then T gives rise to a completely isometric representation of the disk algebra $A(\mathbb{D})$ if and only if $\sigma(T)$ contains the unit circle. For such a T , the identity representation of $C^*(T)$ is never a boundary representation for $\mathcal{B}(T)$.*

We also need some results about the operator S_M .

Lemma 2.4 ([Zhu, Theorem 3.1]). *Suppose M is an invariant subspace of $L_a^2(\mathbb{D})$, $R_M = M_z|_M$, $N = L_a^2(\mathbb{D}) \ominus M$, and $S_M = P_N M_z|_N$, then the following are equivalent:*

- (1) $[R_M^*, R_M] = R_M^* R_M - R_M R_M^*$ is compact.
- (2) $[S_M^*, S_M] = S_M^* S_M - S_M S_M^*$ is compact.
- (3) $\dim M \ominus zM < \infty$.

Lemma 2.5 ([Ri, Corollary 4.6] and [Hed, Theorem 1.3]). *Let M be an invariant subspace of $L_a^2(\mathbb{D})$, $M \neq \{0\}$, and S_M be as in Lemma 2.4. If $\dim M \ominus zM = 1$, then $\sigma(S_M) = Z_*(M)$. If $\dim M \ominus zM > 1$, then $\sigma(S_M) = \bar{\mathbb{D}}$.*

Lemma 2.6. *Let M , N and S_M be as in Lemma 2.4. Then the operator S_M is irreducible.*

Before proving the lemma, let us recall the definition of the Berezin transform. Note that $L_a^2(\mathbb{D})$ is a reproducing kernel Hilbert space; i.e., for any $\lambda \in \mathbb{D}$, the evaluation functional $E_\lambda : f \mapsto f(\lambda)$ is continuous. By the Riesz theorem, for every $\lambda \in \mathbb{D}$, there is a unique function $K_\lambda \in L_a^2(\mathbb{D})$ such that

$$f(\lambda) = \langle f, K_\lambda \rangle, \quad f \in L_a^2(\mathbb{D}).$$

K_λ is called the *reproducing kernel* of $L_a^2(\mathbb{D})$ at λ . Direct computation shows that on the Bergman space $L_a^2(\mathbb{D})$, $K_\lambda(z) = 1/(1 - \bar{\lambda}z)^2$. Set $k_\lambda = K_\lambda/\|K_\lambda\|$. For a

bounded linear operator A on $L_a^2(\mathbb{D})$, the *Berezin transform* of A is defined by

$$\tilde{A}(\lambda) = \langle Ak_\lambda, k_\lambda \rangle, \quad \lambda \in \mathbb{D}.$$

It is well known that for two bounded linear operators A_1 and A_2 on $L_a^2(\mathbb{D})$, $\tilde{A}_1 = \tilde{A}_2$ if and only if $A_1 = A_2$.

For two functions f and g in $L_a^2(\mathbb{D})$, let $f \otimes g$ be the rank-one operator defined by

$$(f \otimes g)h = \langle h, g \rangle f, \quad h \in L_a^2(\mathbb{D}).$$

Proof. Since the reproducing kernel of $L_a^2(\mathbb{D})$ is $K_\lambda(z) = 1/(1 - \bar{\lambda}z)^2$, it is easy to compute that

$$\langle (1 \otimes 1)k_\lambda, k_\lambda \rangle = |\langle k_\lambda, 1 \rangle|^2 = 1/\|K_\lambda\|^2 = (1 - |\lambda|^2)^2.$$

One can also verify that

$$M_z^* k_\lambda = \bar{\lambda} k_\lambda,$$

and hence

$$\begin{aligned} & \langle (id - 2M_z M_z^* + M_z^2 M_z^{*2})k_\lambda, k_\lambda \rangle \\ &= \langle k_\lambda, k_\lambda \rangle - 2\langle M_z^* k_\lambda, M_z^* k_\lambda \rangle + \langle M_z^{*2} k_\lambda, M_z^{*2} k_\lambda \rangle \\ &= (1 - |\lambda|^2)^2. \end{aligned}$$

By the above-mentioned property of the Berezin transform, we have

$$(2.1) \quad 1 \otimes 1 = id - 2M_z M_z^* + M_z^2 M_z^{*2}.$$

Then

$$P_N(1 \otimes 1)P_N = P_N(id - 2M_z M_z^* + M_z^2 M_z^{*2})P_N,$$

and hence

$$(2.2) \quad P_N 1 \otimes P_N 1 = id_N - 2S_M S_M^* + S_M^2 S_M^{*2}.$$

Let $e = P_N 1$, then $e \neq 0$. Assume that there is some projection Q commuting with S_M , and we will show that Q is trivial. Clearly, Q commutes with the right-hand side of (2.2), so it commutes with the left-hand side. That is

$$Q(e \otimes e) = (e \otimes e)Q,$$

and immediately,

$$Qe \otimes e = e \otimes Qe.$$

Therefore, $Qe = ce$ for some constant c . Since

$$ce = Qe = Q(Qe) = Q(ce) = cQe = c^2e$$

and $e \neq 0$, we have $c = 0$ or $c = 1$.

If $c = 0$, then $Qe = 0$, and thus $S_M^n Qe = 0$ for any integer n . Since Q commutes with S_M , $QS_M^n e = 0$. That is, $QP_N z^n = 0$ for any n . But the linear span of $\{P_N z^n\}_{n=0}^\infty$ is dense in $P_N L_a^2(\mathbb{D})$, so we get $Q = 0$.

If $c = 1$, then $Qe = e$, i.e., $(1 - Q)e = 0$. The same argument as above gives that $1 - Q = 0$, and hence $Q = 1$.

The above reasoning shows that there is no nontrivial projection commuting with S_M , so the operator S_M is irreducible. This completes the proof. \square

Lemma 2.7. *Let M , N , and S_M be as in Lemma 2.4. Suppose M satisfies $\dim M \ominus zM < \infty$. Then $\sigma_e(S_M) = \sigma(S_M) \cap \mathbb{T}$.*

Proof. The proof is similar to Arveson’s proof in the Hardy space case (cf. [Arv1]). Let $\text{Comm}(S_M)$ be the commutator ideal of $C^*(S_M)$, i.e., the closed ideal generated by $\{xy - yx : x, y \in C^*(S_M)\}$. Since $\dim M \ominus zM < \infty$, it follows from Lemma 2.4 that

$$[S_M^*, S_M] = S_M^* S_M - S_M S_M^* \in \mathcal{K}.$$

Lemma 2.6 says that the operator S_M is irreducible, so $\text{Comm}(S_M) = \mathcal{K}$ (see for example [Arv1, Corollary 3.3.7]).

Since $C^*(S_M)/\text{Comm}(S_M) = C^*(S_M)/\mathcal{K}$ is an abelian C^* -algebra, there is a natural bijection between the characters of $C^*(S_M)/\text{Comm}(S_M)$ and the points in $\sigma_e(S_M)$. At the same time, since a character of $C^*(S_M)$ must vanish on $\text{Comm}(S_M)$, there is also a natural bijection between characters of $C^*(S_M)$ and characters of $C^*(S_M)/\text{Comm}(S_M)$. Therefore, the characters of $C^*(S_M)$ are in one-to-one correspondence with points in $\sigma_e(S_M)$.

Let $\Gamma : \chi \mapsto \chi(S_M)$ be the canonical bijective map from the characters of $C^*(S_M)$ to the points of $\sigma_e(S_M)$. We will show that Γ is also a bijection from the characters of $C^*(S_M)$ to the points of $\sigma(S_M) \cap \mathbb{T}$, by which the conclusion will follow.

Now if λ is a point in $\sigma(S_M) \cap \mathbb{T}$, then $|\lambda| = 1 = \|S_M\|$, so $\lambda \in \partial W(S_M)$, where $W(S_M) = \{\langle S_M \xi, \xi \rangle : \xi \in N, \|\xi\| = 1\}$ is the numerical range of S_M . By [Arv1, Theorem 3.1.2], there is a unique character χ of $C^*(S_M)$ such that $\chi(S_M) = \lambda$.

Conversely, suppose χ is a character of $C^*(S_M)$ and we must show that $\chi(S_M) \in \sigma(S_M) \cap \mathbb{T}$. But $\chi(S_M) \in \sigma_e(S_M)$ is obvious, so it remains to show $|\chi(S_M)| = 1$. Since χ vanishes on $\text{Comm}(S_M) = \mathcal{K}$, applying χ to formula (2.2) gives

$$1 - 2|\chi(S_M)|^2 + |\chi(S_M)|^4 = 0,$$

and hence $|\chi(S_M)| = 1$. □

We are now ready to prove the main result Theorem 1.2.

Proof of Theorem 1.2. The necessity part. We need to show that if the identity representation of $C^*(S_M)$ is a boundary representation for the subalgebra $\mathcal{B}(S_M)$, then $\dim M \ominus zM = 1$ and $Z_*(M) \cap \mathbb{T}$ is a proper subset of \mathbb{T} .

Since $\|S_M\| \leq \|M_z\| = 1$, S_M is a contraction. Suppose $\dim M \ominus zM > 1$. Then by Lemma 2.5, we have $\sigma(S_M) = \mathbb{D} \supseteq \mathbb{T}$. Then Lemma 2.3 implies that the identity representation of $C^*(S_M)$ is not a boundary representation for the subalgebra $\mathcal{B}(S_M)$, a contradiction. Hence $\dim M \ominus zM = 1$.

Suppose $Z_*(M) \cap \mathbb{T}$ is not a proper subset of \mathbb{T} , i.e., $Z_*(M) \cap \mathbb{T} = \mathbb{T}$. Since we have just proved $\dim M \ominus zM = 1$, it follows from Lemma 2.5 that $\sigma(S_M) = Z_*(M) \supseteq \mathbb{T}$. Using Lemma 2.3 again, we get a contradiction.

The sufficiency part. Suppose $\dim M \ominus zM = 1$ and $Z_*(M) \cap \mathbb{T}$ is a proper subset of \mathbb{T} , we will prove that the identity representation of $C^*(S_M)$ is a boundary representation for the subalgebra $\mathcal{B}(S_M)$.

Since $\dim M \ominus zM = 1$, by Lemma 2.4, the operator $S_M^* S_M - S_M S_M^*$ is compact. Lemma 2.6 says that the operator S_M is irreducible. So according to Corollary 2.2, it suffices to show that if $Z_*(M) \cap \mathbb{T}$ is a proper subset of \mathbb{T} , then there is a polynomial p such that

$$|\sigma_e(p(S_M))| = \sup\{|p(\lambda)| : \lambda \in \sigma_e(S_M)\} < \|p(S_M)\|.$$

The following proof uses the same idea as that in [Arv2, Theorem 2.2.1, Corollary 1]. Since M satisfies $\dim M \ominus zM = 1$, by Lemma 2.7,

$$\sigma_e(S_M) = \sigma(S_M) \cap \mathbb{T} = Z_*(M) \cap \mathbb{T}.$$

If we can show that $Z_*(M) \cap \mathbb{T}$ is not a spectral set for S_M , then the conclusion follows. Suppose S_M has $Z_*(M) \cap \mathbb{T}$ as its spectral set. Since the complement of $Z_*(M) \cap \mathbb{T}$ is connected and $Z_*(M) \cap \mathbb{T}$ has no interior, S_M must be normal, a contradiction. The proof is complete. \square

Remark 2.8. We have completely determined when the identity representation of $C^*(S_M)$ is a boundary representation for the subalgebra $\mathcal{B}(S_M)$. One may wonder for an invariant subspace M of $L_a^2(\mathbb{D})$, when the identity representation of $C^*(R_M)$ is a boundary representation for the subalgebra $\mathcal{B}(R_M)$, where $R_M = M_z|_M$. This is an easy consequence of [GHX, Theorem 3.2] which says that if T is subnormal and $\sigma(T) = \bar{\mathbb{D}}$, then the identity representation of $C^*(T)$ is not a boundary representation for the subalgebra $\mathcal{B}(T)$. Since R_M is subnormal and $\sigma(R_M) = \bar{\mathbb{D}}$ [Zhu, Proposition 2.1], we know that the identity representation of $C^*(R_M)$ is never a boundary representation for the subalgebra $\mathcal{B}(R_M)$.

3. RESULTS ON CO-INVARIANT SUBSPACES OF $L_a^2(\mathbb{B}_n)$

In this section, we will consider boundary representations on co-invariant subspaces of the Bergman space $L_a^2(\mathbb{B}_n)$.

Recall that the Bergman space $L_a^2(\mathbb{B}_n)$ is the subspace of $L^2(\mathbb{B}_n, dV)$ consisting of analytic functions over the unit ball \mathbb{B}_n , where dV is the normalized volume measure. Let M be an invariant subspace of $L_a^2(\mathbb{B}_n)$, i.e., M is closed and invariant under the coordinate multiplication operators $\{M_{z_1}, \dots, M_{z_n}\}$. Let $N = L_a^2(\mathbb{B}_n) \ominus M$ be the co-invariant subspace and $S_i = P_N M_{z_i}|_N$ for $1 \leq i \leq n$.

An interesting problem is to characterize all the boundary representations of $C^*(S_1, \dots, S_n)$ relative to $\mathcal{B}(S_1, \dots, S_n)$, where $\mathcal{B}(S_1, \dots, S_n)$ is the operator norm-closed Banach algebra generated by S_1, \dots, S_n and the identity. In particular, we are mainly concerned with whether the identity representation of $C^*(S_1, \dots, S_n)$ is a boundary representation for the subalgebra $\mathcal{B}(S_1, \dots, S_n)$. This problem has a close connection with the essential normality of the operator tuple (S_1, \dots, S_n) . Recall that the operator tuple (S_1, \dots, S_n) is said to be *essentially normal* if $S_i^* S_j - S_j S_i^*$ is compact for all $1 \leq i, j \leq n$. For brevity, we often say that the co-invariant subspace $N = L_a^2(\mathbb{B}_n) \ominus M$ on which the operator tuple (S_1, \dots, S_n) acts is essentially normal. In an even more general setting than the Bergman space $L_a^2(\mathbb{B}_n)$, essential normality has been explored deeply. See for example [Arv5, Dou, Guo, GW1, GW2, GWa].

The results on essential normality shed some light on the problem of boundary representations. First of all, it is an empirical fact that on many Hilbert spaces of analytic functions, the operator tuple (S_1, \dots, S_n) is irreducible. If the tuple is also essentially normal, then $C^*(S_1, \dots, S_n)$ must contain the algebra \mathcal{K} of all compact operators. As a result of this, we can use Arveson's boundary theorem, Theorem 2.1, to tell us whether the identity representation of $C^*(S_1, \dots, S_n)$ is a boundary representation for the subalgebra $\mathcal{B}(S_1, \dots, S_n)$. The following theorem is an attempt in this direction.

Theorem 3.1. *Let p be a homogeneous polynomial, and let $M = [p]$ be the invariant subspace of $L_a^2(\mathbb{B}_n)$ generated by p . Let $p = p_1^{k_1} \cdots p_l^{k_l}$ be the irreducible factorization*

of p . Suppose there is some k_i such that $k_i > 1$, then the identity representation of $C^*(S_1, \dots, S_n)$ is a boundary representation for the subalgebra $\mathcal{B}(S_1, \dots, S_n)$.

Proof. It is shown in [GW1] that for an invariant subspace M of $L_a^2(\mathbb{B}_n)$ which is generated by a homogeneous polynomial p , the co-invariant subspace $L_a^2(\mathbb{B}_n) \ominus M$ is essentially normal; i.e., $S_i^*S_j - S_jS_i^*$ is compact for all $1 \leq i, j \leq n$. The same paper also shows that the operator tuple (S_1, \dots, S_n) is irreducible, and the essential spectrum is $\sigma_e(S_1, \dots, S_n) = Z(p) \cap \partial\mathbb{B}_n$, so there is a short exact sequence

$$0 \rightarrow \mathcal{K} \rightarrow C^*(S_1, \dots, S_n) \rightarrow C(Z(p) \cap \partial\mathbb{B}_n) \rightarrow 0,$$

where $Z(p)$ is the zero set of p and $C(Z(p) \cap \partial\mathbb{B}_n)$ is the set of continuous functions on $Z(p) \cap \partial\mathbb{B}_n$. Consider the operator S_q with $q = p_1 \cdots p_l$, then $S_q \neq 0$. Since $q|_{Z(p) \cap \partial\mathbb{B}_n} = 0$, S_q is compact by the exactness of the above sequence. Hence the map

$$q : \mathcal{L}(N) \rightarrow \mathcal{L}(N)/\mathcal{K}(N)$$

is not completely isometric on the linear span of $\mathcal{B}(S_1, \dots, S_n) \cup \mathcal{B}(S_1^*, \dots, S_n^*)$. Applying Arveson's boundary theorem, Theorem 2.1, we obtain the desired conclusion. \square

Remark 3.2. As in the $L_a^2(\mathbb{D})$ case, one naturally asks for an invariant subspace M of $L_a^2(\mathbb{B}_n)$, when the identity representation of $C^*(R_1, \dots, R_n)$ is a boundary representation for the subalgebra $\mathcal{B}(R_1, \dots, R_n)$, where $R_i = M_{z_i}|_M$ for $1 \leq i \leq n$. The operator tuple (R_1, \dots, R_n) is obviously subnormal. When M is homogeneous (i.e., M is generated by homogeneous polynomials) and essentially normal (i.e., the operator $R_iR_j^* - R_j^*R_i$ is compact for all $1 \leq i, j \leq n$), $\sigma(R_1, \dots, R_n) = \mathbb{B}_n$ by [GW1]. Consequently, if M is homogeneous and essentially normal, the identity representation of $C^*(R_1, \dots, R_n)$ is never a boundary representation for the subalgebra $\mathcal{B}(R_1, \dots, R_n)$ by [GHX, Theorem 3.2].

Remark 3.3. The results in this paper also hold on the weighted Bergman space without modifying the proofs.

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