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STRICTLY SINGULAR OPERATORS ON L_p SPACES AND INTERPOLATION

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Abstract. We study the class V_p of strictly singular non-compact operators on L_p spaces. This allows us to obtain interpolation results for strictly singular operators on L_p spaces. Given $1 \leqslant p < q \leqslant \infty$, it is shown that if an operator T bounded on L_p and L_q is strictly singular on L_r for some $p \leqslant r \leqslant q$, then it is compact on L_s for every $p < s < q$.

1. Introduction

Given Banach spaces E and F, a bounded operator $T : E \to F$ is strictly singular (or Kato) if the restriction of T to any infinite-dimensional subspace of E is not an isomorphism. This class was introduced by T. Kato in [K] as an extension of compact operators and in connection with the perturbation theory of Fredholm operators. Strictly singular operators form a closed operator ideal which in certain aspects behaves in a different way compared to that of compact operators. Thus, in general, strictly singular operators are not stable under duality (cf. [P], [Whi]), they are not suitable for interpolation properties (cf. [B], [H]) and fail to have invariant subspaces ([R]).

However, in the setting of operators on L_p spaces ($1 \leqslant p \leqslant \infty$) the behaviour of strictly singular operators is somehow closer to that of compact operators. For example, concerning endomorphisms on L_p spaces, it is known that an operator $T : L_p \to L_p$ is strictly singular if and only if $T^* : L_p^* \to L_p^*$ is strictly singular. One implication of this result was given by V. Milman in [M], and it was completely proved by L. Weis in [W1]. This same fact for L_1 and $C(K)$ spaces was already known, since in these cases the class of strictly singular operators coincides with that of weakly compact ones (see [P]). Moreover, recall that the square of a strictly singular operator $T : L_p \to L_p$ is always a compact operator ([M]).

The aim of this paper is to study interpolation properties of strictly singular operators on L_p spaces ($1 \leqslant p \leqslant \infty$). In particular, we present an extension of Krasnoselskii's result [Kr] on interpolation of compact operators on L_p spaces. To

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this end, we first study the properties of the class V_p of strictly singular non-compact operators on an L_p space.

As a starting point, we will show that for $p > 2$ strictly singular non-compact operators behave "locally" as inclusions $i_{2,p} : \ell_2 \hookrightarrow \ell_p$, and from this fact some structural properties of the operator class V_p will follow. Thus, in Section 3 we give a version of Kato's result that $\mathcal{S}(L_2) = K(L_2)$ for operators which are simultaneously bounded on different L_p spaces (see Corollary 3.4). This is deduced from an extrapolation type result for strict singularity (see Theorem 3.3). The connection of an operator $T \in V_p$ with boundedness in the scale of L_q spaces will also be explored (see Theorem 3.7).

In Section 4 we present an extension of Krasnoselskii's result on interpolation of compact operators on L_p spaces to strictly singular operators. Namely, we will show that if an operator is strictly singular in L_r and bounded in some L_s for $1 \leqslant r, s \leqslant \infty$, then the operator is compact in L_p for every p strictly between r and s (Theorem 4.2).

2. Preliminaries

In this section we fix the terminology and include some results that will be needed later. A bounded operator $T : E \to F$ between Banach spaces is called *strictly singular* if the restriction of T to any (closed) infinite-dimensional subspace of E is not an isomorphism. Strictly singular operators form a closed operator ideal that contains the ideal of compact operators. It is well-known that an operator $T : E \to F$ is strictly singular if and only if for every infinite-dimensional subspace X of E, there exists another infinite-dimensional subspace Y of X such that the restriction $T|_Y$ is compact (cf. [LT, Prop. 2.c.4]).

We denote by $\mathcal{S}(E)$ and $K(E)$ the sets of strictly singular and compact operators on a Banach space E. It holds that $K(E) \subset \mathcal{S}(E) \subset \mathcal{L}(E)$. In the case when E is a sequence space ℓ_p $(1 \leqslant p < \infty)$ or c_0, it is well-known that the space of all bounded operators $\mathcal{L}(E)$ contains only a unique non-trivial closed two-sided ideal ([C], [GMF]). From this it follows that $K(\ell_p) = \mathcal{S}(\ell_p)$ and $K(c_0) = \mathcal{S}(c_0)$. The simplest examples of strictly singular non-compact operators are the formal inclusion mappings $i_{p,q} : \ell_p \hookrightarrow \ell_q$, with $p < q$.

Given $1 \leqslant p \leqslant \infty$, let L_p denote the function space $L_p[0,1]$ with the Lebesgue measure μ. In [K] Kato showed that for Hilbert spaces strictly singular and compact operators coincide, so $\mathcal{S}(L_2) = K(L_2)$ (this also follows from results about ideals in $\mathcal{L}(\ell_2)$ given in [C]). However, for every $p \neq 2$ it holds that $\mathcal{S}(L_p) \neq K(L_p)$ ([GMF]). We will denote by V_p the class $\mathcal{S}(L_p) \setminus K(L_p)$. Let us recall some well-known examples of operators in the class V_p for $1 \leqslant p \neq 2 \leqslant \infty$.

Let $1 \leqslant q < 2$. Consider a complemented subspace F_q of L_q isomorphic to ℓ_q (generated by disjointly supported functions), and denote by P_q a projection from L_q onto F_q. Let us take the inclusion $i_{q,2}$ and the operator Q defined by $Qx = \sum_{k=1}^{\infty} x_k r_k(t)$, for $x \in \ell_2$, where (r_k) are the Rademacher functions $(r_k(t) = \operatorname{sign} \sin 2^k \pi t)$. By Khintchine's inequality, the operator Q is an isomorphic embedding of ℓ_2 into L_p for every $1 \leqslant p < \infty$. Clearly, the operator $A_q : L_q \to L_q$ given by

$$A_q = Q\, i_{q,2}\, P_q \tag{1}$$

belongs to V_q.

Now, let $2 < p < \infty$. It is well-known that the orthogonal projection R on the span $[r_k]$ acts from L_p $(p > 1)$ into L_2 which is isomorphic to ℓ_2. Consider the inclusion $i_{2,p}$ and denote by j_p an isometric embedding of ℓ_p into L_p. Then the operator $B_p : L_p \to L_p$

$$B_p = j_p \, i_{2,p} \, R \tag{2}$$

belongs to V_p. Note that the operator $A_q \in \mathcal{L}(L_r)$ for every $r \in [q, \infty)$ and the operator $B_p \in \mathcal{L}(L_r)$ for every $r \in (1, p]$.

There also exist strictly singular and non-compact operators in L_∞ and $C(0,1)$. For instance, consider the operator $T : L_\infty \to L_\infty$ given by $T = JR$, where $J : L_2 \to L_\infty$ is an isometric embedding and $R : L_\infty \hookrightarrow L_2$ is the formal inclusion.

Given $1 \leqslant p < \infty$, for each $\varepsilon > 0$ we will consider the Kadeč-Pełczyński sets ([KP]):

$$M_p(\varepsilon) = \{f \in L_p : \mu(\{t : |f(t)| \geqslant \varepsilon \|f\|_p\}) \geqslant \varepsilon\}.$$

Theorem 2.1. *Let X be a subspace of L_p $(1 < p < \infty)$. The following alternative holds:*

1. *If $X \subset M_p(\varepsilon)$ for some $\varepsilon > 0$, then the inclusion $i|_X$ of L_p into L_1 restricted to X is an isomorphism (in this case we say that X is a strongly embedded subspace).*
2. *If $X \not\subseteq M_p(\varepsilon)$ for any $\varepsilon > 0$, then X contains an almost disjoint normalized sequence; that is, there exists a normalized sequence $(x_n) \subset X$ such that $x_n = u_n + v_n$, where (u_n) is a disjoint sequence, $v_n \to 0$ in L_p, and $|u_n| \wedge |v_n| = 0$. In particular, (x_n) can be taken to be equivalent to the unit vector basis of ℓ_p.*

The next result, due to L. Dor [D] (cf. [AO, Theorem 44]), will be used in the proof of Theorem 3.3.

Theorem 2.2. *Let $1 \leqslant p \neq 2 < \infty$, $0 < \theta \leqslant 1$, and $(f_i)_{i=1}^\infty$ in L_p. Assume that either:*

1. *$1 \leqslant p < 2$, $\|f_i\| \leqslant 1$ for all i, and $\|\sum_{i=1}^n a_i f_i\|_p \geqslant \theta (\sum_{i=1}^n |a_i|^p)^{1/p}$ for scalars $(a_i)_{i=1}^n$ and every $n \in \mathbb{N}$, or*
2. *$2 < p < \infty$, $\|f_i\| \geqslant 1$ for all i, and $\|\sum_{i=1}^n a_i f_i\|_p \leqslant \theta^{-1} (\sum_{i=1}^n |a_i|^p)^{1/p}$ for scalars $(a_i)_{i=1}^n$ and every $n \in \mathbb{N}$.*

Then there exist disjoint measurable sets $(A_i)_{i=1}^\infty$ in $[0,1]$ such that

$$\|f_i \, \chi_{A_i}\|_p \geqslant \theta^{2/|p-2|}.$$

A classical interpolation result for compact operators on L_p spaces proved by Krasnoselskii is the following [Kr] (see also [KZPS]).

Theorem 2.3. *Let $1 \leqslant p_0, p_1, q_0, q_1 \leqslant \infty$. If $T : L_{p_0} \to L_{q_0}$ is a compact operator and $T : L_{p_1} \to L_{q_1}$ is bounded, then $T : L_{p_\theta} \to L_{q_\theta}$ is compact, where $\frac{1}{p_\theta} = \frac{1-\theta}{p_0} + \frac{\theta}{p_1}$ and $\frac{1}{q_\theta} = \frac{1-\theta}{q_0} + \frac{\theta}{q_1}$, for every $\theta \in (0,1)$.*

An analogous result for interpolating strictly singular operators does not hold in general. Indeed, consider the formal inclusion $i : L_\infty \to L_1$ which is strictly singular by a result of Grothendieck (cf. [Ru, Theorem 5.2]) and bounded as an operator $i : L_1 \to L_1$. However, for $1 < p < \infty$, $i : L_p \to L_1$ is not strictly singular (since it is an isomorphism on the span of the Rademacher functions). Apparently, positive results for one-sided interpolation of strictly singular operators are only known in the degenerated case when the initial couple reduces to one single space (see [B, Prop. 2.1], [CMMM], [H, Prop. 1.6]).

Recall that an operator T between Banach spaces is compact if and only if its adjoint T^* is compact (Schauder's theorem). This fact is not true in general for strictly singular operators (cf. [P], [Whi]). However, for endomorphisms on L_p spaces we have the following fact due to V. Milman [M] and L. Weis [W1]:

Theorem 2.4. *Let $1 \leqslant p \leqslant \infty$. An operator $T : L_p \to L_p$ is strictly singular if and only if $T^* : L_p^* \to L_p^*$ is strictly singular.*

We refer the reader to the monographs [AA], [G] and [LT] for unexplained terminology.

3. Strictly singular non-compact operators

Let us start with some preliminary results about the operator sets $V_p = \mathcal{S}(L_p) \setminus K(L_p)$, for $1 \leqslant p \neq 2 < \infty$ (recall that $V_2 = \varnothing$). Notice that unlike $K(L_p)$, the space $\mathcal{S}(L_p)$ is not separable; thus neither is V_p.

Lemma 3.1. *Let $2 < p < \infty$. If an operator $T \in V_p$, then there exists a Hilbertian subspace H of L_p which is complemented, such that the restriction $T|_H$ behaves, up to equivalence, like the inclusion $j_p\, i_{2,p}$.*

Proof. We proceed as in [LST, Lemma 2.10]. Since $T \notin K(L_p)$, there exists a sequence (x_k) in L_p, such that $\|x_k\|_p = 1$, $x_k \xrightarrow{w} 0$ and $\|Tx_k\|_p \geqslant \varepsilon$ for some $\varepsilon > 0$. By the Kadeč-Pełczyński theorem [KP] every weakly null seminormalized sequence in L_p contains a subsequence equivalent to the unit vector basis of ℓ_2 or ℓ_p. Applying this theorem to the sequences (x_k) and (Tx_k), we may suppose that (x_k) (resp. (Tx_k)) is equivalent to the unit vector basis of ℓ_q (resp. ℓ_r) where $q, r \in \{2, p\}$.

The cases (i) $q = r = 2$, (ii) $q = r = p$, and (iii) $q = p$, $r = 2$ are impossible. Indeed, the restriction of T on the subspace $[x_k]$ is an isomorphism in the cases (i) or (ii). This contradicts the assumption that $T \in \mathcal{S}(L_p)$. While, if the case (iii) holds, then we clearly have

$$\left\| \sum_{k=1}^{n} x_k \right\|_p \approx n^{\frac{1}{p}} \quad \text{and} \quad \left\| \sum_{k=1}^{n} Tx_k \right\|_p \approx n^{\frac{1}{2}},$$

where the sign $\approx$ means two-side estimates with constants which do not depend on n. Then it follows that

$$\frac{\left\| T\left(\sum\limits_{k=1}^{n} x_k \right) \right\|_p}{\left\| \sum\limits_{k=1}^{n} x_k \right\|_p} \approx n^{\frac{1}{2}-\frac{1}{p}} \to \infty$$

as $n \to \infty$, which contradicts that T is bounded in L_p.

Hence, (x_k) is equivalent to the unit vector basis of ℓ_2 and (Tx_k) is equivalent to the unit vector basis of ℓ_p. And since any Hilbertian subspace in L_p is complemented if $2 < p < \infty$ ([KP]), we have that $[x_n]$ is complemented in L_p. □

We need an improvement of Lemma 3.1. Recall that two measurable functions f and g are equi-measurable if for every $-\infty < s < \infty$ the distribution functions satisfy

$$\mu(\{t : f(t) > s\}) = \mu(\{t : g(t) > s\}).$$

Lemma 3.2. *Let $2 < p < \infty$. If an operator T belongs to V_p, then there exists a sequence (y_k) in L_p with $\|y_k\|_p \leqslant 1$ such that (y_k) is equivalent to the unit vector basis of ℓ_2, the sequence $(|y_k|)$ is equi-measurable, and (Ty_k) is equivalent to the unit vector basis of ℓ_p.*

Proof. By Lemma 3.1 there exists a sequence (x_n) in $L_p, \|x_n\|_p = 1$, $x_n \overset{w}{\to} 0$ such that (x_n) is equivalent to the unit vector basis of ℓ_2 and (Tx_n) is equivalent to the unit vector basis of ℓ_p. Denote by K the basis constant of the sequence (Tx_n). Using [SS, Theorem 3.2] we can choose a subsequence (x_{n_k}) such that $x_{n_k} = u_k + v_k + w_k$, where

(1) $|u_k|$ are equi-measurable; i.e. there exists a function u equi-measurable with $|u_k|$ for any $k \in \mathbb{N}$ and $\|u\|_p \leqslant 1$. Moreover, $u_k \overset{w}{\to} 0$;

(2) $\operatorname{supp} v_i \cap \operatorname{supp} v_j = \varnothing$ for any $i \neq j$ in $\mathbb{N}$, with $\|v_k\|_p \leqslant 2$, and $v_k \overset{w}{\to} 0$;

(3) $\lim_{k\to\infty} \|w_k\|_p = 0$.

It holds that $\lim_{k\to\infty} \|Tv_k\|_p = 0$. Indeed, otherwise we can select a subsequence (v_{i_k}) such that $\inf_k \|Tv_{i_k}\|_p > 0$. By the Kadeč-Pełczyński theorem [KP] some subsequence of (Tv_{i_k}) is equivalent to the unit vector basis of ℓ_2 or ℓ_p. Both cases are impossible because (v_{i_k}) is equivalent to the unit vector basis of ℓ_p (see Lemma 3.1).

Now, since $\lim_{k\to\infty} \|w_k\|_p = 0$ we have that $\lim_{k\to\infty} \|Tw_k\|_p = 0$, and so

$$\lim_{k\to\infty} (\|Tv_k\|_p + \|Tw_k\|_p) = 0.$$

Thus, we can find an increasing sequence of integers (j_k) such that $\|Tv_{j_k}\|_p + \|Tw_{j_k}\|_p < \frac{1}{2^{k+1}K}$. Thus

$$\sum_{k=1}^{\infty} \|Tx_{n_{j_k}} - Tu_{j_k}\|_p \leqslant \sum_{k=1}^{\infty} (\|Tv_{j_k}\|_p + \|Tw_{j_k}\|_p) < \frac{1}{2K}.$$

Hence, by the stability basis result [LT, Thm. 1.a.9], it follows that (Tu_{j_k}) is also equivalent to the unit vector basis of ℓ_p. And, since $u_k \overset{w}{\to} 0$ and $T \in \mathcal{S}(L_p)$, we must have that (u_{j_k}) is equivalent to the unit vector basis of ℓ_2. ☐

We can present now an extrapolation type result for strict singularity:

Theorem 3.3. *Let $1 < q < r < \infty$. If an operator T is bounded in L_q and L_r, and strictly singular in L_p for some $p \in (q, r)$, then T is compact in L_s for all $s \in (q, r)$.*

Proof. Suppose the contrary. By Krasnoselskii's Theorem 2.3, we deduce that T is not compact in L_s for any $s \in (q, r)$. In particular, T is not compact in L_p, and so $T \in V_p$.

Without loss of generality we can assume that $p > 2$. Indeed, for $p = 2$ the result follows directly from the fact that $\mathcal{S}(L_2) = K(L_2)$, while for $p < 2$ it follows from the dual counterpart for the adjoint operator T^*, since by Schauder's theorem and [W1], compact and strictly singular operators on L_p spaces are stable under taking adjoints.

Now, by Lemma 3.2 there exists a sequence (y_k) in L_p such that $(|y_k|)$ is equi-measurable and (Ty_k) is equivalent to the unit vector basis of ℓ_p. By Dor's Theorem 2.2, there exist a constant $c > 0$ and a sequence of disjoint measurable sets $A_k \subset [0, 1]$ such that $\|(Ty_k)\chi_{A_k}\|_p \geqslant c$ for each $k \in \mathbb{N}$.

Since for every $x \in L_p$ we have

$$\lim_{\varepsilon \to 0} \sup_{\mu(A) \leqslant \varepsilon} \|x\chi_A\|_p = 0,$$

and using the fact that $(|y_k|)$ is equi-measurable, we can find $\varepsilon > 0$ such that

$$\|y_k\chi_A\|_p \leqslant \frac{c}{2\|T\|_p}$$

for every $A \subset [0,1]$ with $\mu(A) \leqslant \varepsilon$, and for every $k \in \mathbb{N}$. Moreover, the equi-measurability of $(|y_k|)$ also implies the existence of measurable subsets $B_k \subset [0,1]$ with $\mu(B_k) \geqslant 1-\varepsilon$, such that $y_k\chi_{B_k} \in L_\infty$ and $\|y_k\chi_{B_k}\|_\infty \leqslant y_1^*(\varepsilon)$ for every $k \in \mathbb{N}$. Now, using Hölder's inequality and the fact that $\|y_k\chi_{B_k}\|_r \leqslant \|y_k\chi_{B_k}\|_\infty \leqslant y_1^*(\varepsilon)$ we have

$$\begin{aligned} \|T(y_k)\chi_{A_k}\|_p &\leqslant \|(T(y_k\chi_{B_k}))\chi_{A_k}\|_p + \|T(y_k\chi_{[0,1]\setminus B_k})\|_p \\ &\leqslant \|T(y_k\chi_{B_k})\|_r\|\chi_{A_k}\|_{\frac{pr}{r-p}} + \|T\|_p\|y_k\chi_{[0,1]\setminus B_k}\|_p \\ &\leqslant \|T\|_r\, y_1^*(\varepsilon)\, \mu(A_k)^{\left(\frac{1}{p}-\frac{1}{r}\right)} + \|T\|_p\frac{c}{2\|T\|_p}. \end{aligned}$$

And, since $c \leqslant \|Ty_k\chi_{A_k}\|_p$ and $\mu(A_k) \to 0$ as $k \to \infty$, we obtain $2c \leqslant c$, which is a contradiction. □

The following corollary can be regarded as a version of Kato's result that $K(L_2) = \mathcal{S}(L_2)$ for operators that are simultaneously bounded on different L_p spaces.

Corollary 3.4. *Let $1 < q < r < \infty$, and let T be an operator bounded in L_q and L_r. The following statements are equivalent:*

(i) $T \in K(L_p)$ for some $p \in (q,r)$; *(iii) $T \in \mathcal{S}(L_p)$ for every $p \in (q,r)$;*
(ii) $T \in K(L_p)$ for every $p \in (q,r)$; *(iv) $T \in \mathcal{S}(L_p)$ for some $p \in (q,r)$.*

Proof. $(i) \Rightarrow (ii)$ follows from Krasnoselskii's Theorem 2.3. $(ii) \Rightarrow (iii) \Rightarrow (iv)$ are trivial. $(iv) \Rightarrow (i)$ follows from Theorem 3.3. □

Notice that these facts are no longer true for operators on L_p spaces of infinite measure:

Example 3.1. There exists a strictly singular non-compact operator T on $L_p(0,\infty)$ for every $1 \leqslant p < 2$. Similarly, there exists a strictly singular non-compact operator S on $L_p(0,\infty)$ for every $2 < p < \infty$.

Proof. For $1 \leqslant p < 2$, let $P : L_p(0,\infty) \to \ell_p$ be the operator given by $P(f) = (\int_{n-1}^n f d\mu)_{n=1}^\infty$, and let $Q : \ell_2 \to L_p(0,\infty)$ be the isomorphic embedding via the Rademacher functions in $[0,1]$. Then, $T = Q\, i_{p,2}\, P$ is bounded on $L_p(0,\infty)$ for every $1 \leqslant p \leqslant 2$. Moreover, T is strictly singular for $1 \leqslant p < 2$ since it factors through the inclusion $i_{p,2}$, but it is not compact on any $L_p(0,\infty)$ since the sequence $(\chi_{[n-1,n]})$ has norm one in every $L_p(0,\infty)$ and $T(\chi_{[n-1,n]}) = r_n$ does not have a convergent subsequence.

Similarly, for $2 < p < \infty$, we consider $R : L_p(0,\infty) \to \ell_2$ the projection onto the span of the Rademacher functions on $[0,1]$, and $J : \ell_p \to L_p(0,\infty)$ given by $J(a_n) = \sum_{n=1}^\infty a_n\chi_{[n-1,n]}$. Clearly, the operator $S = J\, i_{2,p}\, R$ is strictly singular and not compact on $L_p(0,\infty)$ for every $2 < p < \infty$. □

As a consequence of Theorem 3.3 we can obtain a result of V. Caselles and M. González [CG] for regular operators (i.e. those which can be written as a difference of positive operators):

Corollary 3.5. *Let* $1 < p < \infty$*, and let* $T : L_p \to L_p$ *be a regular operator. Then* $T \in \mathcal{S}(L_p)$ *if and only if* $T \in K(L_p)$*.*

Proof. Since T is regular, by a result of Weis [W2, Theorem 2.1], there exists a positive isometry $J : L_p \to L_p$, such that the operator $JTJ^{-1} : L_q \to L_q$ is bounded for every $1 \leqslant q \leqslant \infty$. Hence, since $JTJ^{-1} : L_p \to L_p$ is strictly singular, by Theorem 3.3, we have that JTJ^{-1} belongs to $K(L_p)$. Now, since J is an isometry we have that T belongs to $K(L_p)$. □

Notice that this result is no longer true for $p = 1$. Indeed, let $T : L_1 \to L_1$ be given by $T = Q\, i_{1,2}\, P$, where P is a projection onto some subspace isomorphic to ℓ_1 and $Q : \ell_2 \to L_1$ is the isomorphic embedding via the Rademacher functions. Clearly, T belongs to the set V_1 and is a regular operator like every operator in L_1 (cf. [AA, Theorem 3.9]).

It was proved by V. Milman in [M] that the composition of two strictly singular operators on L_p is compact. We present below a converse to this result.

Proposition 3.6. *Let* $1 < p \neq 2 < \infty$*. Given an operator* $R \in \mathcal{L}(L_p)$*, it holds that* $R \in \mathcal{S}(L_p)$ *if and only if* RT *and* TR *are compact for every* $T \in \mathcal{S}(L_p)$*.*

Proof. The "if" part was proved in [M]. Suppose $p > 2$ and $R \notin \mathcal{S}(L_p)$. Then there exists a subspace Q of L_p, such that the restriction $R|_Q$ is an isomorphism, and by Theorem 2.1, we can suppose that Q is isomorphic to ℓ_2 or ℓ_p and complemented in L_p.

(1) If $Q \approx \ell_2$, then we can consider an operator $T \in \mathcal{L}(L_p)$ defined as follows. Since $R(Q)$ is isomorphic to ℓ_2 and complemented, there is a projection $P : L_p \to R(Q)$. Now, take an isomorphic embedding $J : \ell_p \to L_p$ and define $T = J\, i_{2,p}\, P$. Clearly, there exists a sequence (x_n) in Q, equivalent to the unit vector basis to ℓ_2, such that $TR(x_n)$ does not have any convergent subsequence. Hence, TR is not compact, which is a contradiction.

(2) If $Q \approx \ell_p$, then we consider a projection $P : L_p \to H$ onto some Hilbert subspace of L_p, and the isomorphic embedding J of ℓ_p into $Q \subset L_p$. Hence, if we consider the operator $T = J\, i_{2,p}\, P$, then RT is not compact, which is again a contradiction.

This proves the statement for $p > 2$. By duality arguments (Theorem 2.4) the same fact is proved for $p < 2$. □

Note that the assumption in Proposition 3.6 that RT and TR are compact cannot be relaxed to only one condition RT (or respectively TR) being compact for every $T \in \mathcal{S}(L_p)$.

Let $1 \leqslant p \neq q \leqslant \infty$, and let $T : L_p \to L_p$ be a bounded operator. If $q > p$, then T is also defined acting from L_q. If $q < p$, then T is defined on a dense subset of L_q. Thus, in both cases we can consider the quantity $\|T\|_q$ taking values in $[0, +\infty]$, and we can analyze the boundedness or unboundedness of T from L_q to L_q. Let us denote

$$O(T) = \{q \in [1, +\infty] : T \text{ is bounded in } L_q\}.$$

It follows from M. Riesz's interpolation result that $O(T)$ is a convex subset of $[1, +\infty]$, which may or may not contain its endpoints.

Theorem 3.7. *Let $1 < p < \infty$. If an operator $T \in V_p$, then p is an endpoint of $O(T)$. Moreover, p is the right (respectively left) endpoint of $O(T)$ when $p > 2$ (resp. $p < 2$).*

Proof. It follows from Theorem 3.3 that p is always an endpoint of $O(T)$.

First consider the case $p > 2$. By Lemma 3.2, there exists a sequence (x_k) in L_p which is equivalent to the unit vector basis of ℓ_2 and with $(|x_k|)$ equi-measurable, such that (Tx_k) is equivalent to the unit vector basis of ℓ_p. Actually, since $(|x_k|)$ is equi-measurable and $\|Tx_k\|_p \geqslant \alpha$ for some $\alpha > 0$, we can truncate (x_k) considering $y_k = x_k \chi_{\{|x_k| \leqslant M\}}$. Since

$$\lim_{M\to\infty} \sup_k \|x_k \chi_{\{|x_k|>M\}}\|_p = 0,$$

then for large enough M, we have $\|Ty_k\|_p \geqslant \frac{\alpha}{2}$ for all $k \in \mathbb{N}$. Now, as in the proof of Lemma 3.1, by Theorem 2.1, we have that the sequence (y_k) is equivalent to the unit vector basis of ℓ_2 and (Ty_k) is equivalent to the unit vector basis of ℓ_p.

Now, suppose that p is not the right endpoint of $O(T)$; that is, $T : L_q \to L_q$ is also bounded for some $q > p$. Since (y_k) is also in L_q, and $\|Ty_k\|_q \geqslant \|Ty_k\|_p \geqslant \frac{\alpha}{2}$, by Theorem 2.1, we have that (y_k) is equivalent in L_q to the unit vector basis of ℓ_2 and (Ty_k) is equivalent to the unit vector basis of ℓ_q. However, this yields

$$C_1 n^{\frac{1}{p}} \leqslant \Big\| \sum_{k=1}^n Ty_k \Big\|_p \leqslant \Big\| \sum_{k=1}^n Ty_k \Big\|_q \leqslant C_2 n^{\frac{1}{q}},$$

for certain constants $C_1, C_2 > 0$ and every $n \in \mathbb{N}$. This is a contradiction since $q > p$.

The case when $p < 2$ follows by duality. Indeed, if $T \in V_p$, then by Theorem 2.4, we have $T^* \in V_{p'}$, where $\frac{1}{p} + \frac{1}{p'} = 1$. Since $p' > 2$, by the first part of the proof we have that p' is the right endpoint of $O(T^*)$, which means that p is the left endpoint of $O(T)$. This finishes the proof. □

The examples of operators in V_p presented above always depend on the scalar p. The following result explains this phenomenon.

Proposition 3.8. *Let $1 < q < p < \infty$. The set $V_q \cap V_p$ is not empty if and only if $q < 2 < p$.*

Proof. Let $1 < q < 2 < p < \infty$. Let us consider the operators A_q and B_p defined above in (1) and (2). Also, consider the following operators acting on functions on $[0, 1]$:

$$\begin{cases} Ux(t) = x(2t), & 0 \leqslant t \leqslant \frac{1}{2}, \\ Wx(t) = x(2t-1), & \frac{1}{2} \leqslant t \leqslant 1. \end{cases}$$

Then the operators UA_qU^{-1} and WB_pW^{-1} act in the corresponding function spaces on $\left[0, \frac{1}{2}\right]$ and $\left[\frac{1}{2}, 1\right]$ respectively. Given a measurable function x on $[0, 1]$, denote $x = y + z$, where y and z are the restriction of x on $\left[0, \frac{1}{2}\right]$ and $\left[\frac{1}{2}, 1\right]$, and define the operator

$$T_{p,q}(x) = UA_qU^{-1}(y) + WB_pW^{-1}(z).$$

Since $A_q \in \mathcal{L}(L_r)$ for any $r \in [q, \infty)$ and $B_p \in \mathcal{L}(L_r)$ for any $r \in (1, p]$, $T_{p,q} \in \mathcal{L}(L_r)$ for any $r \in [q, p]$. Moreover, $A_q \in V_q$ and $B_p \in V_p$ clearly imply that $T_{p,q} \in V_q \cap V_p$.

Let us prove the converse. If $T \in V_p$ and $1 < p < 2$, then by Theorem 3.7, T does not belong to $\mathcal{L}(L_q)$. Similarly, if $T \in V_q$ and $q > 2$, then, by Theorem 3.7, $T \notin \mathcal{L}(L_p)$. □

4. INTERPOLATION OF STRICTLY SINGULAR OPERATORS

Let us denote by P_A the operator of multiplication by the characteristic function of a measurable set A, i.e. $P_A x(t) = x(t)\chi_A(t)$. Notice that $\|P_A\|_{L_p} = 1$ for every $A \subset [0,1]$ with positive measure and every $1 \leqslant p \leqslant \infty$.

Proposition 4.1. *Let $1 \leqslant p \leqslant \infty$ and let $T : L_p \to L_p$ be an operator which is not an isomorphism when restricted to any subspace isomorphic to ℓ_p (c_0 when $p = \infty$). Then for every sequence of disjoint measurable sets (A_n) the following holds:*

1. *If $2 \leqslant p \leqslant \infty$, then $\lim\limits_{n\to\infty} \|TP_{A_n}\|_{L_p} = 0$.*
2. *If $1 \leqslant p \leqslant 2$, then $\lim\limits_{n\to\infty} \|P_{A_n}T\|_{L_p} = 0$.*

Proof. Let us first prove the case (1). Suppose the contrary. Then there exist $\alpha > 0$, $x_n \in L_p$, and pairwise disjoint sets $A_n \subset [0,1]$ such that $\|x_n\|_{L_p} \leqslant 1$, $\mathtt{supp}(x_n) \subset A_n$, and $\|Tx_n\|_{L_p} \geqslant \alpha$ for every $n \in \mathbb{N}$.

Let $p = \infty$. As (x_n) is semi-normalized and disjoint, (x_n) is equivalent to the unit vector basis of c_0. In particular, (Tx_n) is weakly null and semi-normalized; hence it has a basic subsequence (Tx_{n_k}). This yields that there exist constants c, C such that for every scalar sequence $(a_k)_{k=1}^n$ it holds that

$$c \sup_{1\leqslant k\leqslant n} |a_k| \leqslant \Big\| \sum_{k=1}^n a_k T x_{n_k} \Big\|_{L_\infty} \leqslant \|T\| \Big\| \sum_{k=1}^n a_k x_{n_k} \Big\|_{L_\infty} \leqslant C \sup_{1\leqslant k\leqslant n} |a_k|,$$

which is a contradiction to the fact that T is not an isomorphism on any subspace isomorphic to c_0.

Similarly, if $p = 2$, both (x_n) and (Tx_n) are weakly null semi-normalized sequences; hence extracting subsequences we can assume that both are equivalent to the unit vector basis of ℓ_2. Again we obtain a contradiction.

Now, suppose $2 < p < \infty$. In this case, (x_n) is equivalent to the unit vector basis of ℓ_p. And, since $\alpha \leqslant \|Tx_n\|_{L_p} \leqslant \|T\|_{L_p}$ for every $n \in \mathbb{N}$ and $Tx_n \to 0$ weakly, we have, by [KP, Corollary 5], that there exists an increasing sequence $(n_k) \subset \mathbb{N}$ such that (Tx_{n_k}) is equivalent to the unit vector basis of ℓ_2 or ℓ_p. Both cases will lead to a contradiction. Indeed, in the first case we would have

$$n^{\frac{1}{2}} \approx \Big\| \sum_{k=1}^n T x_{n_k} \Big\|_{L_p} \leqslant \|T\|_{L_p} \Big\| \sum_{k=1}^n x_{n_k} \Big\|_{L_p} \approx \|T\|_{L_p}\, n^{\frac{1}{p}},$$

which is impossible for large $n \in \mathbb{N}$. In the second case, the sequences (Tx_{n_k}) and (x_{n_k}) are both equivalent to the unit vector basis of ℓ_p. Hence, the operator T is an isomorphism on the span $[x_{n_k}]$ in contradiction with the assumption on T. This finishes the proof of case (1).

To prove (2), we will proceed by duality. First, notice that for $1 \leqslant p \leqslant 2$, if an operator $T : L_p \to L_p$ is not an isomorphism on any subspace isomorphic to ℓ_p, then $T^* : L_p^* \to L_p^*$ is not an isomorphism on a subspace isomorphic to ℓ_p^*. Indeed, suppose that T^* is invertible in a subspace X of L_p^* isomorphic to ℓ_p^*. Then as $p \leqslant 2$ it follows that X and $T^*(X)$ are complemented and isomorphic to ℓ_p^* [KP]. This implies that T^{**} is also invertible in a subspace isomorphic to ℓ_p. In the case $1 < p$, since $T = T^{**}$, the claim is proved. Now, for $p = 1$ recall that if $T : L_1 \to L_1$ is not an isomorphism on a subspace isomorphic to ℓ_1, then T is weakly compact and in particular $T^{**}(L_1) \subseteq L_1$. This proves the claim.

Therefore, by the case (1), we get that $\lim\limits_{n\to\infty} \|T^* P_{A_n}\|_{L_p^*} = 0$ for every disjoint sequence (A_n) in $[0,1]$. And, since $(P_A)^* = P_A$, we obtain

$$\lim_{n\to\infty} \|P_{A_n} T\|_{L_p} = \lim_{n\to\infty} \|T^* P_{A_n}^*\|_{L_p^*} = 0.$$

□

Theorem 4.2. *Let $1 \leqslant r, s \leqslant \infty$, $r \neq s$, and T be an operator bounded on L_s. If $T \in \mathcal{S}(L_r)$, then $T \in K(L_p)$ for every p between r and s.*

Proof. Let us prove first the case $r < \infty$. By Theorem 3.3, it is enough to show that $T \in \mathcal{S}(L_p)$ for some p strictly between r and s. So, let us suppose that $T \notin \mathcal{S}(L_p)$ for any $p \neq 2$. Thus, for every p between r and s, T is an isomorphism on a subspace X_p of L_p which, by [W1], can be taken to be isomorphic either to ℓ_2 or ℓ_p, with both subspaces X_p and $T(X_p)$ complemented in L_p. We distinguish two cases:

(A) Suppose that for some p the subspace X_p is isomorphic to ℓ_2. Let us denote $X = X_p$. Then, by Theorem 2.1, both X and $T(X)$ are strongly embedded subspaces of L_p. Thus, we can distinguish two subcases:

1. If $r < p$, then X and $T(X)$ are also closed subspaces of L_r and isomorphic to ℓ_2 in the norm of L_r. This gives a contradiction to the fact that $T \in \mathcal{S}(L_r)$.
2. If $r > p$, then, since X and $T(X)$ are complemented in L_p, it follows that $T^* : L_{p'} \to L_{p'}$ $(\frac{1}{p} + \frac{1}{p'} = 1)$ is an isomorphism on a complemented subspace Z of $L_{p'}$ isomorphic to ℓ_2. Using again Theorem 2.1, we have that Z and $T^*(Z)$ must be strongly embedded in $L_{p'}$. Now since $r' < p'$, as in case (a), this yields that $T^* : L_{r'} \to L_{r'}$ $(\frac{1}{r} + \frac{1}{r'} = 1)$ is also an isomorphism on a subspace isomorphic to ℓ_2. Now, by [PR, Thm. 3.1], every such subspace contains another complemented subspace, so we get that $T^{**} = T : L_r \to L_r$ is an isomorphism on a subspace isomorphic to ℓ_2. This is a contradiction to the fact that $T \in \mathcal{S}(L_r)$.

(B) Otherwise, suppose that for every p between r and s the subspace X_p is isomorphic to ℓ_p. Then the subspaces X_p and $T(X_p)$ are not included in $M_p(\varepsilon)$ for any $\varepsilon > 0$. Now, assume first $r > 2$, hence we can fix some $p > 2$ between r and s. By Theorem 2.1, we can find a sequence $(x_n) \subset X_p$, such that $\|x_n\|_{L_p} = 1$, $x_n = u_n + v_n$, where (u_n) is a disjoint sequence in L_p and $\lim_{n\to\infty} \|v_n\|_{L_p} = 0$. Hence, we can suppose that the operator T is an isomorphism on the subspace $[u_k]$. In particular there exists a constant $c > 0$ such that $\|T(u_n)\|_{L_p} \geqslant c\|u_n\|_{L_p}$ for every $n \in \mathbb{N}$. Now, let us denote $A_n = \mathtt{supp}(u_n)$ and let $\theta \in (0,1)$ such that $\frac{1}{p} = \frac{1-\theta}{r} + \frac{\theta}{s}$. By the Riesz interpolation theorem we have that

$$\|TP_{A_n}\|_{L_p} \leqslant \|TP_{A_n}\|_{L_r}^{1-\theta} \, \|TP_{A_n}\|_{L_s}^{\theta} \leqslant \|TP_{A_n}\|_{L_r}^{1-\theta} \, \|T\|_{L_s}^{\theta}.$$

Since $\lim_{n\to\infty} \mu(A_n) = 0$, we have, by Proposition 4.1, $\lim\limits_{n\to\infty} \|TP_{A_n}\|_{L_r} = 0$. Therefore, $\lim\limits_{n\to\infty} \|TP_{A_n}\|_{L_p} = 0$. However, we have that

$$\|TP_{A_n}\|_{L_p} \geqslant \frac{\|TP_{A_n}(u_n)\|_{L_p}}{\|u_n\|_{L_p}} = \frac{\|T(u_n)\|_{L_p}}{\|u_n\|_{L_p}} \geqslant c > 0,$$

which is a contradiction.

The proof when $r < 2$ is analogous. Indeed, in this case we can fix some $p < 2$, and by Theorem 2.1, we can find an almost disjoint normalized sequence (y_n) in $T(X_p)$; that is, $y_n = u_n + v_n$, where (u_n) is a disjoint sequence in L_p, $\lim_{n\to\infty} \|v_n\|_{L_p} = 0$ and $|u_n| \wedge |v_n| = 0$ for every $n \in \mathbb{N}$. Moreover, $y_n = T(x_n)$ for some semi-normalized sequence (x_n) in X_p. As in the previous case, if we denote $A_n = \mathtt{supp}(u_n)$, then we have

$$\|P_{A_n}T\|_{L_p} \geqslant \frac{\|P_{A_n}T(x_n)\|_{L_p}}{\|x_n\|_{L_p}} = \frac{\|u_n\|_{L_p}}{\|x_n\|_{L_p}} \geqslant \alpha$$

for some $\alpha > 0$ and n large enough, because $\|v_n\| \to 0$. However, by the Riesz interpolation theorem, we have

$$\|P_{A_n}T\|_{L_p} \leqslant \|P_{A_n}T\|_{L_r}^{1-\theta}\|P_{A_n}T\|_{L_s}^{\theta} \leqslant \|P_{A_n}T\|_{L_r}^{1-\theta}\|T\|_{L_s}^{\theta},$$

for the corresponding $\theta \in (0,1)$. Then apply Proposition 4.1 to conclude.

This finishes the proof for $r < \infty$. The case $r = \infty$ follows by duality. Indeed, if $T : L_\infty \to L_\infty$ is strictly singular and bounded on L_s for some $1 < s < \infty$, then $T^* : L_\infty^* \to L_\infty^*$ is strictly singular and bounded on $L_{s'}$ (with $\frac{1}{s} + \frac{1}{s'} = 1$). Therefore, we have

$$T^*(L_1) = T^*(\overline{L_{s'}}^{\|\,\|_{L_1^{**}}}) \subseteq \overline{T^*(L_{s'})}^{\|\,\|_{L_1^{**}}} \subseteq \overline{L_{s'}}^{\|\,\|_{L_1^{**}}} = L_1.$$

In particular, the operator $T^*|_{L_1} : L_1 \to L_1$ is also strictly singular. Now, by the previous part of the proof we conclude that $T^* \in K(L_q)$ for every q between 1 and s'. Hence, by Schauder's theorem, the operator $T \in K(L_p)$ for every $s < p < \infty$. $\square$

References

[AA] Y.A. Abramovich, C. D. Aliprantis. An invitation to operator theory. Graduate Studies in Mathematics, 50. American Mathematical Society, Providence, RI, 2002. MR1921782 (2003h:47072)

[AO] D. Alspach, E. Odell. L_p spaces. Handbook of the geometry of Banach spaces, Vol. I, 123–159, North-Holland, Amsterdam, 2001. MR1863691 (2003b:46013)

[B] O. J. Beucher. On interpolation of strictly (co-)singular linear operators. Proc. Roy. Soc. Edinb. A 112 (1989), 263–269. MR1014656 (90j:46064)

[C] J. W. Calkin. Two-sided ideals and congruences in the ring of bounded operators in Hilbert space. Annals of Math. (2) 42, 4 (1941), 839–873. MR0005790 (3:208c)

[CG] V. Caselles, M. González. Compactness properties of strictly singular operators in Banach lattices. Semesterbericht Funktionalanalysis. Tübingen, Sommersemester (1987), 175–189.

[CMMM] F. Cobos, A. Manzano, A. Martínez, P. Matos. On interpolation of strictly singular operators, strictly co-singular operators and related operator ideals. Proc. Roy. Soc. Edinb. A 130 (2000), 971–989. MR1800087 (2001j:46113)

[D] L. Dor. On projections in L_1. Annals of Math. (2) 102 (1975), 463–474. MR0420244 (54:8258)

[GMF] I. T. Gohberg, A. S. Markus, I. A. Feldman. On normal solvable operators and related ideals. Amer. Math. Soc. Transl. (2), Vol. 61, Amer. Math. Soc., Providence, RI, 1967, 63–84.

[G] S. Goldberg. Unbounded linear operators. Theory and applications. Dover Publications, Mineola, NY, 2006. MR2446016

[H] S. Heinrich. Closed operator ideals and interpolation. J. Funct. Analysis 35 (1980), 397–411. MR563562 (81f:47045)

[KP] M. I. Kadeč, A. Pełczyński. Bases, lacunary sequences and complemented subspaces in the spaces L_p. Studia Math. 21 (1962), 161–176. MR0152879 (27:2851)

[K] T. Kato. Perturbation theory for nullity deficiency and order quantities of linear operators. J. Analyse. Math. 6 (1958), 273–322. MR0107819 (21:6541)

[Kr] M. A. Krasnoselʹskiĭ. On a theorem of M. Riesz. Dokl. Akad. Nauk SSSR 131, 246–248 (in Russian); translated as Soviet Math. Dokl. 1 (1960), 229–231. MR0119086 (22:9852)

[KZPS] M. A. Krasnoselʹskiĭ, P. P. Zabreĭko, E. I. Pustylʹnik, P. E. Sobolevskiĭ. Integral operators in spaces of summable functions. Noordhoff International Publishing, Leiden, 1976. MR0385645 (52:6505)

[LT] J. Lindenstrauss, L. Tzafriri. Classical Banach spaces. I. Springer-Verlag, Berlin-New York, 1977. MR0500056 (58:17766)

[LST] M. Lindström, E. Saksman, H.-O. Tylli. Strictly singular and cosingular multiplications. Canad. J. Math. 57 (2005), 1249–1278. MR2178561 (2006h:47058)

[M] V. D. Milman. Operators of classes C_0 and C_0^*. Functions theory, functional analysis and appl., Vol. 10 (1970), 15–26 (in Russian).

[P] A. Pełczyński. On strictly singular and strictly cosingular operators, I and II. Bull. Acad. Polon. Sci. Sér. Sci. Math. Astronom. Phys. 13 (1965), 31–41. MR0177300 (31:1563)

[PR] A. Pełczyński, H. P. Rosenthal. Localization techniques in L^p spaces. Studia Math. 52 (1974/75), 263–289. MR0361729 (50:14174)

[R] C. J. Read. Strictly singular operators and the invariant subspace problem. Studia Math. 132 (1999), 203–226. MR1669678 (2000e:47012)

[Ru] W. Rudin. Functional analysis. McGraw-Hill, New York, 1973. MR0365062 (51:1315)

[SS] E. M. Semënov, F. A. Sukochev. The Banach–Saks index. Sbornik Mathematics 195 (2004), 263–285. MR2068953 (2005d:46063)

[W1] L. Weis. On perturbations of Fredholm operators in $L_p(\mu)$-spaces. Proc. Amer. Math. Soc. 67 (1977), 287–292. MR0467377 (57:7236)

[W2] L. Weis. Integral operators and changes of density. Indiana Univ. Math. J. 31 (1982), 83–96. MR642619 (84a:47066)

[Whi] R. J. Whitley. Strictly singular operators and their conjugates. Trans. Amer. Math. Soc. 113 (1964), 252–261. MR0177302 (31:1565)

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