PROCEEDINGS OF THE
AMERICAN MATHEMATICAL SOCIETY
Volume 138, Number 2, February 2010, Pages 655–661
S 0002-9939(09)10092-8
Article electronically published on October 6, 2009

REMOVABLE SETS FOR THE FLUX OF CONTINUOUS VECTOR FIELDS

SÉBASTIEN DE VALERIOLA AND LAURENT MOONENS

(Communicated by Tatiana Toro)

Abstract. We show that any closed set E having a σ-finite $(n-1)$-dimensional Hausdorff measure does *not* support the nonzero distributional divergence of a continuous vector field; in particular it has the property that any C^1 function in $\mathbb{R}^n$ that is harmonic outside it is harmonic in $\mathbb{R}^n$. We also exhibit a compact set E having Hausdorff dimension $n-1$, supporting the nonzero distributional divergence of a continuous vector field yet having the property that any C^1 function that is harmonic outside E is harmonic in $\mathbb{R}^n$.

Introduction

The problem of removing singularities for the equation $\operatorname{div} v = 0$ can be stated as follows: given a collection $\mathcal{B}$ of locally bounded, measurable vector fields in $\mathbb{R}^n$, give conditions on a compact set E guaranteeing that each vector field $v \in \mathcal{B}$ satisfying $\operatorname{div} v = 0$ outside E (in the distributional sense) satisfies $\operatorname{div} v = 0$ in $\mathbb{R}^n$ (in the distributional sense, too).

This problem has been studied for various classes of functions. When $\mathcal{B}$ is the collection of all (locally) bounded, measurable vector fields, the compact removable sets are *exactly* those having zero $(n-1)$-dimensional Hausdorff measure (see [2], [8] and [11]). In this short paper, we focus on the case when $\mathcal{B}$ is the collection of all *continuous* vector fields in $\mathbb{R}^n$. Theorem 5.2 (second part) in [11] states that removable sets in this context have Hausdorff dimension at most $n-1$. On the other hand, it is a rather easy observation (see [9, Lemma 8.18] for example) that any compact set E satisfying $\mathcal{H}^{n-1}(E) < \infty$ is removable. In this paper, we extend this property to closed sets having σ-finite $(n-1)$-dimensional Hausdorff measure (Theorem 12). In particular, each function u that is C^1 in $\mathbb{R}^n$ and harmonic outside a closed set having σ-finite $(n-1)$-dimensional Hausdorff measure is harmonic in $\mathbb{R}^n$ (apply the removability property to the vector field $v = \nabla u$).

We then show that the dimension condition does *not* characterize removable sets. In particular, we exhibit (Theorem 13) a compact set E that has Hausdorff dimension $n-1$, is not removable, yet satisfies the following condition: any C^1 function $u : \mathbb{R}^n \to \mathbb{R}$ harmonic outside E is harmonic in $\mathbb{R}^n$.

Received by the editors January 7, 2009, and, in revised form, June 10, 2009.
2000 *Mathematics Subject Classification.* Primary 49Q15; Secondary 35B60.
The second author is an *aspirant* of the Fonds de la Recherche scientifique — FNRS (Belgium).

1. Preliminaries and notation

In the sequel, we work in Euclidean space $\mathbb{R}^n$, $n \geqslant 1$. The Euclidean norm of $x = (x_1, \ldots, x_n) \in \mathbb{R}^n$ is denoted $|x|$; for $r > 0$ let $B[x,r] = \{y \in \mathbb{R}^n : |x-y| \leqslant r\}$ and $Q[x,r] = \{y \in \mathbb{R}^n : |y_i - x_i| \leqslant r, 1 \leqslant i \leqslant n\}$. We write $B[r]$ and $Q[r]$ instead of $B[0,r]$ and $Q[0,r]$, respectively. A *cube* is a set of the form $Q[x,r]$ for $x \in \mathbb{R}^n$ and $r > 0$.

Given $E \subseteq \mathbb{R}^n$, we will write $E \subset\subset U$ to indicate that the closure of E is a compact subset of the open set $U \subseteq \mathbb{R}^n$. We further denote by $\mathcal{L}^n(E)$ or $|E|$ the Lebesgue (outer) measure of E; in particular we define $\boldsymbol{\alpha}(n) := |B[1]|$. The notation $d(E)$ stands for the *diameter* of E, and for $0 \leqslant s \leqslant n$ we let $\mathcal{H}^s(E)$ (resp. $\mathcal{S}^s(E)$) denote the (normalized by $\boldsymbol{\alpha}(s)2^{-s}$) s-dimensional Hausdoff measure (resp. the s-dimensional spherical Hausdorff measure) of E. One has $\mathcal{H}^s(E) \leqslant \mathcal{S}^s(E) \leqslant 2^s\mathcal{H}^s(E)$, as is easily checked. We also say that E is *thin* in case $\mathcal{H}^{n-1} \llcorner E$ is σ-finite.

Given $f : X \to Y$ and $A \subseteq X$, we let $f \restriction A$ denote the *restriction* of f to A. The restriction of an outer measure μ on $\mathbb{R}^n$ to $E \subseteq \mathbb{R}^n$ is the outer measure $\mu \llcorner E$ defined by $\mu \llcorner E(B) := \mu(B \cap E)$.

We denote by $\|f\|_\infty$ and $\|f\|_1$ the L^∞- and L^1-norms of $f : \mathbb{R}^n \to \mathbb{R}^m$, respectively. Given $f \in L^1(\mathbb{R}^n)$, we denote by $\|Dg\|$ its total variation. For $0 \leqslant k \leqslant \infty$, we give $C^k(\mathbb{R}^n, \mathbb{R}^m)$, $C^k_c(\mathbb{R}^n, \mathbb{R}^m)$ and $C^k(\mathbb{R}^n)$, $C^k_c(\mathbb{R}^n)$ their usual meanings.

2. Charges and fluxes

The *flux* of any (locally integrable) $v : \mathbb{R}^n \to \mathbb{R}^n$ is the distribution $F_v : C_c^\infty(\mathbb{R}^n) \to \mathbb{R}$ defined by $F_v(\varphi) = -\int_{\mathbb{R}^n} v(x) \cdot \nabla\varphi(x)\,dx$. We observe an interesting continuity property of the flux of a *continuous* vector field.

Proposition 1. *Given $v \in C^0(\mathbb{R}^n, \mathbb{R}^n)$, $\varepsilon > 0$, $r > 0$ and $C > 0$, there exists $\delta > 0$ such that $F_v(\varphi) \leqslant \varepsilon$ provided $\varphi \in C_c^\infty(\mathbb{R}^n)$ satisfies $\{\varphi \neq 0\} \subseteq B[r]$, $\|\nabla\varphi\|_1 \leqslant C$ and $\|\varphi\|_1 \leqslant \delta$.*

Proof. Fix $v \in C^0(\mathbb{R}^n, \mathbb{R}^n)$, $\varepsilon > 0$ and choose $w \in C^1_c(\mathbb{R}^n, \mathbb{R}^n)$ satisfying $|v(x) - w(x)| \leqslant \varepsilon/2C$ for each $x \in B[r]$. Let $\theta := 1 + \|\operatorname{div} w\|_\infty$ and say $\delta := \varepsilon/(2\theta)$. For $\varphi \in C_c^\infty(\mathbb{R}^n)$ satisfying $\{\varphi \neq 0\} \subseteq B[r]$, $\|\nabla\varphi\|_1 \leqslant C$ and $\|\varphi\|_1 \leqslant \delta$, compute using the integration by parts formula:

$$F_v(\varphi) = \int_{\mathbb{R}^n} [v(x) - w(x)] \cdot \nabla\varphi(x)\,dx + \int_{\mathbb{R}^n} \varphi(x)\ \operatorname{div} w(x)\,dx \leqslant \frac{\varepsilon}{2C}\|\nabla\varphi\|_1 + \theta\|\varphi\|_1 \leqslant \varepsilon.$$

The proof is complete. □

Remark 2. According to the approximation property of BV functions (see [5, Section 5.2.2]), it is obvious that given $v \in C^0(\mathbb{R}^n, \mathbb{R}^n)$, F_v extends to the linear functional

$$F_v : BV_c(\mathbb{R}^n) \to \mathbb{R}; g \mapsto -\int_{\mathbb{R}^n} v \cdot d[Dg]$$

on the space $BV_c(\mathbb{R}^n)$ of compactly supported functions having bounded variation in $\mathbb{R}^n$. Furthermore for any $\varepsilon > 0$, $r > 0$ and $C > 0$ there exists $\delta > 0$ such that $F(g) \leqslant \varepsilon$ provided $g \in BV_c(\mathbb{R}^n)$ satisfies $\{g \neq 0\} \subseteq B[r]$, $\|Dg\| \leqslant C$ and $\|g\|_1 \leqslant \delta$.

Given $r > 0$ and $C > 0$, let

$$BV_{r,C} := \{g \in BV_c(\mathbb{R}^n) : \operatorname{supp} g \subseteq B[r], \|Dg\| \leqslant C\}$$

and denote by $\mathfrak{B}$ the collection of all absorbing, balanced convex sets $\mathbf{U} \subseteq BV_c(\mathbb{R}^n)$ having the property that for all $r > 0$ and $C > 0$, the set $\mathbf{U} \cap BV_{r,C}$ is a relative L^1-neigborhood of g in $BV_{r,C}$. According to [3, Theorem 3.3], for example, $\mathfrak{B}$ is a neighborhood basis at 0 for a locally convex topology $\mathfrak{T}$ on $BV_c(\mathbb{R}^n)$ which is characterized by the following two properties:

(A) $\mathfrak{T}$ is weaker than the L^1 topology on $BV_{r,C}$ for all $r > 0, C > 0$;
(B) for any locally convex space X, a linear map $f : BV_c(\mathbb{R}^n) \to X$ is $\mathfrak{T}$-continuous whenever the restriction $f \restriction BV_{r,C}$ is L^1-continuous on $BV_{r,C}$ for all $r > 0, C > 0$.

Given $v \in C^0(\mathbb{R}^n, \mathbb{R}^n)$, condition (B) together with Remark 2 implies in particular that F_v is $\mathfrak{T}$-continuous on $BV_c(\mathbb{R}^n)$.

Definition 3. A $\mathfrak{T}$-continuous linear functional F on $BV_c(\mathbb{R}^n)$ is called a *charge*.

The following theorem, due to T. De Pauw and W.F. Pfeffer [4], is very deep and identifies fluxes and charges.

Theorem 4 (De Pauw-Pfeffer)**.** *Given a charge F, there exists a vector field $v \in C^0(\mathbb{R}^n, \mathbb{R}^n)$ for which $F = F_v$.*

On the other hand, N.C. Phuc and M. Torres [11] give a sufficient condition for a Radon measure to define a charge.

Theorem 5. *Assume that μ is a Radon measure in $\mathbb{R}^n$ and that for each compact set $K \subseteq \mathbb{R}^n$ one has*

$$\lim_{\delta \to 0} \sup_{0<r<\delta} \sup_{x \in K} \frac{\mu(B[x,r])}{r^{n-1}} = 0.$$

Then the linear functional

$$C_c^\infty(\mathbb{R}^n) \to \mathbb{R}, \varphi \mapsto \int_{\mathbb{R}^n} \varphi \, d\mu$$

extends in a unique way to a charge.

3. Removing thin sets

The following lemma will be useful in the sequel.

Lemma 6. *Assume that $E \subseteq \mathbb{R}^n$ is bounded and satisfies $\mathcal{H}^{n-1}(E) < \infty$. For all $\varepsilon > 0$ and $\eta > 0$, there exists an open set U containing E and satisfying the following conditions:*

(A) $|U| \leqslant \varepsilon$,
(B) $\|D\chi_U\| \leqslant \eta + \boldsymbol{\beta}(n)\mathcal{H}^{n-1}(E)$,

where $\boldsymbol{\beta}(n) := 2^{n-1}$.

Proof. First, observe that the dominated convergence theorem yields $|B(E,r)| \leqslant \varepsilon$ in case $r > 0$ is sufficiently small, say $r \leqslant \delta$. Continue by choosing a countable family $(B_k)_{k \geqslant 1}$ of open balls with diameter less than δ such that $U := \bigcup_{k=1}^\infty B_k \supseteq E$ and $2^{1-n}\boldsymbol{\alpha}(n-1) \sum_{k=1}^\infty d(B_k)^{n-1} \leqslant \mathcal{S}^{n-1}(E) + \eta$. Without loss of generality, we can assume that $B_k \cap E \neq \emptyset$ for each $k \geqslant 1$; in particular we have $U \subseteq B(E,\delta)$ and hence $|U| \leqslant \varepsilon$. On the other hand,

$$\|D\chi_U\| \leqslant \sum_{k=1}^{\infty} \|D\chi_{B_k}\| = 2^{1-n}\boldsymbol{\alpha}(n-1)\sum_{k=1}^{\infty} d(B_k)^{n-1} \leqslant 2^{n-1}\mathcal{H}^{n-1}(E) + \eta,$$

and the proof is complete. □

Lemma 7. *If $E \subseteq \mathbb{R}^n$ is closed and thin, then the collection of all $g \in BV_c(\mathbb{R}^n)$ that vanishes in a neighborhood of E is $\mathfrak{T}$-dense in $BV_c(\mathbb{R}^n)$.*

Proof. First, write E as an increasing union $\bigcup_{k=1}^{\infty} E_k$ where $\mathcal{H}^{n-1}(E_k) < \infty$ for each $k \geqslant 1$. Replacing if necessary E_k by $E_k \cap B(k)$ we may assume that E_k is bounded for each $k \geqslant 1$. We further define for each $k \geqslant 1$ a constant $C_k := \|g\|_\infty[1 + \sum_{j=1}^{k} \mathcal{H}^{n-1}(E_j)] + \|Dg\|$ and choose $r > 0$ so that $\operatorname{supp} g \subseteq B[r]$.

Now fix $\mathbf{U}$ a $\mathfrak{T}$-neighborhood of 0 in $BV_c(\mathbb{R}^n)$. According to the definition of $\mathfrak{T}$, there exists $\delta_k > 0$ such that any $h \in BV_c(\mathbb{R}^n)$ satisfying $\operatorname{supp} h \subset B[r]$, $\|Dh\| \leqslant C_k$ and $\|h\|_1 \leqslant \delta_k$ belongs to $2^{-k}\mathbf{U}$.

For each $k \geqslant 1$, choose according to Lemma 6 an open set $U_k \supseteq E_k \cap \operatorname{supp} g$ satisfying $\|g\|_\infty |U_k| \leqslant \delta_k$ and $\|D\chi_{U_k}\| \leqslant 2^{-k} + \boldsymbol{\beta}(n)\mathcal{H}^{n-1}(E_k)$. Observe that by compactness, there exists $l \geqslant 1$ such that $E \cap \operatorname{supp}(g) \subseteq \bigcup_{k=1}^{l} U_k$.

Define $g_1 := g\chi_{U_1}$, and for $k \geqslant 2$, let $g_k := g\chi_{U_k \setminus \bigcup_{j=1}^{k-1} U_j}$. Fix $k \geqslant 1$. We obviously have $\operatorname{supp} g_k \subseteq B[r]$ and $\|g_k\|_1 \leqslant \delta_k$, while

$$\|Dg_k\| \leqslant \|g\|_\infty \sum_{j=1}^{k} \|D\chi_{U_j}\| + \|Dg\| \leqslant \|g\|_\infty \left[1 + \boldsymbol{\beta}(n)\sum_{j=1}^{k} \mathcal{H}^{n-1}(E_j)\right] + \|Dg\| \leqslant C_k,$$

where we used [10, Proposition 1.8.4] for example to estimate $\|D\chi_{U_k \setminus \bigcup_{j=1}^{k-1} U_j}\|$. Consequently we get $g_k \in 2^{-k}\mathbf{U}$.

We finish the argument by observing that $g' := \sum_{k=1}^{l} g_k$ equals g on a neighborhood of $E \cap \operatorname{supp} g$ (and hence equals g on a neighborhood of E) and belongs to $\sum_{k=1}^{l} 2^{-k}\mathbf{U} \subseteq \mathbf{U}$. □

Observation 8. It is not sufficient that E be thin for the conclusion of Lemma 7 to hold, as the following example shows.

Example 9. Assume that $n = 1$ and let E denote the set of rational numbers (which is thin but not closed). If $g \in BV_c(\mathbb{R})$ vanishes on an open set $U \supset E$, then choose according to [1, Theorem 3.27] a function $\tilde{g} = g$ a.e. whose pointwise variation equals $\|Dg\|$. We claim that $\{\tilde{g} \neq 0\}$ is countable, for otherwise given any $k \geqslant 1$, one would be able to find a finite set of $B_k \subset \{\tilde{g} \neq 0\}$ such that $\sum_{x \in E_k} \tilde{g}(x) \geqslant k$. This would imply that the pointwise variation of $\tilde{g}$ is infinite because $\{\tilde{g} = 0\}$ is dense in $\mathbb{R}$. Consequently $g = 0$ a.e.

Definition 10. The *support* of a charge F, denoted $\operatorname{supp} F$, is the complementary set of the largest open set $U \subseteq \mathbb{R}^n$ having the following property: $F(g) = 0$ provided $g \in BV_c(\mathbb{R}^n)$ satisfies $\{g \neq 0\} \subset\subset U$.

Observation 11. Given a charge F, the support of F coincides with the support (defined in the usual way) of the distribution $F \restriction C_c^\infty(\mathbb{R}^n)$ uniquely determining F.

An immediate corollary of Lemma 7 is the following.

Theorem 12. *If F is a charge and if $\operatorname{supp} F$ is thin, then $F \equiv 0$.*

4. A NONREMOVABLE SET WITH DIMENSION $n-1$

Given a cube Q, we denote by $\sigma(Q)$ the length of its edges. For $0<\lambda<1/2$, we let $\mathcal{E}(Q,\lambda)$ stand for the collection of 2^n cubes contained in Q whose edges have length $\lambda\sigma(Q)$, arranged in such a way that each cube of $\mathcal{E}(Q,\lambda)$ has a common vertex with Q.

Fix a sequence $\lambda=(\lambda_k)_{k\geqslant 1}\subseteq(0,1/2)$. Write $E_0:=[0,1]^n$ and $\mathcal{E}_0=\{E_0\}$. For each $k\geqslant 1$, define inductively

$$\mathcal{E}_k(\lambda_1,\dots,\lambda_k):=\bigcup\{\mathcal{E}(Q,\lambda_k):Q\in\mathcal{E}_{k-1}(\lambda_1,\dots,\lambda_{k-1}\}$$

and $E_k(\lambda_1,\dots,\lambda_k):=\bigcup\mathcal{E}_k(\lambda_1,\dots,\lambda_k)$ (or briefly $\mathcal{E}_k$ and E_k when the underlying sequence is clear). In particular, each cube in $\mathcal{E}_k$ has edges of length $\sigma_k:=\lambda_1\dots\lambda_k$. Finally we let $E(\lambda)=\bigcap_{k=1}^\infty E_k$.

Associated to λ, we choose $h:[0,\infty)\to[0,\infty)$ as a nondecreasing function satisfying $h(0)=0$ and $h(\sigma_k)=2^{-kn}$ for each $k\geqslant 1$. Recall that the Hausdorff h-measure of $E(\lambda)$ is positive and finite (see [7, Section 4.9]).

Theorem 13. *There is a compact set $E\subseteq\mathbb{R}^n$ having Hausdorff dimension $n-1$ and satisfying the following properties:*

(A) *there exists $v\in C^0(\mathbb{R}^n,\mathbb{R}^n)$ and a* nonzero *Radon measure μ supported in E such that $\operatorname{div}v=\mu$ (in the distribution sense);*
(B) *if $u\in C^1(\mathbb{R}^n)$ is harmonic outside E, then u is harmonic in $\mathbb{R}^n$.*

In particular, $\mathcal{H}^{n-1}\llcorner E$ is not σ-finite.

Proof. For $k\geqslant 3$, let $\ell_k=2^{-\frac{n}{n-1}}\log k/\log(k-1)$ and assume that $\lambda_k=\ell_k$ for k sufficiently large. Then:

(1) $\mathcal{H}^{n-1}\llcorner E(\lambda)$ is not σ-finite,
(2) $\dim_{\mathcal{H}}(E(\lambda))=n-1$;

the proofs of both facts are standard.

Given $k\geqslant 1$, we associate to E_k a Radon measure $\mu_k:=|E_k|^{-1}\mathcal{L}^n\llcorner E_k$ supported in E_k. Observe that $\mu_k(\mathbb{R}^n)=1$ for each k. Hence, [7, Theorem 1.23] ensures that there exists a subsequence of (μ_k) weakly*-converging to a Radon measure μ supported in $E(\lambda)$.

We now turn to showing that

$$\lim_{\delta\to 0}\sup_{0<r<\delta}\sup_{x\in\mathbb{R}^n}\frac{\mu(B[x,r])}{r^{n-1}}=0. \tag{1}$$

Observe that it suffices to show (1) with $\mathbb{R}^n$ replaced by $E(\lambda)$ and $B[x,r]$ replaced by $Q[x,r]$.

Claim 14. Fix $x\in E(\lambda)$ and k sufficiently large for $\lambda_k=\ell_k$ to hold. If $\sigma_{k+1}\leqslant r\leqslant\sigma_k$, then

$$\frac{\mu(Q[x,r])}{r^{n-1}}\leqslant 2^{-kn}.$$

Proof of the claim. Observe that $Q[x,r]$ is contained in 2^n cubes $K_1,\dots,K_{2^n}$ having x as a common vertex and edges of length σ_k. In particular, we get for $1\leqslant i\leqslant 2^n$ and some arbitrary but fixed $Q_*\in\mathcal{E}_{k-1}$:

$$|E_l\cap K_i|\leqslant|E_l\cap Q|\quad\text{for any}\quad Q\in\mathcal{E}(Q_*,\lambda_k),$$

and consequently

$$\mu_l(Q[x,r]) \leqslant 2^{-ln}\sigma_l^{-n}|E_l \cap Q_*| = 2^{-ln}2^{(k+1)n}.$$

This yields

$$\frac{\mu_l(Q[x,r])}{r^{n-1}} \leqslant \frac{2^{(k-l+1)n}}{\sigma_{k+1}^{n-1}} \leqslant 2^{-ln} \leqslant 2^{-kn};$$

the conclusion follows for $\mu(Q[x,r]) \leqslant \overline{\lim}_l \mu_l(Q[x,r])$. □

Claim 14 together with Theorems 4 and 5 yields a vector field $v \in C^0(\mathbb{R}^n, \mathbb{R}^n)$ such that $\operatorname{div} v = \mu$ in the distribution sense. Moreover, the equality $\mu(\mathbb{R}^n) = 1$ ensures that μ is nonzero. This proves property (A) of Theorem 13.

The proof of property (B) uses a recent result by A. Ruiz de Villa and X. Tolsa [12]. Observe that for k_0 sufficiently large

$$\tau := \sum_{k=k_0}^{\infty} \frac{1}{(2^{kn}\sigma_k^{n-1})^2} = \sum_{k=k_0}^{\infty} \left(\frac{1}{\log k}\right)^{2(n-1)} = \infty.$$

According to [12, Theorem 6], we get

$$\kappa^c(E(\lambda)) := \sup\{|\langle \Delta u, 1\rangle| : u \in C^1(\mathbb{R}^n), \operatorname{supp}(\Delta u) \subseteq E, \|\nabla u\|_\infty \leqslant 1\} = \tau^{-1/2} = 0.$$

It follows from [6, Proposition 2.2] that the above equality is equivalent to property (B). The proof is complete. □

Acknowledgement

The authors want to thank the referee for suggesting the topological argument (Lemma 7) which considerably simplifies the proof of Theorem 12.

References

[1] Luigi Ambrosio, Nicola Fusco, and Diego Pallara. *Functions of bounded variation and free discontinuity problems*. Oxford Mathematical Monographs. The Clarendon Press, Oxford University Press, New York, 2000. MR1857292 (2003a:49002)

[2] Thierry De Pauw. On the exceptional sets of the flux of a bounded vectorfield. *J. Math. Pures Appl. (9)*, 82(9):1191–1217, 2003. MR2012808 (2004m:42027)

[3] Thierry De Pauw, Laurent Moonens, and Washek F. Pfeffer. Charges in middle dimensions. *J. Math. Pures Appl.*, to appear (DOI 10.1016/j.matpur.2009.04.001), 2009.

[4] Thierry De Pauw and Washek F. Pfeffer. Distributions for which div $v = F$ has a continuous solution. *Comm. Pure Appl. Math.*, 61(2):230–260, 2008. MR2368375 (2009e:46035)

[5] Lawrence C. Evans and Ronald F. Gariepy. *Measure theory and fine properties of functions*. Studies in Advanced Mathematics, CRC Press, Boca Raton, FL, 1992. MR1158660 (93f:28001)

[6] P. Mattila and P. V. Paramonov. On geometric properties of harmonic Lip_1-capacity. *Pacific J. Math.*, 171(2):469–491, 1995. MR1372240 (97b:31005)

[7] Pertti Mattila. *Geometry of sets and measures in Euclidean spaces*, Fractals and rectifiability, volume 44 of Cambridge Studies in Advanced Mathematics. Cambridge University Press, Cambridge, 1995. MR1333890 (96h:28006)

[8] Laurent Moonens. Removable singularities for the equation div $v = 0$. *Real Anal. Exchange* (30th Summer Symposium Conference):125–132, 2006. MR2323837 (2008h:35013)

[9] Laurent Moonens. *From Kurzweil-Henstock integration to charges in Euclidean spaces*. Faculté des Sciences, École doctorale en Mathématique, Université catholique de Louvain. CIACO, Louvain-la-Neuve, 2008.

[10] Washek F. Pfeffer. *Derivation and integration*, volume 140 of Cambridge Tracts in Mathematics. Cambridge University Press, Cambridge, 2001. MR1816996 (2001m:26018)

[11] Nguyen Cong Phuc and Monica Torres. Characterizations of the existence and removable singularities of divergence-measure vector fields. *Indiana Univ. Math. J.*, 57(4):1573–1597, 2008. MR2440874 (2009f:35026)

[12] Alex Ruiz de Villa and Xavier Tolsa. Characterization and semiadditivity of the C^1 harmonic capacity. Preprint (available on arXiv:0812.2421), 2008.

Département de Mathématique, Université catholique de Louvain, Chemin du Cyclotron, 2, 1348 Louvain-la-neuve, Belgium

E-mail address: `sebastien.devaleriola@uclouvain.be`

Département de Mathématique, Université catholique de Louvain, Chemin du Cyclotron, 2, 1348 Louvain-la-neuve, Belgium

E-mail address: `laurent.moonens@uclouvain.be`