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SUM OF MULTIPLE q-ZETA VALUES

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Abstract. The generating function of the sums of multiple q-zeta values with fixed weights, depths and 1-heights, 2-heights, ..., r-heights is represented in terms of specializations of basic hypergeometric functions.

1. Introduction

In this paper, let $\mathbf{k} = (k_1, k_2, \ldots, k_n)$ be a sequence of positive integers. For such a $\mathbf{k} = (k_1, k_2, \ldots, k_n)$, as in [4], we define the weight, depth and i-height of $\mathbf{k}$ by $\mathrm{wt}(\mathbf{k}) = k_1 + k_2 + \cdots + k_n$, $\mathrm{dep}(\mathbf{k}) = n$ and $i\text{-ht}(\mathbf{k}) = \#\{l \mid k_l > i\}$, respectively. A sequence $\mathbf{k} = (k_1, k_2, \ldots, k_n)$ is admissible if $k_1 > 1$.

Throughout this paper, we assume that q is a real number with $0 < q < 1$. For a positive integer n, its q-analogue is defined by

$$[n] = [n]_q = 1 + q + q^2 + \cdots + q^{n-1} = \frac{1-q^n}{1-q}.$$

For a sequence $\mathbf{k} = (k_1, k_2, \ldots, k_n)$, the multiple q-zeta value $\zeta[\mathbf{k}]$ is defined by the following q-series (see [1, 8]):

$$\zeta[\mathbf{k}] = \sum_{m_1 > m_2 > \cdots > m_n > 0} \frac{q^{m_1(k_1-1)+m_2(k_2-1)+\cdots+m_n(k_n-1)}}{[m_1]^{k_1}[m_2]^{k_2}\cdots[m_n]^{k_n}}. \tag{1.1}$$

If $\mathbf{k}$ is admissible, the right hand side of (1.1) absolutely converges. Taking the limit $q \to 1^-$, we obtain the usual multiple zeta value $\zeta(\mathbf{k})$:

$$\lim_{q\to 1^-} \zeta[\mathbf{k}] = \zeta(\mathbf{k}) = \sum_{m_1 > m_2 > \cdots > m_n > 0} \frac{1}{m_1^{k_1} m_2^{k_2} \cdots m_n^{k_n}}.$$

As in [6], for a sequence $\mathbf{k} = (k_1, k_2, \ldots, k_n)$, the multiple q-polylogarithms (of one variable) $\mathrm{Li}_{\mathbf{k}}[t]$ are defined by

$$\mathrm{Li}_{\mathbf{k}}[t] = \mathrm{Li}_{k_1,k_2,\ldots,k_n}[t] = \sum_{m_1 > m_2 > \cdots > m_n > 0} \frac{t^{m_1}}{[m_1]^{k_1}[m_2]^{k_2}\cdots[m_n]^{k_n}}. \tag{1.2}$$

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The right hand side of (1.2) is absolutely convergent for $|t| < 1$. Taking the limit $q \to 1^-$, we get the multiple polylogarithms (of one variable)

$$\lim_{q \to 1^-} \mathrm{Li}_{\mathbf{k}}[t] = \sum_{m_1 > m_2 > \cdots > m_n > 0} \frac{t^{m_1}}{m_1^{k_1} m_2^{k_2} \cdots m_n^{k_n}}.$$

Ohno and Zagier [5] studied the generating function of the sums of multiple zeta values with fixed weights, depths and 1-heights. They proved that these sums can be written as polynomials of Riemann zeta values with rational coefficients. We studied the generating function of the sums of multiple zeta values with fixed weights, depths and 1-heights, 2-heights,..., r-heights for any natural number r in [4]. We found that this generating function can be represented by specializations of the generalized hypergeometric function ${}_{r+1}F_r$. In [1], Bradley partially gave a q-analogue of the result of Ohno and Zagier [5], and in [6], Okuda and Takeyama generalized this result to multiple q-zeta values completely.

In this paper, we give a q-analogue of our results of [4], which generalizes the result of Okuda and Takeyama in [6]. We represent the generating function of the sums of multiple q-zeta values with fixed weights, depths and 1-heights, 2-heights, ..., r-heights in terms of specializations of basic hypergeometric functions. In Section 2, we list some preliminary facts related to basic hypergeometric functions. In Section 3, after treating the case of the sum of multiple q-polylogarithms, we obtain our main theorem (Theorem 3.2) about the sum of multiple q-zeta values.

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2. q-ANALOGUES

Let us recall the definition of basic hypergeometric functions from the book [2]. The q-shifted factorial $(a;q)_n$ is defined by

$$(a;q)_n = \begin{cases} 1, & n = 0, \\ (1-a)(1-aq)\cdots(1-aq^{n-1}), & n = 1, 2, \ldots. \end{cases}$$

To simplify the notation, we denote the products $(a_1;q)_n (a_2;q)_n \cdots (a_m;q)_n$ by $(a_1, a_2, \ldots, a_m; q)_n$. For a natural number r, the basic hypergeometric series is defined by

$$(2.1) \qquad {}_{r+1}\phi_r \left[\begin{matrix} a_1, a_2, \ldots, a_{r+1} \\ b_1, \ldots, b_r \end{matrix} ; q, t \right] = \sum_{n=0}^{\infty} \frac{(a_1, a_2, \ldots, a_{r+1}; q)_n}{(q, b_1, \ldots, b_r; q)_n} t^n,$$

where $b_j \neq q^{-m}$ for $m = 0, 1, \ldots$ and $j = 1, 2, \ldots, r$. The right hand side of (2.1) is absolutely convergent for $|t| < 1$, which gives the basic hypergeometric function.

The q-Stirling numbers $S_q(n,k)$ of the second kind are defined by the following recurrence ([3, 7]):

$$S_q(n,k) = \begin{cases} q^{k-1} S_q(n-1, k-1) + [k] S_q(n-1, k), & \text{if } 0 < k \leq n, \\ 1, & \text{if } n = k = 0, \\ 0, & \text{otherwise.} \end{cases}$$

As in [2, Exercise 1.12], let $\mathcal{D}_q$ be the q-difference operator

$$(\mathcal{D}_q f)(t) = \frac{f(t) - f(qt)}{(1-q)t}.$$

Let $\mathcal{D}_q^n u = \mathcal{D}_q(\mathcal{D}_q^{n-1} u)$ and $\mathcal{D}_q^0 u = u$.

Lemma 2.1. *For a nonnegative integer n, we have*

$$\mathcal{D}_q^n \,{}_{r+1}\phi_r \left[\begin{matrix} a_1, a_2, \dots, a_{r+1} \\ b_1, \dots, b_r \end{matrix}; q, at\right] = \frac{(a_1, a_2, \dots, a_{r+1}; q)_n a^n}{(b_1, \dots, b_r; q)_n (1-q)^n} \,{}_{r+1}\phi_r \left[\begin{matrix} a_1 q^n, a_2 q^n, \dots, a_{r+1} q^n \\ b_1 q^n, \dots, b_r q^n \end{matrix}; q, at\right]. \tag{2.2}$$

One can prove the above lemma by using induction on n.

We define another q-difference operator Θ_q by $\Theta_q = t\mathcal{D}_q$. In other words, we have

$$(\Theta_q f)(t) = \frac{f(t) - f(qt)}{1-q}.$$

Let $\Theta_q^n u = \Theta_q(\Theta_q^{n-1} u)$ and $\Theta_q^0 u = u$.

By using induction and the recurrence of q-Stirling numbers of the second kind, we get the following fact.

Lemma 2.2. *For a nonnegative integer n, we have*

$$\Theta_q^n = \sum_{m=0}^{n} S_q(n,m) t^m \mathcal{D}_q^m. \tag{2.3}$$

Finally, following [2, Exercise 1.31], we have:

Lemma 2.3. *Let r be a positive integer and $a_1 \cdots a_{r+1} b_1 \cdots b_r \neq 0$. Let $b_j \neq q^{-m}$ for $m = 0, 1, \dots$ and $j = 1, 2, \dots, r$. Set $v(t) = {}_{r+1}\phi_r \left[\begin{smallmatrix} a_1, a_2, \dots, a_{r+1} \\ b_1, \dots, b_r \end{smallmatrix}; q, \frac{b_1 \cdots b_r}{a_1 \cdots a_{r+1} q^r} t\right]$. We have that $v(t)$ satisfies the following q-difference equation:*

$$\Theta_q \left(\Theta_q + \frac{q - b_1}{b_1(1-q)}\right) \cdots \left(\Theta_q + \frac{q - b_r}{b_r(1-q)}\right) v(t) = t \left(\Theta_q + \frac{1 - a_1}{a_1(1-q)}\right) \cdots \left(\Theta_q + \frac{1 - a_{r+1}}{a_{r+1}(1-q)}\right) v(t).$$

3. Main results

3.1. **Main theorems.** We abbreviate $1-q$ as q_1. Let r be a fixed positive integer. For given nonnegative integers $k, n, h_1, h_2, \dots, h_r$, we define $I_j(k, n, h_1, h_2, \dots, h_r)$ as a set

$$\{\mathbf{k} = (k_1, \dots, k_n) \mid \mathrm{wt}(\mathbf{k}) = k, \mathrm{dep}(\mathbf{k}) = n, 1\text{-}\mathrm{ht}(\mathbf{k}) = h_1, \dots, r\text{-}\mathrm{ht}(\mathbf{k}) = h_r, k_1 \geq j+2\}$$

and

$$G_j(k, n, h_1, h_2, \dots, h_r; t) = \sum_{\mathbf{k} \in I_j(k, n, h_1, h_2, \dots, h_r)} \mathrm{Li}_{\mathbf{k}}[t],$$

where $j = -1, 0, \dots, r-1$. In the above definition, the sum is assigned to be zero if $I_j(k, n, h_1, h_2, \dots, h_r)$ is empty and $G_{-1}(\underbrace{0, 0, \dots, 0}_{r+2}; t) = 1$. For $j = 0, 1, \dots, r-1$, we also study the sum $G_j(k, n, h_1, \dots, h_r)$ of multiple q-zeta values, where

$$G_j(k, n, h_1, \dots, h_r) = \sum_{\mathbf{k} \in I_j(k, n, h_1, \dots, h_r)} \zeta[\mathbf{k}].$$

We define the following generating functions:

$$\Phi_j(x_1, x_2, \ldots, x_{r+2}; t) = \sum_{k,n,h_1,h_2,\ldots,h_r \geq 0} G_j(k, n, h_1, h_2, \ldots, h_r; t)$$
$$\cdot x_1^{k-n-\sum_{j=1}^r h_j} x_2^{n-h_1} x_3^{h_1-h_2} x_4^{h_2-h_3} \cdots x_{r+1}^{h_{r-1}-h_r} x_{r+2}^{h_r}$$

and

$$\Psi_j(u_1, u_2, \ldots, u_{r+2}) = \sum_{k,n,h_1,\ldots,h_r \geq 0} G_j(k, n, h_1, \ldots, h_r)$$
$$\cdot u_1^{k-n-\sum_{j=1}^r h_j} u_2^{n-h_1} u_3^{h_1-h_2} \cdots u_{r+1}^{h_{r-1}-h_r} u_{r+2}^{h_r}.$$

Here $x_1, \ldots, x_{r+2}$ and $u_1, \ldots, u_{r+2}$ are variables with relations

$$\text{(3.1)} \qquad \begin{cases} x_1 = \frac{u_1}{1+q_1 u_1}, \\ x_j = \frac{1}{u_1^{r+2-j}} \left\{ \sum_{k=j}^{r+1} \binom{k-2}{j-2} (-q_1 u_1)^{k-j} (u_1^{r+2-k} u_k - u_{r+2}) \right. \\ \qquad \left. + \frac{u_{r+2}}{(1+q_1 u_1)^{j-1}} \right\}, \ j = 2, \ldots, r+2. \end{cases}$$

It is easy to show that (3.1) is equivalent to

$$\text{(3.2)} \qquad \begin{cases} u_1 = \frac{x_1}{1-q_1 x_1}, \\ u_j = \frac{1}{x_1^{r+2-j}} \left\{ \sum_{i=j-1}^{r} \binom{i-1}{j-2} (q_1 x_1)^{i-j+1} \left(x_1^{r+1-i} x_{i+1} - x_{r+2}\right) \right. \\ \qquad \left. + \frac{x_{r+2}}{(1-q_1 x_1)^{j-1}} \right\}, \ j = 2, \ldots, r+2. \end{cases}$$

Let us denote $\Phi_j = \Phi_j(x_1, x_2, \ldots, x_{r+2}; t)$ for short. We represent these generating functions via basic hypergeometric functions in the following theorem.

Theorem 3.1. *For any integer j with $-1 \leq j \leq r-1$, we have*

$$\Phi_j = \frac{1}{x_{r+2} - x_1 x_{r+1}} \left\{ \sum_{i=0}^{r-1-j} A_i^{(j)} B_i t^i \right.$$
$$\left. \times \ {}_{r+1}\phi_r \left[\begin{matrix} a_1 q^i, \ldots, a_{r+1} q^i \\ b q^i, \underbrace{q^{i+1}, \ldots, q^{i+1}}_{r-1} \end{matrix}; q, \frac{b}{a_1 \cdots a_{r+1} q} t \right] - A_0^{(j)} \right\} + \delta_{j,-1}.$$

We explain the notation appearing in the above theorem. First, $\delta_{i,j}$ is the Kronecker delta defined by $\delta_{i,j} = 1$ for $i = j$ and 0 otherwise. Let $\alpha_1, \alpha_2, \ldots, \alpha_{r+1}$ be variables determined by

$$\text{(3.3)} \qquad \begin{cases} \alpha_1 + \alpha_2 + \cdots + \alpha_{r+1} = x_2 - x_1, \\ \sum_{1 \leq i_1 < \ldots < i_j \leq r+1} \alpha_{i_1} \cdots \alpha_{i_j} = x_{j+1} - x_1 x_j, \ \ 2 \leq j \leq r+1, \end{cases}$$

and let

$$\text{(3.4)} \qquad b = \frac{q}{1 - q_1 x_1} = q(1 + q_1 u_1), \ a_i = \frac{1}{1 + q_1 \alpha_i}, \ \ i = 1, \ldots, r+1.$$

Finally, we set

$$B_i = \frac{(a_1, \dots, a_{r+1}; q)_i}{(b, \underbrace{q, \dots, q}_{r-1}; q)_i} \left(\frac{b}{a_1 \dots a_{r+1} q q_1} \right)^i, \ i = 0, 1, \dots, r, \tag{3.5}$$

and

$$A_i^{(j)} = \sum_{m=i}^{r-1-j} (x_{r+2-m} - x_1 x_{r+1-m}) S_q(m, i) + x_1 x_{j+2} S_q(r-1-j, i). \tag{3.6}$$

Here $-1 \le j \le r-1$, $0 \le i \le r-1-j$ in (3.6).

Using Theorem 3.1, we obtain the main result of this paper, which represents $\Psi_0 = \Psi_0(u_1, u_2, \dots, u_{r+2})$ by the specializations of the basic hypergeometric function ${}_{r+1}\phi_r$.

Theorem 3.2. *We have*

$$\Psi_0 = \frac{[1+q_1 u_1]^2}{u_{r+2} - u_1 u_{r+1}} \left\{ \sum_{j=0}^{r-1} A_j \widetilde{B}_j \, {}_{r+1}\phi_r \left[\begin{matrix} a_1 q^j, \dots, a_{r+1} q^j \\ b q^j, \underbrace{q^{j+1}, \dots, q^{j+1}}_{r-1} \end{matrix} ; q, \frac{b}{a_1 \cdots a_{r+1}} \right] - A_0 \right\}.$$

In the above, for $0 \le i \le r-1$, we define

$$\widetilde{B}_i = \frac{(a_1, \dots, a_{r+1}; q)_i}{(b, \underbrace{q, \dots, q}_{r-1}; q)_i} \left(\frac{b}{a_1 \dots a_{r+1} q_1} \right)^i \tag{3.7}$$

and

$$A_i = \sum_{m=i}^{r-1} c_m S_q(m, i) + \frac{u_1 u_2}{(1+q_1 u_1)^2} S_q(r-1, i) \tag{3.8}$$

with

$$\begin{aligned} c_m = \sum_{k=r-m+1}^{r+1} & \left[\binom{k-2}{r-m} + \frac{q_1 u_1}{1+q_1 u_1} \binom{k-2}{r-m-1} \right] (-q_1)^{k-r+m-2} \\ & \cdot \left(\frac{u_k}{1+q_1 u_1} - u_1^{k-r-2} u_{r+2} + \frac{q_1 u_{k+1}}{1+q_1 u_1} \right). \end{aligned} \tag{3.9}$$

We remark that by taking the limit $q \to 1$ in Theorem 3.2, we indeed come back to the main result of [4]. Moreover one can get a similar formula for any Ψ_j.

To end this subsection, we list two computations.

Examples. (1) For $r = 1$, we have

$$\Psi_0(u_1, u_2, u_3) = \frac{u_3}{u_3 - u_1 u_2} \left\{ {}_2\phi_1 \left[\begin{matrix} a_1, a_2 \\ b \end{matrix} ; q, \frac{b}{a_1 a_2} \right] - 1 \right\},$$

where

$$b = q[1 + (1-q) u_1]$$

and

$$\begin{cases} a_1 + a_2 = \frac{2 + (1-q)(u_1+u_2) + (1-q)^2 (u_1 u_2 - u_3)}{1 + (1-q) u_2}, \\ a_1 a_2 = \frac{1 + (1-q) u_1}{1 + (1-q) u_2}. \end{cases}$$

By using Heine's q-analogue of Gauss' summation formula, we obtain a q-analogue of Ohno-Zagier's relation deduced by Okuda and Takeyama in [6].

(2) For $r=2$, we have

$$\Psi_0(u_1,u_2,u_3,u_4)=\frac{q[u_3+(1-q)(u_1u_3-u_4)]}{1-qu_1}\,{}_3\phi_2\left[\begin{matrix}a_1q,a_2q,a_3q\\ bq,q^2\end{matrix};q,\frac{b}{a_1a_2a_3}\right]+{}_3\phi_2\left[\begin{matrix}a_1,a_2,a_3\\ b,q\end{matrix};q,\frac{b}{a_1a_2a_3}\right]-1,$$

where

$$b=q[1+(1-q)u_1],$$

and

$$\begin{cases} a_1+a_2+a_3=\frac{3+(1-q)(u_1+2u_2)+(1-q)^2(u_1u_2-u_3)}{1+(1-q)u_2},\\ a_1a_2+a_2a_3+a_3a_1=\frac{3+(1-q)(2u_1+u_2)+(1-q)^2(u_1u_2-u_3)+(1-q)^3(u_4-u_1u_3)}{1+(1-q)u_2},\\ a_1a_2a_3=\frac{1+(1-q)u_1}{1+(1-q)u_2}.\end{cases}$$

3.2. **Sum of multiple q-polylogarithms.** We prove Theorem 3.1 in this subsection. The multiple q-polylogarithms satisfy the following relations:

$$\mathcal{D}_q\,\mathrm{Li}_{k_1,\ldots,k_n}[t]=\begin{cases}\frac{1}{t}\,\mathrm{Li}_{k_1-1,k_2,\ldots,k_n}[t], & \text{if } k_1\geq 2,\\ \frac{1}{1-t}\,\mathrm{Li}_{k_2,\ldots,k_n}[t], & \text{if } k_1=1.\end{cases}\tag{3.10}$$

Since the formula (3.10) is the same as that of the differential operator $\frac{d}{dt}$ operating on the multiple polylogarithms (of one variable) and recalling that $\Theta_q=t\mathcal{D}_q$, we can get the following proposition after the same argument as in [4].

Proposition 3.3. *Let $y_0=\Phi_{r-1}$ and $y_j=\Phi_{r-1-j}-\Phi_{r-j}, j=1,2,\ldots,r$. We have*

(1):

$$y_j=\frac{x_{r+2-j}}{x_{r+2}}\Theta_q^j y_0-\frac{x_1x_{r+2-j}}{x_{r+2}}\Theta_q^{j-1}y_0+\delta_{j,r},\ j=1,\ldots,r;\tag{3.11}$$

(2): *y_0 satisfies the following q-difference equation:*

$$\left\{\Theta_q^r(\Theta_q-x_1)-t\left[\Theta_q^{r+1}+(x_2-x_1)\Theta_q^r+\sum_{j=0}^{r-1}(x_{r+2-j}-x_1x_{r+1-j})\Theta_q^j\right]\right\}y_0=tx_{r+2}.$$

To prove Theorem 3.1, we prepare the following lemma.

Lemma 3.4. *Let m be a positive integer, and $p_i(t)=c_{i0}+c_{i1}t\in\mathbb{C}[t]$ be a degree at most 1 polynomial for $i=0,1,\ldots,m$. Assume that $\sum_{i=0}^{m}[n]^ic_{m-i,0}\neq 0$ for any positive integer n. Then the initial problem*

$$\begin{cases}\left\{p_0(t)\Theta_q^m+p_1(t)\Theta_q^{m-1}+\cdots+p_{m-1}(t)\Theta_q+p_m(t)\right\}y=0,\\ y(0)=0\end{cases}$$

has a unique holomorphic solution $y(t)=0$ for $|t|<1$.

Proof. It is sufficient to prove the uniqueness. Assume that $y(t)$ is a solution. Then we set $y(t)=\sum_{n=1}^{\infty} a_n t^n$. By the q-difference equation satisfied by $y(t)$, we get

$$\sum_{n=1}^{\infty}\left(\sum_{i=0}^{m}[n]^i c_{m-i,0}\right)a_n t^n+\sum_{n=2}^{\infty}\left(\sum_{i=0}^{m}[n-1]^i c_{m-i,1}\right)a_{n-1}t^n=0.$$

Hence $a_n=0$ for any $n\geq 1$. □

Now we come to the proof of Theorem 3.1. For $j=r-1$, since $B_0=1$ and $A_0^{(r-1)}=(x_{r+2}-x_1x_{r+1})S_q(0,0)+x_1x_{r+1}S_q(0,0)=x_{r+2}$, it is enough to prove that

$$\Phi_{r-1}=\frac{x_{r+2}}{x_{r+2}-x_1x_{r+1}}\left\{{}_{r+1}\phi_r\left[\begin{matrix}a_1,\dots,a_{r+1}\\ b,\underbrace{q,\dots,q}_{r-1}\end{matrix};q,\frac{b}{a_1\cdots a_{r+1}q}t\right]-1\right\},$$

which follows from Lemma 2.3, Lemma 3.4, Proposition 3.3 (2) and $\Phi_{r-1}(0)=0$. The general case is a consequence of (3.11), (2.3), (2.2) and the result for $j=r-1$. □

3.3. **Sum of multiple q-zeta values.** We prove the main theorem (Theorem 3.2) in this subsection. For an admissible sequence $\mathbf{k}=(k_1,k_2,\dots,k_n)$, it is known that the specialization of multiple polylogarithms at 1 is just $\zeta(\mathbf{k})$. So in [4], we only need to set $t=1$ in Φ_0 to obtain the generating function of sums of multiple zeta values, while the situation here is more complicated. From [6], we have

$$\mathrm{Li}_{\mathbf{k}}[q]=\sum_{a_1=2}^{k_1}\sum_{a_2=1}^{k_2}\cdots\sum_{a_n=1}^{k_n}\binom{k_1-2}{a_1-2}\left\{\prod_{j=2}^{n}\binom{k_j-1}{a_j-1}\right\}$$

$$\cdot(1-q)^{\sum_{j=1}^{n}(k_j-a_j)}\zeta[a_1,\dots,a_n]. \tag{3.12}$$

Thus we need all the Φ_j's and some matrix computations.

In this subsection, for any j with $0\leq j\leq r-1$, we abbreviate $\Phi_j(x_1,\dots,x_{r+2};q)$ as $\Phi_j(q)$ and $\Psi_j(u_1,\dots,u_{r+2})$ as Ψ_j. We set

$$X_l=\frac{x_1^{r+1-l}u_{l+1}}{x_{r+2}},\quad l=1,\dots,r+1, \tag{3.13}$$

(3.14)

$$Y_l=\sum_{i=l}^{r}\binom{i-2}{l-2}(q_1x_1)^{i-l}\frac{x_1^{r+1-i}x_{i+1}-x_{r+2}}{x_{r+2}}+(1-q_1x_1)^{1-l},\ l=2,\dots,r,$$

and

$$Z_{lj}=\sum_{i=l}^{j}\binom{i-2}{l-2}(q_1x_1)^{i-l}\frac{x_1^{r+1-i}x_{i+1}}{x_{r+2}},\quad 2\leq l\leq j\leq r. \tag{3.15}$$

Then using (3.12), we obtain the following lemma.

Lemma 3.5. *Let A and D be two $r \times r$ matrices defined by*

$$A = \begin{pmatrix} Y_2 & Y_3 & Y_4 & \cdots & Y_r & 1 \\ Y_2 - Z_{22} & Y_3 & Y_4 & \cdots & Y_r & 1 \\ Y_2 - Z_{23} & Y_3 - Z_{33} & Y_4 & \cdots & Y_r & 1 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ Y_2 - Z_{2,r-1} & Y_3 - Z_{3,r-1} & Y_4 - Z_{4,r-1} & \cdots & Y_r & 1 \\ Y_2 - Z_{2r} & Y_3 - Z_{3r} & Y_4 - Z_{4r} & \cdots & Y_r - Z_{rr} & 1 \end{pmatrix}$$

and

$$D = \mathrm{diag}(X_2^{-1}, X_3^{-1}, \ldots, X_r^{-1}, 1 - q_1 x_1).$$

We have

$$\begin{pmatrix} \Phi_0(q) \\ \Phi_1(q) \\ \vdots \\ \Phi_{r-1}(q) \end{pmatrix} = AD \begin{pmatrix} \Psi_0 - \Psi_1 \\ \Psi_1 - \Psi_2 \\ \vdots \\ \Psi_{r-2} - \Psi_{r-1} \\ \Psi_{r-1} \end{pmatrix}.$$

Proof. Using (3.12) and changing the order of the sums, we get

$$\Phi_j(q) = \sum_{k,n,h_1,\ldots,h_r \geq 0} \sum_{(a_1,\ldots,a_n) \in I_0(k,n,h_1,\ldots,h_r)} \sum_{k_i' \geq a_i, i=1,\ldots,n, k_1' \geq j+2} \binom{k_1'-2}{a_1-2} \prod_{l=2}^{n} \binom{k_l'-1}{a_l-1}$$
$$\cdot q_1^{\sum_{l=1}^n (k_l'-a_l)} x_1^{k'-n-\sum_{l=1}^r h_l'} x_2^{n-h_1'} x_3^{h_1'-h_2'} \cdots x_{r+1}^{h_{r-1}'-h_r'} x_{r+2}^{h_r'} \zeta[a_1, \ldots, a_n],$$

where $k', h_1', \ldots, h_r'$ are nonnegative integers such that $(k_1', \ldots, k_n') \in I_j(k', n, h_1', \ldots, h_r')$. Since

$$x_1^{k'-n-\sum_{j=1}^r h_j'} x_2^{n-h_1'} x_3^{h_1'-h_2'} \cdots x_{r+1}^{h_{r-1}'-h_r'} x_{r+2}^{h_r'}$$
$$= \left(\frac{x_{r+2}}{x_1^{r+1}}\right)^n \prod_{m=1}^{n} x_1^{k_m'} \left(\frac{x_1^r x_2}{x_{r+2}}\right)^{\delta_{k_m',1}} \left(\frac{x_1^{r-1} x_3}{x_{r+2}}\right)^{\delta_{k_m',2}} \cdots \left(\frac{x_1 x_{r+1}}{x_{r+2}}\right)^{\delta_{k_m',r}},$$

we have

$$\Phi_j(q) = \sum_{k,n,h_1,\ldots,h_r \geq 0} \sum_{(a_1,\ldots,a_n) \in I_0(k,n,h_1,\ldots,h_r)} S^{(j)}(a_1, \ldots, a_n) x_1^k \left(\frac{x_{r+2}}{x_1^{r+1}}\right)^n \zeta[a_1, \ldots, a_n],$$

where for any fixed $(a_1, \ldots, a_n) \in I_0(k, n, h_1, \ldots, h_r)$, we set

$$S^{(j)}(a_1, \ldots, a_n) = \sum_{k_i' \geq a_i, i=1,\ldots,n, k_1' \geq j+2} \binom{k_1'-2}{a_1-2} \prod_{l=2}^{n} \binom{k_l'-1}{a_l-1} (q_1 x_1)^{\sum_{l=1}^n (k_l'-a_l)}$$
$$\cdot \prod_{m=1}^{n} \left(\frac{x_1^r x_2}{x_{r+2}}\right)^{\delta_{k_m',1}} \left(\frac{x_1^{r-1} x_3}{x_{r+2}}\right)^{\delta_{k_m',2}} \cdots \left(\frac{x_1 x_{r+1}}{x_{r+2}}\right)^{\delta_{k_m',r}}.$$

We compute these terms. Setting

$$S_1^{(j)} = \sum_{i \geq a_1, i \geq j+2} \binom{i-2}{a_1-2} (q_1 x_1)^{i-a_1} \left(\frac{x_1^r x_2}{x_{r+2}}\right)^{\delta_{i,1}} \left(\frac{x_1^{r-1} x_3}{x_{r+2}}\right)^{\delta_{i,2}} \cdots \left(\frac{x_1 x_{r+1}}{x_{r+2}}\right)^{\delta_{i,r}}$$

and

$$S_m = \sum_{i\geq a_m} \binom{i-1}{a_m-1} (q_1x_1)^{i-a_m} \left(\frac{x_1^r x_2}{x_{r+2}}\right)^{\delta_{i,1}} \left(\frac{x_1^{r-1}x_3}{x_{r+2}}\right)^{\delta_{i,2}} \cdots \left(\frac{x_1x_{r+1}}{x_{r+2}}\right)^{\delta_{i,r}}$$

for $m = 2,\ldots,n$, we have $S^{(j)}(a_1,\ldots,a_n) = S_1^{(j)} S_2 \cdots S_n$. For $a_1 \geq r+1$, we have

$$S_1^{(j)} = \sum_{i\geq a_1} \binom{i-2}{a_1-2} (q_1x_1)^{i-a_1} = (1-q_1x_1)^{1-a_1};$$

for $a_1 = l$ with $j+2 \leq l \leq r$, we get

$$S_1^{(j)} = \sum_{i=l}^{r} \binom{i-2}{l-2} (q_1x_1)^{i-l} \frac{x_1^{r+1-i}x_{i+1}}{x_{r+2}} + \sum_{i\geq r+1} \binom{i-2}{l-2} (q_1x_1)^{i-l} = Y_l;$$

and for $a_1 = l$ with $2 \leq l \leq j+1$, we have $S_1^{(j)} = Y_l - Z_{l,j+1}$. For S_m, if $a_m = l$ with $1 \leq l \leq r$, then we have $S_m = X_l$; and if $a_m \geq r+1$, we get $S_m = (1-q_1x_1)^{-a_m}$. Hence if we set $\widetilde{X}_l = X_l(1-q_1x_1)^l$ for $l = 1,\ldots,r$, then for $a_1 \geq r+1$, we have

$$\begin{aligned} S^{(j)}(a_1,\ldots,a_n) =&(1-q_1x_1)^{1-a_1} X_1^{n-h_1} X_2^{h_1-h_2} \cdots X_r^{h_{r-1}-h_r} \\ &\cdot \frac{(1-q_1x_1)^{-\sum_{l=2}^n a_l}}{(1-q_1x_1)^{-[(n-h_1)+2(h_1-h_2)+\cdots+r(h_{r-1}-h_r)]}} \\ =&\frac{1-q_1x_1}{(1-q_1x_1)^k} \widetilde{X}_1^{n-h_1} \widetilde{X}_2^{h_1-h_2} \cdots \widetilde{X}_r^{h_{r-1}-h_r}; \end{aligned}$$

for $a_1 = l$ with $j+2 \leq l \leq r$, we have

$$S^{(j)}(a_1,\ldots,a_n) = \frac{Y_l X_l^{-1}}{(1-q_1x_1)^k} \widetilde{X}_1^{n-h_1} \widetilde{X}_2^{h_1-h_2} \cdots \widetilde{X}_r^{h_{r-1}-h_r};$$

and for $a_1 = l$ with $2 \leq l \leq j+1$, we get

$$S^{(j)}(a_1,\ldots,a_n) = \frac{(Y_l - Z_{l,j+1})X_l^{-1}}{(1-q_1x_1)^k} \widetilde{X}_1^{n-h_1} \widetilde{X}_2^{h_1-h_2} \cdots \widetilde{X}_r^{h_{r-1}-h_r}.$$

Now the result follows from the fact that $I_0(k,n,h_1,\ldots,h_r)$ is the disjoint union

$$I_{r-1}(k,n,h_1,\ldots,h_r) \sqcup \bigsqcup_{i=0}^{r-2} \left(I_i(k,n,h_1,\ldots,h_r) \setminus I_{i+1}(k,n,h_1,\ldots,h_r)\right),$$

and the formulas for $S^{(j)}(a_1,\ldots,a_n)$. □

We begin the proof of Theorem 3.2. Theorem 3.1 gives that

$$\Phi_j(q) = \frac{1}{x_{r+2} - x_1x_{r+1}} \left\{ \sum_{i=0}^{r-1-j} A_i^{(j)} \widetilde{B}_i f_i - A_0^{(j)} \right\},$$

where $f_i = {}_{r+1}\phi_r\left[\begin{matrix} a_1q^i,\dots,a_{r+1}q^i \\ bq^i,\underbrace{q^{i+1},\dots,q^{i+1}}_{r-1}\end{matrix};q,\frac{b}{a_1\dots a_{r+1}}\right]$. Computing the inverse of the matrices A and D in Lemma 3.5, we get

$$\Psi_0 = \frac{(1+q_1u_1)^2}{u_{r+2}-u_1u_{r+1}}\left\{\sum_{j=0}^{r-1}\left[A_j^{(0)}+x_{r+2}\sum_{i=1}^{r-1}(X_{i+1}(1-q_1x_1)-Y_{i+1})\frac{c_{i,j+1}}{x_1^{r-i}}\right]\widetilde{B}_jf_j - A_0\right\}.$$

Here we set

$$c_{i,j+1} = \sum_{m=1}^{\min(i,r-j)}\binom{i-1}{m-1}(-q_1)^{i-m}a_{m,j+1},$$

with

$$a_{ij} = \begin{cases} S_q(r-i,j-1)-x_1S_q(r-1-i,j-1), & \text{if } i+j\le r+1,\\ 0, & \text{if } i+j>r+1.\end{cases}$$

We omit the details of the computations.

Let $\widetilde{A}_j = A_j^{(0)}+\Sigma_1$ with

$$\Sigma_1 = x_{r+2}\sum_{i=1}^{r-1}(X_{i+1}(1-q_1x_1)-Y_{i+1})\frac{c_{i,j+1}}{x_1^{r-i}}.$$

Then we only need to prove that $\widetilde{A}_j = A_j$ for any $0\le j\le r-1$. In the rest of the proof we fix one j with $0\le j\le r-1$.

We have that for any $2\le l\le r$,

$$X_l(1-q_1x_1)-Y_l = \frac{u_1^{r+1-l}(1+q_1u_1)^{l-1}}{u_{r+2}}\sum_{k=l+1}^{r+1}(-q_1)^{k-l}(u_1u_k-u_{k+1}),$$

which induces

$$\Sigma_1 = \frac{1}{1+q_1u_1}\sum_{i=2}^{r}\sum_{k=i+1}^{r+1}\sum_{m=1}^{\min(i-1,r-j)}\binom{i-2}{m-1}(-q_1)^{k-m-1}(u_1u_k-u_{k+1})a_{m,j+1}.$$

Changing the order of the sums, we get

$$\Sigma_1 = \frac{1}{1+q_1u_1}\sum_{m=1}^{r-j}\sum_{k=m+2}^{r+1}\binom{k-2}{m}(-q_1)^{k-m-1}(u_1u_k-u_{k+1})a_{m,j+1}.$$

In the above formula, $a_{m,j+1} = S_q(r-m,j)-x_1S_q(r-m-1,j)$, which implies that

$$\begin{aligned}\Sigma_1 = \sum_{m=j}^{r-1}&\left\{\sum_{k=r-m+1}^{r+1}\left[\binom{k-2}{r-m}+\frac{q_1u_1}{1+q_1u_1}\binom{k-2}{r-m-1}\right]\right.\\ &\left.\times\frac{(-q_1)^{k-r+m-1}}{1+q_1u_1}(u_1u_k-u_{k+1})\right\}S_q(m,j)\\ &+\left[\frac{u_1}{(1+q_1u_1)^2}\sum_{k=2}^{r+1}(-q_1)^{k-1}(u_1u_k-u_{k+1})\right]S_q(r-1,j).\end{aligned}$$

By the definition of $A_j^{(0)}$, and after some computations, we obtain

$$A_j^{(0)} = \sum_{m=j}^{r-1} \left\{ \sum_{k=r-m+1}^{r+1} \left[\binom{k-2}{r-m} + \frac{q_1 u_1}{1+q_1 u_1} \binom{k-2}{r-m-1} \right] \cdot (-q_1)^{k-r+m-2} u_1^{k-r-2} (u_1^{r-k+2} u_k - u_{r+2}) \right\} S_q(m,j) + \left[\frac{u_1}{1+q_1 u_1} \sum_{k=2}^{r+1} (-q_1)^{k-2} u_k + \frac{(-q_1)^r u_1 u_{r+2}}{(1+q_1 u_1)^2} \right] S_q(r-1,j).$$

Finally, we get

$$\widetilde{A}_j = A_j^{(0)} + \Sigma_1 = \sum_{m=j}^{r-1} \tilde{c}_m S_q(m,j) + c S_q(r-1,j),$$

where

$$\tilde{c}_m = \sum_{k=r-m+1}^{r+1} \left[\binom{k-2}{r-m} + \frac{q_1 u_1}{1+q_1 u_1} \binom{k-2}{r-m-1} \right] \cdot (-q_1)^{k-r+m-2} u_1^{k-r-2} (u_1^{r-k+2} u_k - u_{r+2}) + \sum_{k=r-m+1}^{r+1} \left[\binom{k-2}{r-m} + \frac{q_1 u_1}{1+q_1 u_1} \binom{k-2}{r-m-1} \right] \frac{(-q_1)^{k-r+m-1}}{1+q_1 u_1} (u_1 u_k - u_{k+1})$$

and

$$c = \frac{u_1}{1+q_1 u_1} \sum_{k=2}^{r+1} (-q_1)^{k-2} u_k + \frac{(-q_1)^r u_1 u_{r+2}}{(1+q_1 u_1)^2} + \frac{u_1}{(1+q_1 u_1)^2} \sum_{k=2}^{r+1} (-q_1)^{k-1} (u_1 u_k - u_{k+1}).$$

It is trivial to find that $c = \frac{u_1 u_2}{(1+q_1 u_1)^2}$, so what we need to show is $\tilde{c}_m = c_m$. However this is clear. We have finished the proof of Theorem 3.2. □

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