

## GLOBAL GORENSTEIN DIMENSIONS

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*Dedicated to our advisor, Salah-Eddine Kabbaj*

ABSTRACT. In this paper, we prove that the global Gorenstein projective dimension of a ring  $R$  is equal to the global Gorenstein injective dimension of  $R$  and that the global Gorenstein flat dimension of  $R$  is smaller than the common value of the terms of this equality.

### 1. INTRODUCTION

Throughout this paper,  $R$  denotes a non-trivial associative ring with identity and all modules are, if not specified otherwise, left  $R$ -modules. All the results, except Proposition 2.6, are formulated for left modules, and the corresponding results for right modules hold as well. For an  $R$ -module  $M$ , we use  $\text{pd}_R(M)$ ,  $\text{id}_R(M)$ , and  $\text{fd}_R(M)$  to denote, respectively, the classical projective, injective, and flat dimensions of  $M$ . We use  $l.\text{gldim}(R)$  and  $r.\text{gldim}(R)$  to denote, respectively, the classical left and right global dimensions of  $R$ , and  $\text{wgldim}(R)$  to denote the weak global dimension of  $R$ . Recall that the left finitistic projective dimension of  $R$  is the quantity  $l.\text{FPD}(R) = \sup\{\text{pd}_R(M) \mid M \text{ is an } R\text{-module with } \text{pd}_R(M) < \infty\}$ .

Furthermore, we use  $\text{Gpd}_R(M)$ ,  $\text{Gid}_R(M)$ , and  $\text{Gfd}_R(M)$  to denote, respectively, the Gorenstein projective, injective, and flat dimensions of  $M$  (see [3, 4, 8]).

The main result of this paper is an analog of a classical equality that is used to define the global dimension of  $R$ ; see [12, Theorems 9 and 10]. For Noetherian rings the following theorem is proved in [4, Theorem 12.3.1].

**Theorem 1.1.** *The following equality holds:*

$$\sup\{\text{Gpd}_R(M) \mid M \text{ is an } R\text{-module}\} = \sup\{\text{Gid}_R(M) \mid M \text{ is an } R\text{-module}\}.$$

We call the common value of the quantities in the theorem the *left Gorenstein global dimension* of  $R$  and denote it by  $l.\text{Ggldim}(R)$ . Similarly, we set

$$l.\text{wGgldim}(R) = \sup\{\text{Gfd}_R(M) \mid M \text{ is an } R\text{-module}\}$$

and call this quantity the *left weak Gorenstein global dimension* of  $R$ .

**Corollary 1.2.** *The following inequalities hold:*

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- (1)  $l.wGgldim(R) \leq \sup\{l.Ggldim(R), r.Ggldim(R)\}$ ,
- (2)  $l.FPD(R) \leq l.Ggldim(R) \leq l.gldim(R)$ ,
- (3)  $l.wGgldim(R) \leq wgldim(R)$ .

Equalities hold in (2) and (3) if  $wgldim(R) < \infty$ .

The theorem and its corollary are proved in Section 2.

## 2. PROOFS OF THE MAIN RESULTS

The proofs use the following results:

**Lemma 2.1.** *If  $\sup\{Gpd_R(M) \mid M \text{ is an } R\text{-module}\} < \infty$ , then, for a positive integer  $n$ , the following are equivalent:*

- (1)  $\sup\{Gpd_R(M) \mid M \text{ is an } R\text{-module}\} \leq n$ ,
- (2)  $id_R(P) \leq n$  for every  $R$ -module  $P$  with finite projective dimension.

*Proof.* Use [6, Theorem 2.20] and [12, Theorem 9.8]. □

The proof of the main theorem depends on the notions of strong Gorenstein projectivity and injectivity, which were introduced in [1] as follows:

**Definition 2.2** ([1, Definition 2.1]). An  $R$ -module  $M$  is called strongly Gorenstein projective if there exists an exact sequence of projective  $R$ -modules

$$\mathbf{P} = \dots \xrightarrow{f} P \xrightarrow{f} P \xrightarrow{f} P \xrightarrow{f} \dots$$

such that  $M \cong \text{Ker } f$  and such that  $\text{Hom}_R(-, Q)$  leaves the sequence  $\mathbf{P}$  exact whenever  $Q$  is a projective  $R$ -module.

Strongly Gorenstein injective modules are defined dually.

*Remark 2.3.* It is easy to see that an  $R$ -module  $M$  is strongly Gorenstein projective if and only if there exists a short exact sequence of  $R$ -modules  $0 \rightarrow M \rightarrow P \rightarrow M \rightarrow 0$ , where  $P$  is projective, and  $\text{Ext}_R^i(M, Q) = 0$  for some integer  $i > 0$  and for every  $R$ -module  $Q$  with finite projective dimension (or for every projective  $R$ -module  $Q$ ).

Strongly Gorenstein injective modules are characterized in similar terms.

The principal role of these modules is to characterize the Gorenstein projective and injective modules, as follows:<sup>1</sup>

**Lemma 2.4** ([1, Theorems 2.7]). *An  $R$ -module is Gorenstein projective (resp., injective) if and only if it is a direct summand of a strongly Gorenstein projective (resp., injective)  $R$ -module.*

*Proof of Theorem 1.1.* For every integer  $n$  we need to show:

$$\begin{aligned} Gpd_R(M) \leq n \text{ for every } R\text{-module } M &\iff \\ &Gid_R(M) \leq n \text{ for every } R\text{-module } M. \end{aligned}$$

We prove only the direct implication; the converse one has a dual proof.

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<sup>1</sup> In [1], the base ring is assumed to be commutative. However, for the result needed here, one can show easily that this assumption is not necessary.

Assume first that  $M$  is strongly Gorenstein projective. By Remark 2.3 there is a short exact sequence  $0 \rightarrow M \rightarrow P \rightarrow M \rightarrow 0$  with  $P$  is projective. The Horseshoe Lemma (see [10, Remark, page 187]) gives a commutative diagram

$$\begin{array}{ccccccc}
 & & 0 & & 0 & & 0 \\
 & & \downarrow & & \downarrow & & \downarrow \\
 0 & \rightarrow & M & \rightarrow & P & \rightarrow & M \rightarrow 0 \\
 & & \downarrow & & \downarrow & & \downarrow \\
 0 & \rightarrow & I_0 & \rightarrow & I_0 \oplus I_0 & \rightarrow & I_0 \rightarrow 0 \\
 & & \downarrow & & \downarrow & & \downarrow \\
 & & \vdots & & \vdots & & \vdots \\
 & & \downarrow & & \downarrow & & \downarrow \\
 0 & \rightarrow & I_n & \rightarrow & E_n & \rightarrow & I_n \rightarrow 0 \\
 & & \downarrow & & \downarrow & & \downarrow \\
 & & 0 & & 0 & & 0
 \end{array}$$

where  $I_i$  is injective for  $i = 0, \dots, n - 1$ . Since  $P$  is projective,  $\text{id}_R(P) \leq n$  (by Lemma 2.1); hence  $E_n$  is injective. On the other hand, from [7, Theorem 2.2],  $\text{pd}_R(E) \leq n$  for every injective  $R$ -module  $E$ . Then,  $\text{Ext}_R^i(E, I_n) = 0$  for all  $i \geq n+1$ . Then, from Remark 2.3,  $I_n$  is strongly Gorenstein injective, and so  $\text{Gid}_R(M) \leq n$ . This implies, from [6, Proposition 2.19], that  $\text{Gid}_R(G) \leq n$  for any Gorenstein projective  $R$ -module  $G$ , since every Gorenstein projective  $R$ -module is a direct summand of a strongly Gorenstein projective  $R$ -module (Lemma 2.4).

Finally, consider an  $R$ -module  $M$  with  $\text{Gpd}_R(M) \leq m \leq n$ . We can assume that  $\text{Gpd}_R(M) \neq 0$ . Then, there exists a short exact sequence  $0 \rightarrow K \rightarrow N \rightarrow M \rightarrow 0$  such that  $N$  is Gorenstein projective and  $\text{Gpd}_R(K) \leq m - 1$  [6, Proposition 2.18]. By induction,  $\text{Gid}_R(K) \leq n$  and  $\text{Gid}_R(N) \leq n$ . Therefore, using [6, Theorems 2.22 and 2.25] and the long exact sequence of  $\text{Ext}$ , we get that  $\text{Gid}_R(M) \leq n$ .  $\square$

*Proof of Corollary 1.2.* (1) We may assume that  $\sup\{l.\text{Ggldim}(R), r.\text{Ggldim}(R)\} < \infty$ . Then, the character module,  $I^* = \text{Hom}_{\mathbb{Z}}(I, \mathbb{Q}/\mathbb{Z})$ , of every injective right  $R$ -module  $I$  has finite projective dimension (by [7, Theorem 2.2] and [12, Theorem 3.52]). Then, similarly to the proof of [6, Proposition 3.4], we get that every Gorenstein projective  $R$ -module is Gorenstein flat. Therefore,  $l.w\text{Ggldim}(R) \leq \sup\{l.\text{Ggldim}(R), r.\text{Ggldim}(R)\}$ .

(2) and (3) The inequality  $l.\text{FPD}(R) \leq l.\text{Ggldim}(R)$  follows from [6, Theorem 2.28].

The inequalities  $l.\text{Ggldim}(R) \leq l.\text{gldim}(R)$  and  $l.w\text{Ggldim}(R) \leq w\text{gldim}(R)$  hold true since every projective (resp., flat) module is Gorenstein projective (resp., Gorenstein flat).

If  $w\text{gldim}(R) < \infty$ , then, from [10, Corollary 3],  $l.\text{FPD}(R) = l.\text{Ggldim}(R) = l.\text{gldim}(R)$  and, from [1, Corollary 3.8],  $l.w\text{Ggldim}(R) = w\text{gldim}(R)$ .  $\square$

*Remark 2.5.* It is well-known that there are examples of rings for which the left and right global dimensions differ (see [5, pages 74-75] and [9]). Then, by Corollary 1.2, the same examples show that there are also examples of rings for which the left and right Gorenstein global dimensions differ. However, as the classical case [12, Corollary 9.23], we have  $l.\text{Ggldim}(R) = r.\text{Ggldim}(R)$  if  $R$  is Noetherian [4, Theorem 12.3.1].

For the case where  $l.\text{Ggldim}(R) = 0$  or  $r.\text{Ggldim}(R) = 0$ , we have the following result, which is [2, Theorem 2.2] in a non-commutative setting. Recall that a ring is called quasi-Frobenius if it is Noetherian and both left and right self-injective (see [11]).

**Proposition 2.6.** *The following are equivalent:*

- (1)  $R$  is quasi-Frobenius,
- (2)  $l.\text{Ggldim}(R) = 0$ ,
- (3)  $r.\text{Ggldim}(R) = 0$ .

*Proof.* The implications  $1 \Rightarrow 2$  and  $1 \Rightarrow 3$  are well-known (see, for example, [4, Exercise 5, page 257]).

The implication  $2 \Rightarrow 1$  follows from Lemma 2.1 and the Faith-Walker Theorem [11, Theorem 7.56]. The implication  $3 \Rightarrow 1$  is proved similarly.  $\square$

We finish with a generalization of a result of Iwanaga; see [4, Proposition 9.1.10].

**Corollary 2.7.** *Assume that  $l.\text{Ggldim}(R) \leq n$  holds for some non-negative integer  $n$ . If for an  $R$ -module  $M$  one of the numbers  $\text{pd}_R(M)$ ,  $\text{id}_R(M)$ , or  $\text{fd}_R(M)$  is finite, then all of them are less than or equal to  $n$ .*

*Proof.* If  $\text{pd}_R(M)$  is finite, then [6, Proposition 2.27] and the assumption give  $\text{pd}_R(M) = \text{Gpd}_R(M) \leq n$ . The argument for  $\text{id}_R(M) < \infty$  is similar. Finally, Corollary 1.2(2) and the assumption give  $l.\text{FPD}(R) \leq n$ , and then  $\text{fd}_R(M) < \infty$  implies  $\text{pd}_R(M) < \infty$  by [10, Proposition 6].  $\square$

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