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ON THE JAMES AND VON NEUMANN-JORDAN CONSTANTS IN BANACH SPACES

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Abstract. Recently Alonso, Martín and Papini conjectured that the value of the von Neumann-Jordan constant is less than or equal to that of the James constant. This paper presents an affirmative answer to such a conjecture. Moreover, we obtain a sharp estimate for the von Neumann-Jordan constant.

1. Introduction

Both the James constant $J(X)$ and the von Neumann-Jordan constant $C_{\mathrm{NJ}}(X)$ play an important role in the description of various geometric structures. It is therefore worthwhile to clarify the relation between them. Kato, Maligranda and Takahashi [13] are the first who discussed their relation by the following inequality:

$$C_{\mathrm{NJ}}(X) \leq \frac{[J(X)]^2}{1+[J(X)-1]^2}.$$

Since then, many authors have conducted worthwhile research on improving the above estimate. In 2003 Maligranda [15] formulated the following conjecture:

$$C_{\mathrm{NJ}}(X) \leq 1+[J(X)]^2/4.$$

Later on, some weak inequalities were obtained by several authors (see e.g. Maligranda et al. [16], Saejung [17], Takahashi [18]).

Recently Alonso, Martín and Papini [2] presented an inequality:

$$C_{\mathrm{NJ}}(X) \leq 2\left(1+J(X)-\sqrt{2J(X)}\right),$$

which is a strong improvement of Maligranda's conjecture. Wang and Pang [19] strengthened this inequality as

$$C_{\mathrm{NJ}}(X) \leq J(X)+\sqrt{J(X)-1}\left(\sqrt{1+(1-\sqrt{J(X)-1}\,)^2}-1\right),$$

but it is weaker than the inequality $C_{\mathrm{NJ}}(X) \leq J(X)$, conjectured by Alonso, Martín and Papini [2].

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This paper is devoted to an investigation of the relation between the James and von Neumann-Jordan constants. We first state an inequality concerning the constants $A_2(X)$ and $J(X)$, which enables us to clarify the relation between $J(X)$ and $J(X^*)$. Another inequality related to the constants $E(X)$ and $J(X)$ is also obtained. This allows us to get a better estimate for the von Neumann-Jordan constant.

2. Definitions and notation

Let X be a real Banach space with $\dim X \geq 2$ and denote by S_X and B_X the unit sphere and the unit ball, respectively. The *James constant*

$$J(X) = \sup\big\{\|x+y\| \wedge \|x-y\| : x, y \in S_X\big\}$$

and the *von Neumann-Jordan constant*

$$C_{\mathrm{NJ}}(X) = \sup\left\{\frac{\|x+y\|^2 + \|x-y\|^2}{2(\|x\|^2 + \|y\|^2)} : \; x \in S_X, y \in B_X\right\}$$

have been extensively studied. For more details, we refer to [4, 5, 6, 7, 10, 11, 12, 13, 14, 15, 16, 17, 18, 20, 19]. We also need some other constants:

$$\begin{aligned} A_2(X) &= \sup\left\{\frac{\|x+y\| + \|x-y\|}{2} : x, y \in S_X\right\}, \\ E(X) &= \sup\big\{\|x+y\|^2 + \|x-y\|^2 : \; x, y \in S_X\big\}, \\ g(X) &= \inf\big\{\max(\|x+y\|, \|x-y\|) : x, y \in S_X\big\}. \end{aligned}$$

The first constant was defined by Baronti, Casini and Papini [3], and the other two were defined by Gao [9, 10]. It is worthwhile to mention that $J(X)g(X) = 2$ (see [10, 13]).

The *modulus of convexity* $\delta_X(\epsilon) : [0,2] \to [0,1]$ is defined as

$$\delta_X(\epsilon) = \inf\left\{1 - \frac{\|x+y\|}{2} : \; x, y \in S_X, \; \|x-y\| \geq \epsilon\right\}.$$

Obviously, $\delta_X(\epsilon)$ is nondecreasing on $[0,2]$. Moreover, the function $\delta_X(\epsilon)/\epsilon$ is also nondecreasing on $(0,2]$ (see [8]). Recall that a Banach space X is called *uniformly nonsquare* if for any $x, y \in S_X$ there exists a $\delta > 0$ such that either $\|x-y\|/2 \leq 1-\delta$, or $\|x+y\|/2 \leq 1-\delta$. It is well known that X is uniformly nonsquare if and only if $J(X) < 2$. The equality

$$J(X) = 2\big[1 - \delta_X(J(X))\big] \tag{1}$$

holds whenever X is uniformly nonsquare (see [4]).

For simplicity we shall respectively denote $A_2(X), E(X), g(X)$ and $J(X)$ by A, E, g and J if it is required.

3. Some preliminary estimates

Let us first state an estimate of $A_2(X)$ in terms of $J(X)$. The first inequality between them was stated by Alonso and Llorens-Fuster [1] as

$$A_2(X) \leq 1 + \frac{J(X)}{2},$$

which has been improved by Wang and Pang [19] as

$$A_2(X) \leq 1 + \sqrt{J(X) - 1}.$$

The following is a further improvement of the above.

Theorem 1. *For any Banach space X,*

$$A_2(X) \leq \frac{3J(X)-2}{J(X)}. \tag{2}$$

Proof. We may assume that X is uniformly nonsquare, since in the case $J = 2$ the inequality (2) is trivial. To show (2), we consider two cases for any $x, y \in S_X$.

Case 1. $J \leq \|x-y\| \leq 2$. Let $\|x-y\| = \epsilon$. According to the monotonicity of the function $\delta_X(\epsilon)/\epsilon$, one gets

$$\begin{aligned}\|x+y\| + \|x-y\| &\leq \epsilon + 2(1-\delta_X(\epsilon)) \leq \epsilon + 2\left(1 - \frac{\delta_X(J)}{J}\epsilon\right)\\ &= \epsilon + (2-(g-1)\epsilon) \leq \max_{J\leq\epsilon\leq 2}(2-g)\epsilon + 2\\ &= 6-2g,\end{aligned}$$

where the first equality follows from (1).

Case 2. $\|x-y\| \leq J$. If $\|x+y\| \leq J$, then $\|x+y\| + \|x-y\| \leq 2J$. Since

$$J + \frac{2}{J} - 3 = \frac{(J-2)(J-1)}{J} \leq 0,$$

we have $J \leq 3 - 2/J = 3 - g$ and

$$\|x+y\| + \|x-y\| \leq 2J \leq 6-2g.$$

Conversely, if $\|x+y\| \geq J$, let $\|x+y\| = \epsilon$. Then we get as in Case 1 that

$$\|x+y\| + \|x-y\| \leq 6-2g.$$

Therefore, from both cases, we get (2). □

Corollary 2. *For any Banach space X,*

$$A_2(X) - J(X) \leq (\sqrt{2}-1)^2.$$

Proof. According to (2), we have

$$\begin{aligned}A_2(X) - J(X) &\leq \frac{-J^2(X) + 3J(X) - 2}{J(X)}\\ &\leq \max_{\sqrt{2}\leq t\leq 2} \frac{-t^2+3t-2}{t} = (\sqrt{2}-1)^2,\end{aligned}$$

which gives the result. □

Kato, Maligranda and Takahashi [13] discussed the relation between $J(X)$ and $J(X^*)$ and proved the following inequality:

$$2(J(X)-1) \leq J(X^*) \leq J(X)/2 + 1.$$

This gives an answer to the question posed by Gao and Lau [11]. Wang and Pang [19] improved this inequality as

$$1 + (J(X)-1)^2 \leq J(X^*) \leq 1 + \sqrt{J(X)-1}.$$

Applying Theorem 1, we can give a further improvement of the above estimates.

Theorem 3. *For any Banach space X,*

$$\frac{2}{3-J(X)} \leq J(X^*) \leq \frac{3J(X)-2}{J(X)}. \tag{3}$$

Proof. It follows from Theorem 1 that

$$\begin{aligned}\frac{2}{3-J(X)} \leq \frac{2}{3-A_2(X)} &= \frac{2}{3-A_2(X^*)}\\ &\leq J(X^*) \leq A_2(X^*)\\ &= A_2(X) \leq 3-2/J(X),\end{aligned}$$

where one should note that $A_2(X)=A_2(X^*)$ [3, Proposition 2.2]. $\square$

Corollary 4. *For any Banach space X,*

$$|J(X)-J(X^*)| \leq (\sqrt{2}-1)^2.$$

Proof. From (3) it follows that

$$\begin{aligned}J(X)-J(X^*) &\leq J(X) - \max\left\{\sqrt{2}, \frac{2}{3-J(X)}\right\}\\ &= \min\left\{J(X)-\sqrt{2},\ \frac{-J^2(X)+3J(X)-2}{3-J(X)}\right\}\\ &\leq \max_{\sqrt{2}\leq t\leq 2} \min\left\{f(t),\, g(t)\right\}\\ &= (\sqrt{2}-1)^2,\end{aligned}$$

where $f(t)=t-\sqrt{2}$ and $g(t)=(-t^2+3t-2)/(3-t)$. On the other hand, also from (3), it follows that

$$\begin{aligned}J(X^*)-J(X) &\leq \frac{-J^2(X)+3J(X)-2}{J(X)}\\ &\leq \max_{\sqrt{2}\leq t\leq 2} \frac{-t^2+3t-2}{t} = (\sqrt{2}-1)^2,\end{aligned}$$

which completes the proof. $\square$

Next let us turn to the constant $E(X)$. Alonso, Martín and Papini [2] proved that

$$2J^2(X) \leq E(X) \leq 4J(X), \tag{4}$$

which can be strengthened as follows.

Theorem 5. *For any Banach space X,*

$$E(X) \leq \frac{4(J^2(X)+4(J(X)-1)^2)}{J^2(X)}. \tag{5}$$

Proof. Since $J(X)g(X)=2$, it suffices to show that

$$E(X) \leq 4(1+(2-g(X))^2).$$

Assume again that X is uniformly nonsquare and consider two cases for any $x, y \in S_X$.

Case 1. $J \le \|x-y\| \le 2$. Let $\|x-y\| = \epsilon$. The monotonicity of $\delta_X(\epsilon)/\epsilon$ yields

$$\begin{aligned}\|x+y\|^2 + \|x-y\|^2 &\le \epsilon^2 + 4(1-\delta_X(\epsilon))^2\\ &\le \epsilon^2 + 4\left(1 - \frac{\delta_X(J)}{J}\epsilon\right)^2\\ &= \epsilon^2 + (2-(g-1)\epsilon)^2.\end{aligned}$$

Since the function

$$\varphi(\epsilon) := \epsilon^2 + (2-(g-1)\epsilon)^2$$

is increasing on $[J,2]$, we have

$$\|x+y\|^2 + \|x-y\|^2 \le \max_{J\le\epsilon\le 2}\varphi(\epsilon) = 4(1+(2-g)^2).$$

Case 2. $\|x-y\| \le J$. If $\|x+y\| \le J$, then $\|x+y\|^2 + \|x-y\|^2 \le 2J^2$. Note that

$$J^4 - 8J^2 + 8J = J(J-2)(J+1+\sqrt{5})(J+1-\sqrt{5}) \le 0,$$

which implies that

$$J^2 \le \frac{8J(J-1)}{J^2} \le \frac{2(J^2+4(J-1)^2)}{J^2} = 2(1+(2-g)^2).$$

Consequently,

$$\|x+y\|^2 + \|x-y\|^2 \le 2J^2 \le 4(1+(2-g)^2).$$

On the contrary, if $\|x+y\| \ge J$, let $\|x+y\| = \epsilon$. We get as in Case 1 that

$$\|x+y\|^2 + \|x-y\|^2 \le 4(1+(2-g)^2).$$

Therefore, from both cases, we get (5). □

Remark 1. It is easy to see that

$$J^3 - (J^2 + 4(J-1)^2) = (J-1)(J-2)^2 \ge 0,$$

from which we have

$$\frac{J^2+4(J-1)^2}{J^2} \le J.$$

So Theorem 5 improves the inequality (4).

4. Estimate for the von Neumann-Jordan constant

Following the ideas in [17, 20], we can rewrite

$$C_{\mathrm{NJ}}(X) = \sup\{C_{\mathrm{NJ}}(t,X) : 0 \le t \le 1\},$$

where

$$C_{\mathrm{NJ}}(t,X) := \sup\left\{\frac{\|x+ty\|^2 + \|x-ty\|^2}{2(1+t^2)} : x, y \in S_X\right\}.$$

Now let us state the main results.

Theorem 6. *For any Banach space X,*

$$C_{\mathrm{NJ}}(X) \le 1 + \frac{2(J(X)-1)}{\sqrt{J^2(X)+(2-J(X))^2}+2-J(X)}. \tag{6}$$

Proof. Let $x, y \in S_X$ and observe first that for every $0 \le t \le 1$,

$$\|x \pm ty\| \le t\|x \pm y\| + (1-t).$$

This together with (2) and (5) yields

$$\begin{aligned} C_{\mathrm{NJ}}(t, X) &= \sup\left\{\frac{\|x+ty\|^2 + \|x-ty\|^2}{2(1+t^2)} : x, y \in S_X\right\} \\ &\le \frac{Et^2 + 4At(1-t) + 2(1-t)^2}{2(1+t^2)} \\ &\le 1 + 2(2-g)\frac{(1-g)t^2 + t}{1+t^2} := F(t). \end{aligned}$$

Let $t_0 = \sqrt{(g-1)^2+1} - (g-1)$. Then $t_0 \in [0, 1]$ satisfies the equation $(1-g)t+1 = (1+t^2)/2$. It is not difficult to deduce that

$$\begin{aligned} C_{\mathrm{NJ}}(X) &\le \max\{F(t) : 0 \le t \le 1\} = F(t_0) \\ &= 1 + \frac{2-g}{\sqrt{1+(g-1)^2} + g - 1}. \end{aligned}$$

Thus (6) follows from $J(X)g(X) = 2$. □

In the paper [2] Alonso, Martín and Papini posed two questions: (1) Does the inequality $C_{\mathrm{NJ}}(X) \le J(X)$ hold for any space? (2) Does the identity $C_{\mathrm{NJ}}(X) = J(X)$ hold only when both constants are equal to 2? By using Theorem 6 we can affirmatively answer these two questions.

Corollary 7. *For any Banach space X,*

$$C_{\mathrm{NJ}}(X) \le J(X).$$

Moreover the equality holds if and only if X is not uniformly nonsquare.

Proof. Since $\sqrt{2} \le J \le 2$, one gets

$$1 + \frac{2(J-1)}{\sqrt{J^2+(2-J)^2} + 2 - J} - J = \frac{(J-1)(J - \sqrt{J^2+(2-J)^2}\,)}{\sqrt{J^2+(2-J)^2} + 2 - J}.$$

It is not hard to see that the right side above is less than or equal to zero and the equality holds if and only if $J = 2$, and thus the result follows. □

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