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A NOTE ON THE BUCHSBAUM-RIM MULTIPLICITY OF A PARAMETER MODULE

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Abstract. In this article we prove that the Buchsbaum-Rim multiplicity $e(F/N)$ of a parameter module N in a free module $F = A^r$ is bounded above by the colength $\ell_A(F/N)$. Moreover, we prove that once the equality $\ell_A(F/N) = e(F/N)$ holds true for some parameter module N in F, then the base ring A is Cohen-Macaulay.

1. Introduction

Let $(A, \mathfrak{m})$ be a Noetherian local ring with the maximal ideal $\mathfrak{m}$ and $d = \dim A > 0$. Let $F = A^r$ be a free module of rank $r > 0$, and let M be a submodule of F such that F/M has finite length and $M \subseteq \mathfrak{m}F$.

In their article [5] from 1964 Buchsbaum and Rim introduced and studied a multiplicity associated to a submodule of finite colength in a free module. This multiplicity, which generalizes the notion of Hilbert–Samuel multiplicity for ideals, is nowadays called the Buchsbaum-Rim multiplicity. In more detail, it first turns out that the function

$$\lambda(n) := \ell_A(\mathcal{S}_n(F)/\mathcal{R}_n(M))$$

is eventually a polynomial of degree $d + r - 1$, where $\mathcal{S}_A(F) = \bigoplus_{n \geq 0} \mathcal{S}_n(F)$ is the symmetric algebra of F and $\mathcal{R}(M) = \bigoplus_{n \geq 0} \mathcal{R}_n(M)$ is the image of the natural homomorphism from $\mathcal{S}_A(M)$ to $\mathcal{S}_A(F)$. The polynomial $P(n)$ corresponding to $\lambda(n)$ can then be written in the form

$$P(n) = \sum_{i=0}^{d+r-1} (-1)^i e_i \binom{n + d + r - 2 - i}{d + r - 1 - i}$$

with integer coefficients e_i. The *Buchsbaum-Rim multiplicity of M in F*, denoted by $e(F/M)$, is now defined to be the coefficient e_0.

Buchsbaum and Rim also introduced in their article the notion of a parameter module (matrix), which generalizes the notion of a parameter ideal (system of parameters). The module N in F is said to be *a parameter module in F* if the following three conditions are satisfied: (i) F/N has finite length, (ii) $N \subseteq \mathfrak{m}F$,

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and (iii) $\mu_A(N) = d + r - 1$, where $\mu_A(N)$ is the minimal number of generators of N.

Buchsbaum and Rim utilized in their study the relationship between the Buchsbaum-Rim multiplicity and the Euler-Poincaré characteristic of a certain complex and proved the following:

Theorem 1.1 (Buchsbaum-Rim [5, Corollary 4.5])**.** *Let $(A, \mathfrak{m})$ be a Noetherian local ring of dimension $d > 0$. Then the following statements are equivalent:*

(1) *A is a Cohen-Macaulay local ring.*
(2) *For any rank $r > 0$, the equality $\ell_A(F/N) = e(F/N)$ holds true for every parameter module N in $F = A^r$.*

Then it is natural to ask the following:

Question 1.2. (1) Does the inequality $\ell_A(F/N) \geq e(F/N)$ hold true for any parameter module N in F?
(2) Does the equality $\ell_A(F/N) = e(F/N)$ for some parameter module N in F imply that the ring A is Cohen-Macaulay?

The purpose of this article is to give a complete answer to Question 1.2. Our results can be summarized as follows:

Theorem 1.3. *Let $(A, \mathfrak{m})$ be a Noetherian local ring of dimension $d > 0$.*

(1) *For any rank $r > 0$, the two inequalities*

$$\ell_A(F/N) \geq e(F/N) \text{ and } \ell_A(A/I(N)) \geq e(F/N)$$

always hold true for every parameter module N in $F = A^r$, where $I(N)$ is the 0-th Fitting ideal of F/N.

(2) *The following statements are equivalent:*
   (*i*) *A is a Cohen-Macaulay local ring.*
   (*ii*) *For some rank $r > 0$, there exists a parameter module N in $F = A^r$ such that the equality $\ell_A(F/N) = e(F/N)$ holds true.*
   (*iii*) *For some rank $r > 0$, there exists a parameter module N in $F = A^r$ such that the equality $\ell_A(A/I(N)) = e(F/N)$ holds true.*

   When this is the case, the equality $\ell_A(F/N) = \ell_A(A/I(N)) = e(F/N)$ holds true for all parameter modules N in $F = A^r$ of any rank $r > 0$.

Note that the equality $\ell_A(F/N) = \ell_A(A/I(N))$ is known by [1, 2.10].

The proof of our Theorem 1.3 will be completed in section 4. Section 2 is of a preliminary character. In that section we will recall the definition and some basic facts about the generalized Koszul complex. In order to prove Theorem 1.3, we will investigate in section 3 the higher Euler-Poincaré characteristics of the generalized Koszul complex and show that they are non-negative. Finally, in section 4, we will obtain Theorem 1.3 as a corollary of a more general result (Theorem 4.1).

2. Preliminaries

In this section we will recall the definition and some basic facts about the generalized Koszul complex introduced in [3, 8] (for more details, see also [7, Appendix A2.6]).

Let A be a commutative Noetherian ring, and let $n \geq r > 0$ be integers. Let $\mathfrak{a} = (a_{ij})$ be an $r \times n$ matrix over A, and let $I_r(\mathfrak{a})$ denote the ideal generated by the maximal minors of $\mathfrak{a}$. Let F and G be free modules with bases $\{f_1,\dots,f_r\}$ and $\{e_1,\dots,e_n\}$, respectively. Let S be the symmetric algebra of F, and let S_ℓ be the ℓ-th symmetric power of F. Let $\wedge$ be the exterior algebra of G, and let $\wedge^\ell$ be the ℓ-th exterior power of G. Associated with the i-th row $[a_{i1}\cdots\ a_{in}]$ of $\mathfrak{a}$, there is a differentiation homomorphism $\delta_i:\wedge\to\wedge$ given by

$$\delta_i(f_{j_1}\wedge\cdots\wedge f_{j_p})=\sum_{k=1}^{p}(-1)^{k-1}a_{ij_k}f_{j_1}\wedge\cdots\wedge\widehat{f_{j_k}}\wedge\cdots\wedge f_{j_p}.$$

Let $f_i:S\to S$ and $f_i^{-1}:S\to S$ denote the multiplication and division maps by f_i, respectively, i.e.,

$$f_i^{-1}(f_1^{\mu_1}\cdots f_i^{\mu_i}\cdots f_r^{\mu_r})=\begin{cases} f_1^{\mu_1}\cdots f_i^{\mu_i-1}\cdots f_r^{\mu_r} & (\mu_i>0)\\ 0 & (\mu_i=0).\end{cases}$$

Then the generalized Koszul complex $K_\bullet(\mathfrak{a};t)$ associated to a matrix $\mathfrak{a}$ and an integer t is the complex

$$K_\bullet(\mathfrak{a};t):\quad \cdots\to K_{p+1}(\mathfrak{a};t)\overset{d_{p+1}}{\to}K_p(\mathfrak{a};t)\overset{d_p}{\to}K_{p-1}(\mathfrak{a};t)\to\cdots$$

defined by

$$K_p(\mathfrak{a};t)=\begin{cases}\wedge^{r+p-1}\otimes_A S_{p-t-1} & (p\ge t+1)\\ \wedge^p\otimes_A S_{t-p} & (p\le t)\end{cases}$$

and

$$d_{p+1}=\begin{cases}\sum_{j=1}^r\delta_j\otimes f_j^{-1} & (p>t)\\ \delta_r\circ\cdots\circ\delta_1\otimes 1 & (p=t)\\ \sum_{j=1}^r\delta_j\otimes f_j & (p<t).\end{cases}$$

The generalized Koszul complex $K_\bullet(\mathfrak{a};t)$ is a free complex of A-modules. We note that it is of length $n-r+1$ when $-1\le t\le n-r+1$. Also recall that $K_\bullet(\mathfrak{a};t)$ coincides with the ordinary Koszul complex for any t in the case $r=1$, whereas $K_\bullet(\mathfrak{a};0)$ is the Eagon-Northcott complex and $K_\bullet(\mathfrak{a};1)$ is the Buchsbaum-Rim complex. Moreover, the generalized Koszul complex has the following important properties (see [8, 10] and [7, Appendix A2.6]):

Proposition 2.1. *Let $\mathfrak{a}$ be an $r\times n$ matrix over A with $n\ge r>0$. Then*

(1) [8, Theorem 1] *For any $t,p\in\mathbb{Z}$, $I_r(\mathfrak{a})H_p(K_\bullet(\mathfrak{a};t))=(0)$.*
(2) [7, Theorem A2.10] *If the grade of $I_r(\mathfrak{a})$ is at least $n-r+1$, then $K_\bullet(\mathfrak{a};t)$ is acyclic for all $-1\le t\le n-r+1$. Furthermore, if $\mathfrak{a}$ is a generic matrix, then $K_\bullet(\mathfrak{a};t)$ is acyclic for all $t\ge -1$.*

3. Higher Euler-Poincaré characteristics

In this section we will investigate higher Euler-Poincaré characteristics of a generalized Koszul complex.

Throughout this section, let $(A,\mathfrak{m})$ be a Noetherian local ring of dimension $d>0$. Let $F=A^r$ be a free module of rank $r>0$ with a basis $\{f_1,\dots,f_r\}$. Let M be a submodule of F generated by $c_1,c_2,\dots,c_n$, where $n=\mu_A(M)$ is the minimal number of generators of M. Writing $c_j=c_{1j}f_1+\cdots+c_{rj}f_r$ for some $c_{ij}\in A$, we have an $r\times n$ matrix (c_{ij}) associated to M. We call this matrix the matrix of M and denote it by $\widetilde{M}$. Let $I(M)=\mathrm{Fitt}_0(F/M)$ be the 0-th Fitting ideal of F/M. We assume that F/M has finite length and $M\subseteq\mathfrak{m}F$. Then $I(M)$ is an $\mathfrak{m}$-primary ideal, because $\sqrt{I(M)}=\sqrt{\mathrm{Ann}_A(F/M)}$. Hence each homology module $H_p(K_\bullet(\widetilde{M};t))$ has finite length by Proposition 2.1(1). So the Euler-Poincaré characteristics of $K_\bullet(\widetilde{M};t)$ can be defined as follows:

Definition 3.1. For any integer $q \geq 0$, we set

$$\chi_q(K_\bullet(\widetilde{M};t)) := \sum_{p \geq q} (-1)^{p-q} \ell_A(H_p(K_\bullet(\widetilde{M};t)))$$

and call it the q-th partial Euler-Poincaré characteristic of $K_\bullet(\widetilde{M};t)$. When $q = 0$, we simply write $\chi(K_\bullet(\widetilde{M};t))$ for $\chi_0(K_\bullet(\widetilde{M};t))$ and call it the Euler-Poincaré characteristic of $K_\bullet(\widetilde{M};t)$.

Buchsbaum and Rim studied in [5] the Euler-Poincaré characteristic of the Buchsbaum-Rim complex in analogy with the Euler-Poincaré characteristic of the ordinary Koszul complex in the case of usual multiplicities. In 1985 Kirby investigated in [9] Euler-Poincaré characteristics of the complex $K_\bullet(\widetilde{M};t)$ for all t and proved the following:

Theorem 3.2 (Buchsbaum-Rim, Kirby)**.** *For any integer $t \in \mathbb{Z}$, we have*

$$\chi(K_\bullet(\widetilde{M};t)) = \begin{cases} e(F/M) & (n = d + r - 1), \\ 0 & (n > d + r - 1), \end{cases}$$

where $n = \mu_A(M)$ is the minimal number of generators of M. In particular, $\chi(K_\bullet(\widetilde{M};t)) \geq 0$ for all $t \in \mathbb{Z}$.

The last statement holds for the higher Euler-Poincaré characteristics, too:

Theorem 3.3. *For any $q \geq 0$ and any $t \geq -1$, we have*

$$\chi_q(K_\bullet(\widetilde{M};t)) \geq 0.$$

Proof. We use ideas from [6]. Let $\widetilde{M} = (c_{ij}) \in \mathrm{Mat}_{r \times n}(A)$ be the matrix of M, and let $X = (X_{ij})$ be the generic matrix of the same size $r \times n$. Let $A[X] = A[X_{ij} \mid 1 \leq i \leq r,\ 1 \leq j \leq n]$ be a polynomial ring over A, and let $B = A[X]_{(\mathfrak{m},X)}$. We will consider the ring A as a B-algebra via the substitution homomorphism $\phi : B \to A\ ; X_{ij} \mapsto c_{ij}$. Let

$$\mathfrak{b} = \operatorname{Ker} \phi = (X_{ij} - c_{ij} \mid 1 \leq i \leq r,\ 1 \leq j \leq n)B.$$

We note here that $K_\bullet(X;t) \otimes_B A \cong K_\bullet(\widetilde{M};t)$, because the generalized Koszul complex is compatible with the base change. Let $C_t(X) := H_0(K_\bullet(X;t))$. By Proposition 2.1(2), the complex $K_\bullet(X;t)$ is a B-free resolution of the B-module $C_t(X)$ for any $t \geq -1$. By tensoring with A and taking the homology, we have that

$$\begin{aligned} H_p(K_\bullet(\widetilde{M};t)) &\cong H_p(K_\bullet(X;t) \otimes_B A) \\ &\cong \mathrm{Tor}_p^B(C_t(X), A) \end{aligned}$$

for all $p \geq 0$. On the other hand, since the ideal $\mathfrak{b}$ in B is generated by a regular sequence of length rn, the ordinary Koszul complex $K_\bullet(\mathfrak{b})$ associated to the sequence $\widetilde{\mathfrak{b}}$ is a B-free resolution of A. Hence, by tensoring with $C_t(X)$, we can compute the Tor as follows:

$$\mathrm{Tor}_p^B(C_t(X), A) \cong H_p(K_\bullet(\mathfrak{b}) \otimes_B C_t(X)).$$

Therefore, for any $p \geq 0$, we have

$$H_p(K_\bullet(\widetilde{M};t)) \cong H_p(K_\bullet(\mathfrak{b}) \otimes_B C_t(X)).$$

It follows that for any $t \geq -1$ and any $q \geq 0$ we have the equality

$$\chi_q(K_\bullet(\widetilde{M};t)) = \chi_q(K_\bullet(\mathfrak{b}) \otimes_B C_t(X)).$$

Here the right hand side is non-negative by Serre's Theorem ([12, Ch. IV, Appendix II]). Therefore $\chi_q(K_\bullet(\widetilde{M};t)) \geq 0$. □

4. Proof of Theorem 1.3

Theorem 1.3 will be a consequence of the following more general result:

Theorem 4.1. *Let $(A, \mathfrak{m})$ be a Noetherian local ring of dimension $d > 0$.*

(1) *For any rank $r > 0$, the inequality $\ell_A(H_0(K_\bullet(\widetilde{N};t))) \geq e(F/N)$ holds true for any integer $t \geq -1$ and any parameter module N in $F = A^r$.*

(2) *The following statements are equivalent:*

   (*i*) *A is a Cohen-Macaulay local ring.*

   (*ii*) *For some rank $r > 0$, there exists an integer $-1 \leq t \leq d$ and a parameter module N in $F = A^r$ such that the equality $\ell_A(H_0(K_\bullet(\widetilde{N};t))) = e(F/N)$ holds true.*

   When this is the case, the equality $\ell_A(H_0(K_\bullet(\widetilde{N};t))) = e(F/N)$ holds true for any integer $-1 \leq t \leq d$ and any parameter module N in $F = A^r$ of any rank $r > 0$.

Proof. (1): Let N be a parameter module in $F = A^r$, and let $t \geq -1$. By Theorem 3.2 we obtain that

$$\begin{aligned} e(F/N) &= \chi(K_\bullet(\widetilde{N};t)) \\ &= \ell_A(H_0(K_\bullet(\widetilde{N};t))) - \chi_1(K_\bullet(\widetilde{N};t)). \end{aligned}$$

Since $\chi_1(K_\bullet(\widetilde{N};t)) \geq 0$ by Theorem 3.3, the desired inequality follows.

(2): Assume that A is Cohen-Macaulay. Let N be any parameter module in $F = A^r$ of any rank $r > 0$. Let $n = \mu_A(N) = d + r - 1$. Then $\operatorname{grade} I(N) = \operatorname{ht} I(N) = d = n - r + 1$. Hence, by Proposition 2.1(2), $K_\bullet(\widetilde{N};t)$ is acyclic for all $-1 \leq t \leq n - r + 1 = d$. Therefore, by Theorem 3.2, we have $e(F/N) = \chi(K_\bullet(\widetilde{N};t)) = \ell_A(H_0(K_\bullet(\widetilde{N};t)))$. This proves the implication (i) ⇒ (ii) and also the last assertion.

It remains to show the implication (ii) ⇒ (i). Assume that there exist integers $r > 0$, $-1 \leq t \leq d$, and a parameter module N in $F = A^r$ such that $\ell_A(H_0(K_\bullet(\widetilde{N};t))) = e(F/N)$. Arguing as in the proof of Theorem 3.3 and using the same notation, we get

$$\begin{aligned} \chi_1(K_\bullet(\mathfrak{b}) \otimes_B C_t(X)) &= \chi_1(K_\bullet(\widetilde{N};t)) \\ &= \ell_A(H_0(K_\bullet(\widetilde{N};t))) - e(F/N) \\ &= 0. \end{aligned}$$

We observe here that $\sqrt{\operatorname{Ann}_B C_t(X)} = \sqrt{I_r(X)}$ (see [11, Lemma 2.7]). Thus $\dim_B C_t(X) = \dim B/I_r(X) = d + (n+1)(r-1) = rn$ (see [2, (5.12), Corollary]). Therefore $\mathfrak{b}$ is a parameter ideal of $C_t(X)$. Hence we have the equality

$$\ell_B(C_t(X)/\mathfrak{b}C_t(X)) - e(\mathfrak{b}; C_t(X)) = \chi_1(K_\bullet(\mathfrak{b}) \otimes_B C_t(X)) = 0,$$

where $e(\mathfrak{b}; C_t(X))$ is the multiplicity of the module $C_t(X)$ with respect to an ideal $\mathfrak{b}$. Since $\ell_B(C_t(X)/\mathfrak{b}C_t(X)) = e(\mathfrak{b}; C_t(X))$, this implies that $C_t(X)$ is a Cohen-Macaulay B-module. On the other hand, $\operatorname{pd}_B C_t(X) = d$, because the complex

$K_\bullet(X;t)$ is a minimal B-free resolution of $C_t(X)$ of length $n-r+1=d$. Hence, by the Auslander-Buchsbaum formula, we have

$$\begin{aligned} d+rn &= \operatorname{pd}_B C_t(X) + \operatorname{depth}_B C_t(X) \\ &= \operatorname{depth} B \\ &\le \dim B \\ &= d+rn. \end{aligned}$$

Thus $\operatorname{depth} B = \dim B$ so that B is Cohen-Macaulay. Therefore A is also a Cohen-Macaulay local ring. □

Taking $t=0,1$ in Theorem 4.1 now readily gives Theorem 1.3.

We want to close this article with a question. For that, let us first recall the notion of a Buchsbaum local ring, which was introduced by Stückrad and Vogel (for more details on Buchsbaum rings, we refer the reader to [13]). Let A be a Noetherian local ring. Then A is said to be a *Buchsbaum ring* if the difference

$$\ell_A(A/Q) - e(A/Q)$$

between the colength and multiplicity of a parameter ideal Q in A is independent of the choice of Q. This difference, which is an invariant of a Buchsbaum ring A, is denoted by $I(A)$. The ring A is Cohen-Macaulay if and only if it is Buchsbaum and $I(A)=0$. In this sense, the notion of a Buchsbaum ring is a natural generalization of that of a Cohen-Macaulay ring. In Theorem 1.3, the inequality $\ell_A(F/N) \ge e(F/N)$, for any parameter module N in F, is an analogue of the well-known inequality $\ell_A(A/Q) \ge e(A/Q)$ for any parameter ideal Q in A. Also, the characterization of the Cohen-Macaulay property of A based on the equality $\ell_A(F/N) = e(F/N)$ generalizes the usual one using parameter ideals. With these remarks in mind, it is natural to ask the following question:

Question 4.2. Let F be a fixed free module of rank $r>0$. Is it then true that the difference

$$\ell_A(F/N) - e(F/N)$$

between the colength and multiplicity of a parameter module N in F is independent of the choice of N if the ring A is Buchsbaum?

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