

EQUIVALENCES INDUCED BY INFINITELY GENERATED TILTING MODULES

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ABSTRACT. We generalize Brenner and Butler’s Theorem as well as Happel’s Theorem on the equivalences induced by a finitely generated tilting module over Artin algebras, to the case of an infinitely generated tilting module over an arbitrary associative ring establishing the equivalences induced between subcategories of module categories and also at the level of derived categories.

1. INTRODUCTION

Tilting theory started in the context of finitely generated modules over Artin algebras and was further generalized over arbitrary associative rings with unit and to infinitely generated modules (see [6], [8], [9], [1]).

One of the most important features in classical tilting theory is the famous Brenner and Butler’s Theorem [5] establishing two equivalences between suitable categories of finitely generated modules.

A finitely generated tilting module T over an Artin algebra Λ gives rise to a torsion pair $(\mathcal{T}, \mathcal{F})$, where \mathcal{T} is the class of modules generated by T . If D denotes the standard duality and Γ is the endomorphism ring of T , then $D(T)$ is a cotilting Γ -module with an associated torsion pair $(\mathcal{X}, \mathcal{Y})$, where \mathcal{Y} is the class of modules cogenerated by $D(T)$. The Brenner and Butler Theorem states that the functor $\text{Hom}_\Lambda(T, -)$ induces an equivalence between the categories \mathcal{T} and \mathcal{Y} with inverse the functor $- \otimes_\Gamma T$, and the functor $\text{Ext}_\Lambda^1(T, -)$ induces an equivalence between \mathcal{F} and \mathcal{X} with inverse the functor $\text{Tor}_1^\Gamma(-, T)$. (See [16] and [17].)

Moreover, \mathcal{T} is the kernel of the functor $\text{Ext}_\Lambda^1(T, -)$, \mathcal{Y} is the kernel of $\text{Tor}_1^\Gamma(-, T)$, \mathcal{F} is the kernel of $\text{Hom}_\Lambda(T, -)$ and \mathcal{X} is the kernel of $- \otimes_\Gamma T$.

Later on, Happel [15] observed that the natural context in which to interpret the above equivalences is that of derived categories. He proved that the total right derived functor of the functor $\text{Hom}_\Lambda(T, -)$ induces a derived equivalence between the bounded derived category of finitely generated Λ -modules and the bounded derived category of finitely generated Γ -modules.

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Colby and Fuller [6] proved a “Tilting Theorem” for finitely presented tilting modules over an arbitrary associative ring, generalizing Brenner and Butler’s Theorem, and Colpi [7] extended the Tilting Theorem to the wider context of Grothendieck categories.

The first instance of a generalization of Brenner and Butler’s Theorem to infinitely generated tilting modules appears in two papers by Facchini [11], [12] where he studied the equivalences induced by the tilting module ∂ , a divisible module introduced by Fuchs [13] over commutative domains. The theorems proved by Facchini provide a link between the Brenner and Butler tilting equivalences and the equivalences established by Harrison and Matlis between subcategories of modules over a commutative domain R .

If Q is the quotient field of a commutative domain R and K is the module Q/R , then Harrison and Matlis’ Theorem states that the functor $\text{Hom}_R(K, -)$ induces an equivalence between the category of h -divisible torsion modules and the category of torsion-free cotorsion modules. Moreover, the functor $\text{Ext}^1(K, -)$ gives an equivalence between the category of h -reduced torsion R -modules and the category of special cotorsion modules. Thus the similarity with tilting equivalences was evident, and the papers by Facchini showed the advantage of working with a tilting module, namely the module ∂ rather than the module K , even though the formal definition of an infinitely generated tilting module was not yet available.

In this paper we generalize both Brenner and Butler’s Theorem and Facchini’s results to the case of an arbitrary (infinitely generated) tilting module over an associative ring R . If $\text{Mod-}R$ is the category of all right R -modules and $T \in \text{Mod-}R$ is a tilting module, then T induces a torsion pair $(\mathcal{T}, \mathcal{F})$ in $\text{Mod-}R$, where \mathcal{T} is the class of modules generated by T . If S is the endomorphism ring of T , we prove that the dual T^d of T with respect to an injective cogenerator of $\text{Mod-}R$ is a partial cotilting right S -module inducing a torsion pair $(\mathcal{T}_{T^d}, \mathcal{F}_{T^d})$ in $\text{Mod-}S$.

By Theorem 4.5, we prove that the functor $\text{Hom}_R(T, -)$ induces an equivalence between the category \mathcal{T} and the intersection of \mathcal{F}_{T^d} with a suitable subcategory \mathcal{M} of $\text{Mod-}S$, namely the double perpendicular category of the module T^d (see the definition in Section 4). Secondly, the functor $\text{Ext}_R^1(T, -)$ induces an equivalence between \mathcal{F} and the intersection of \mathcal{T}_{T^d} with the subcategory \mathcal{M} . Moreover, the inverses of these equivalences are given by the functors $- \otimes_S T$ and $\text{Tor}_1^S(-, T)$.

The subcategories of $\text{Mod-}S$ equivalent to \mathcal{T} and \mathcal{F} in the above equivalences cannot be interpreted as Gabriel quotients of $\text{Mod-}S$, since there are no Serre subcategories arising in the process. Thus again, as in the case of finitely generated tilting modules, the situation can be better illustrated in the context of derived categories, where the equivalences involved can be formulated in a concise and more expressive way. In fact, if $\mathcal{D}(R)$ and $\mathcal{D}(S)$ are the (unbounded) derived categories of the categories $\text{Mod-}R$ and $\text{Mod-}S$, respectively, we prove that the total right derived functor of the functor $\text{Hom}_R(T, -)$, that is, the functor $\mathbb{R}\text{Hom}_R(T, -)$, induces an equivalence between $\mathcal{D}(R)$ and the quotient category of $\mathcal{D}(S)$ modulo the full triangulated subcategory $\text{Ker}(- \overset{\mathbb{L}}{\otimes}_S T)$, namely the kernel of the total left derived functor of the functor $- \otimes_S T$.

2. PRELIMINARIES

In what follows all rings are associative with unit. We recall some definitions and results.

For a ring R , $\text{Mod-}R$ ($R\text{-Mod}$) will denote the category of all right (left) R -modules. For an R -module M we denote by $\text{p.d.}M$ and $\text{i.d.}M$ the projective and injective dimensions of M , respectively.

If λ is a cardinal, $M^{(\lambda)}$ and M^λ will denote the direct sum and the direct product of λ copies of M , respectively.

Let $\mathcal{C} \subseteq \text{Mod-}R$. Define

$$\begin{aligned}\mathcal{C}^\perp &= \{X \in \text{Mod-}R \mid \text{Ext}_R^i(C, X) = 0 \text{ for all } C \in \mathcal{C}, \text{ for all } i \geq 1\}, \\ {}^\perp\mathcal{C} &= \{X \in \text{Mod-}R \mid \text{Ext}_R^i(X, C) = 0 \text{ for all } C \in \mathcal{C}, \text{ for all } i \geq 1\}.\end{aligned}$$

Definition 2.1 ([8], [1]). An R -module T is *1-tilting* provided

- (T1) $\text{p.d.}T \leq 1$,
- (T2) $\text{Ext}_R^i(T, T^{(\lambda)}) = 0$ for each $i \geq 1$ and every cardinal λ , and
- (T3) there exists an exact sequence

$$0 \rightarrow R \rightarrow T_0 \rightarrow T_1 \rightarrow 0$$

such that $T_i \in \text{Add } T$ for each $0 \leq i \leq 1$.

Here, $\text{Add } T$ denotes the class of all direct summands of arbitrary direct sums of copies of T .

If T is a 1-tilting module, T^\perp is called a *1-tilting class*.

Definition 2.2 ([8]). An R -module T is *1-partial tilting* if T satisfies (T1), (T2) and T^\perp is closed under direct sums.

We have also dual definitions.

Definition 2.3 ([9], [1]). A module C is *1-cotilting* provided

- (C1) $\text{i.d.}C \leq 1$,
- (C2) $\text{Ext}_R^i(C^\lambda, C) = 0$ for each $i \geq 1$ and every cardinal λ , and
- (C3) there exists an exact sequence

$$0 \rightarrow C_1 \rightarrow C_0 \rightarrow W \rightarrow 0$$

such that $C_i \in \text{Prod } C$ for each $0 \leq i \leq 1$ and W is an injective R -cogenerator.

Here, $\text{Prod } C$ denotes the class of all direct summands of arbitrary direct products of copies of C .

If C is a 1-cotilting module, ${}^\perp C$ is called a *1-cotilting class*.

Definition 2.4 ([9]). An R -module C is *1-partial cotilting* if C satisfies (C1), (C2) and ${}^\perp C$ is closed under direct products.

If T and U are 1-tilting (1-cotilting) modules, then T is *equivalent* to U if $T^\perp = U^\perp$ (${}^\perp T = {}^\perp U$), which is the case if and only if $\text{Add } T = \text{Add } U$ ($\text{Prod } T = \text{Prod } U$).

We recall some results on infinitely generated 1-tilting and 1-cotilting modules which give a better understanding of their properties.

- By [8, 1.3] a module T is 1-tilting if and only if $T^\perp = \text{Gen}T$, where $\text{Gen}T$ is the class of modules generated by T .
- By [3] if T is a 1-tilting module, then the tilting class T^\perp is of finite type; that is, there is a set \mathcal{S} of finitely presented modules of projective dimension at most 1, such that $\mathcal{S}^\perp = T^\perp$.
- By [2] 1-cotilting modules are pure injective.

- As a consequence of the above results, we have that every 1-tilting right R -module T induces a torsion pair $(\mathcal{T}, \mathcal{F})$ in $\text{Mod-}R$, where $\mathcal{T} = \text{Gen}T = T^\perp$ and $\mathcal{F} = \text{Ker}(\text{Hom}_R(T, -))$.
Every 1-cotilting right R -module C induces a torsion pair $(\mathcal{T}, \mathcal{F})$ in $\text{Mod-}R$ where $\mathcal{F} = \text{Cogen } C = {}^\perp C$ and $\mathcal{T} = \text{Ker}(\text{Hom}_R(-, C))$.

3. INFINITELY GENERATED 1-TILTING MODULES

In this section we adapt the results proved by Facchini in [11] and [12] for the case of the tilting module ∂ defined over a commutative domain to the case of a tilting module over an arbitrary associative ring.

First of all we have to make a suitable choice of a representative in the equivalence class of a 1-tilting module.

Proposition 3.1. *Let R be a ring and let T_R be a 1-tilting module. Up to equivalence we can assume that T fits in an exact sequence of the form*

$$0 \rightarrow R \rightarrow T \rightarrow T_1 \rightarrow 0,$$

where T_1 is a direct summand of T .

Proof. From condition (T3) in the definition of tilting modules, we have an exact sequence

$$0 \rightarrow R \xrightarrow{\iota} T_0 \rightarrow T_1 \rightarrow 0,$$

where $T_0, T_1 \in \text{Add } T$. Consider the module $T' = T_0 \oplus (T_1)^{(\omega)}$ and let $j: T_0 \rightarrow T'$ be the natural embedding of T_0 in T' . Then we have an exact sequence

$$0 \rightarrow R \xrightarrow{j \circ \iota} T' \rightarrow T_1 \oplus (T_1)^{(\omega)} \rightarrow 0,$$

where $T_1 \oplus (T_1)^{(\omega)} \cong (T_1)^{(\omega)}$ is isomorphic to a direct summand of T' . Thus we also have an exact sequence

$$0 \rightarrow R \rightarrow T' \rightarrow T'_1 \rightarrow 0,$$

with T'_1 a direct summand of T' . Now T' is a 1-tilting module. In fact, T' satisfies conditions (T1) and (T2) since $T' \in \text{Add } T$; it satisfies also (T3), by the above sequence. Moreover, T' and T are equivalent, since $T^\perp \subseteq T'^\perp$ and $T'^\perp = \text{Gen}T' \subseteq \text{Gen}T = T^\perp$. □

Definition 3.2. We say that T_R is a *good* 1-tilting module if it is a 1-tilting module such that the short exact sequence of condition (T3) has the form

$$(a) \quad 0 \rightarrow R \xrightarrow{\mu} T \rightarrow T_1 \rightarrow 0,$$

where T_1 is a direct summand of T .

We denote by S the endomorphism ring of T . As in [11] we fix the following notation:

- (1) $\mu(1_R) = w \in T$.
- (2) Let $\phi \in S$ be the composition $T \rightarrow T_1 \rightarrow T$; then ϕ is an endomorphism of T such that $\text{Ker}\phi = wR$ and $\phi(T)$ is a direct summand of T .
- (3) e is a fixed idempotent endomorphism of T such that $e(T) = \phi(T)$.

Lemma 3.3. *Let T_R be a good 1-tilting module. There is a short exact sequence of left S -modules*

$$(b) \quad 0 \rightarrow I \rightarrow S \rightarrow T \rightarrow 0$$

such that

- (1) I is the left ideal $\{f \in S \mid f(w) = 0\}$ and also $I = S\phi$;
- (2) I is isomorphic to Se ;
- (3) $\text{End}_S(T) \cong R$;
- (4) ${}_S T$ is a cyclically presented partial 1-tilting S -module.

Proof. (1) and (2) follow by applying the functor $\text{Hom}_R(-, T)$ to the exact sequence (a).

So ${}_S T$ is cyclically presented; (3) and (4) follow by [8, Lemma 2.15]. \square

Proposition 3.4 ([8], [12]). *Let T_R be a good 1-tilting module. The following hold:*

- (1) *The natural homomorphism (the counit of the adjunction)*

$$\phi: \text{Hom}_R(T, M) \otimes_S T \rightarrow M$$

is an isomorphism if and only if M is in the tilting class T^\perp .

- (2) $\text{Tor}_1^S(\text{Hom}_R(T, M), T) = 0$, for every right R -module M .

Proof. (1) is proved in [8, Corollary 2.18].

(2) The proof is the same as in [11, Proposition 4.2], but we repeat the argument because our context is different. Let N be a right S -module; applying the functor $N \otimes_S -$ to the exact sequence (b), we get that $\text{Tor}_1^S(N, T)$ is the kernel of the map $N \otimes_S I \rightarrow N$. Since $I = S\phi \cong Se$ we have that $\text{Tor}_1^S(N, T)$ is isomorphic to the kernel of the abelian group morphism $Ne \rightarrow N$ defined by $xe \mapsto x\phi$, for every $x \in N$; hence $\text{Tor}_1^S(N, T)$ is isomorphic to $\{x \in N \mid x\phi = 0\}e$.

So we need to show that, for every right R -module M , if $Y = \{g \in \text{Hom}_R(T, M) \mid g\phi = 0\}$, then $Ye = 0$. Now $g\phi = 0$ if and only if $\text{Kerg} \supseteq \phi(T) = eT$ if and only if $ge = 0$. \square

4. EQUIVALENCES BETWEEN SUBCLASSES OF MODULES

For every right R -module M we denote by M^d the dual of M with respect to an injective cogenerator W of $\text{Mod-}R$, that is, $M^d = \text{Hom}_R(M, W)$.

Proposition 4.1. *Let T_R be a good 1-tilting module. The right S -module T^d satisfies the following properties:*

- (1) $[\text{Tor}_i^S(-, T)]^d \cong \text{Ext}_S^i(-, T^d)$. In particular, $\text{i.d.}(T^d)_S \leq 1$.
- (2) $\text{Tor}_i^S(T^d, T) \cong [\text{Ext}_S^i(T, T)]^d = 0$.
- (3) T^d is a partial 1-cotilting right S -module.

Proof. (1) This follows by a well-known Ext-Tor relation and $\text{i.d.}(T^d)_S \leq 1$, since $\text{p.d.}{}_S T \leq 1$, so $\text{Tor}_i^S(-, T) = 0$, for every $i \geq 2$.

(2) ${}_S T$ is a finitely presented left S -module and $\text{p.d.}{}_S T \leq 1$; hence $\text{Tor}_i^S(T^d, T) \cong [\text{Ext}_S^i(T, T)]^d$ and $\text{Ext}_S^i(T, T) = 0$, for every $i \geq 1$, since ${}_S T$ is a partial 1-tilting module, by Lemma 3.3.

(3) By (1) $\text{i.d.}(T^d)_S \leq 1$. Let $\{N_i\}_i$ be a family of right S -modules such that $\text{Ext}_S^1(N_i, T^d) = 0$, for every i . By (1) we have $\text{Tor}_1^S(N_i, T) = 0$. We show that $\text{Ext}_S^1(\prod_i N_i, T^d) = 0$. By (1) again, we have $\text{Ext}_S^1(\prod_i N_i, T^d) \cong [\text{Tor}_1^S(\prod_i N_i, T)]^d$. Since ${}_S T$ is finitely presented and $\text{p.d.}{}_S T \leq 1$, $\text{Tor}_1^S(-, T)$ commutes with direct products; hence $\text{Ext}_S^1(\prod_i N_i, T^d) \cong [\prod_i (\text{Tor}_1^S(N_i, T))]^d$. But, as noted above, $\text{Tor}_1^S(N_i, T) = 0$. Thus, ${}^\perp T^d$ is closed under direct products. To conclude that T^d is a partial 1-cotilting module, it is enough to check that $\text{Ext}_S^1(T^d, T^d) = 0$. Now, $\text{Ext}_S^1(T^d, T^d) \cong [\text{Tor}_1^S(T^d, T)]^d$ and $\text{Tor}_1^S(T^d, T) = 0$ by (2). \square

Recall that if M is an R -module over a ring R , the preradical Rej_M is the subfunctor of the identity functor defined by $\text{Rej}_M(X) = \bigcap \text{Ker}\{f \mid f \in \text{Hom}_R(X, M)\}$, for every R -module X . Rej_M is always a radical and if it is also idempotent, then it is a torsion radical (see [21]). In this case the associated torsion class consists of the modules X such that $\text{Hom}_R(X, M) = 0$ and the torsion-free class is Cogen M .

Proposition 4.2. *In the same notation as in Proposition 4.1, the partial 1-cotilting S -module T^d satisfies the following conditions:*

- (1) T_S^d is a direct summand of a 1-cotilting right S -module C such that ${}^\perp C = {}^\perp T^d$.
- (2) The preradical Rej_{T^d} is an idempotent radical inducing a torsion pair $(\mathcal{T}_{T^d}, \mathcal{F}_{T^d})$ in $\text{Mod-}S$, where $\mathcal{F}_{T^d} = \text{Cogen } T^d \subseteq {}^\perp T^d$.

Proof. (1) See [10, Theorem 2.11].

(2) This follows by [10, Lemma 2.6] and $\text{Cogen } T^d \subseteq {}^\perp T^d$, since T^d is a partial 1-cotilting module, by Proposition 4.1. □

We now define the subcategories of $\text{Mod-}S$ which will play a crucial role in establishing the equivalences which will be proved by Theorem 4.5.

First, we recall a notion of perpendicular categories. If \mathcal{C} is a category of R -modules we define the *right perpendicular category* \mathcal{C}_\perp as

$$\mathcal{C}_\perp = \{M \mid \text{Hom}_R(\mathcal{C}, M) = \text{Ext}_R^1(\mathcal{C}, M) = 0\}$$

and analogously the *left perpendicular category* as

$${}^\perp \mathcal{C} = \{M \mid \text{Hom}_R(M, \mathcal{C}) = \text{Ext}_R^1(M, \mathcal{C}) = 0\}.$$

We will also use the following definitions.

Definition 4.3. Let \mathcal{C} be a subcategory of an abelian category \mathcal{A} .

- (1) \mathcal{C} has the 2 out of 3 property if for every short exact sequence

$$0 \rightarrow L \rightarrow M \rightarrow N \rightarrow 0$$

in \mathcal{A} with two terms in \mathcal{C} , then the third term is also in \mathcal{C} .

- (2) \mathcal{C} is a Serre subcategory if for every short exact sequence

$$0 \rightarrow L \rightarrow M \rightarrow N \rightarrow 0$$

in \mathcal{A} , M is in \mathcal{C} if and only if L and N are in \mathcal{C} .

Proposition 4.4. *Let*

$$\mathcal{E} = \{N \in \text{Mod-}S \mid N \otimes_S T = \text{Tor}_1^S(N, T) = 0\}.$$

The following hold:

- (1) $\mathcal{E} = \{N \in \text{Mod-}S \mid \text{Ext}_S^1(N, T^d) = \text{Hom}_S(N, T^d) = 0\}$, that is, $\mathcal{E} = {}^\perp \{T^d\}$.
- (2) \mathcal{E} is closed under direct sums, direct summands and has the 2 out of 3 property.
- (3) $\mathcal{E} = {}^\perp T^d \cap \mathcal{T}_{T^d}$.

Proof. (1) The equality follows by the usual homological formulas.

(2) This follows by a direct check.

(3) The equality follows by (1). □

Consider now the right perpendicular category \mathcal{M} of \mathcal{E} , that is,

$$\mathcal{M} = \mathcal{E}_\perp = \{M \in \text{Mod-}S \mid \text{Hom}_S(\mathcal{E}, M) = 0 = \text{Ext}_S^1(\mathcal{E}, M)\}.$$

The next theorem, inspired by Facchini's Theorems, is the generalization of the equivalences proved by Brenner and Butler [5] in the case of a classical 1-tilting module (that is finitely generated) over Artin algebras.

Theorem 4.5 ([11], [12]). *Let R be a ring, T_R a good 1-tilting module and let $(\mathcal{T}, \mathcal{F})$ be the tilting torsion pair associated to T . Let $S = \text{End}_R(T)$. The following hold:*

(1) *There is an equivalence*

$$\text{Mod-}R \supseteq \mathcal{T} \xrightarrow{\text{Hom}_R(T, -)} \mathcal{Y} \subseteq \text{Mod-}S,$$

where $\mathcal{Y} = \mathcal{F}_{T^d} \cap \mathcal{M}$ with inverse $- \otimes_S T$.

(2) *There is an equivalence*

$$\text{Mod-}R \supseteq \mathcal{F} \xrightarrow{\text{Ext}_R^1(T, -)} \mathcal{X} \subseteq \text{Mod-}S,$$

where $\mathcal{X} = \mathcal{T}_{T^d} \cap \mathcal{M}$ with inverse $\text{Tor}_1^S(-, T)$.

Proof. The proof of the two equivalences is essentially the same as [11] and [12], with the suitable translation of the terminology.

In those papers the modules named I -divisible are the modules in \mathcal{T}_{T^d} , that is, the right S -modules N such that $\text{Hom}_S(N, T^d) = 0$ or equivalently, $N \otimes_S T = 0$. The modules called I -reduced are the modules in \mathcal{F}_{T^d} . Moreover, the modules in the class \mathcal{E} are called I -divisible and I -torsion-free.

(1) This is proved by the same arguments as in [11], once it is observed that the modules named I -cotorsion in that paper are the modules in the class $\mathcal{F}_{T^d} \cap \mathcal{M}$.

First one shows as in [11, Theorem 7.1] that for every $M \in \text{Mod-}R$, $\text{Hom}_R(T, M) \in \mathcal{F}_{T^d} \cap \mathcal{M}$. Then one uses that, by Proposition 3.4 (1), $\phi: \text{Hom}_R(T, M) \otimes_S T \rightarrow M$ is an isomorphism if and only if M is in the tilting torsion class \mathcal{T} . Finally one verifies that $\eta: N \rightarrow \text{Hom}_R(T, N \otimes_S T)$ is an isomorphism if and only if N is a right S -module in the class $\mathcal{F}_{T^d} \cap \mathcal{M}$. This is obtained by following the proofs of [11, Theorem 7.2, 7.3].

(2) This is proved as in [12] by noticing that there, slightly differently from the definitions in [11], the modules named I -cotorsion are the modules in the class \mathcal{M} .

First, as in [12, Lemma 1], one proves that $\text{Ext}_R^1(T, M)$ is in the class $\mathcal{T}_{T^d} \cap \mathcal{M}$, for every right R -module M . Secondly, one shows that if M is in the torsion-free class \mathcal{F} , then the natural homomorphism $\xi: \text{Tor}_1^S(\text{Ext}_R^1(T, M), T) \rightarrow M$ is an isomorphism (see [12, Lemma 1]).

Then, one proves that $\text{Tor}_1^S(N, T) \in \mathcal{F}$, for every $N \in \mathcal{T}_{T^d} \cap \mathcal{M}$, and that the natural homomorphism $\theta: N \rightarrow \text{Ext}_R^1(T, \text{Tor}_1^S(N, T))$ is an isomorphism if and only if $N \in \mathcal{T}_{T^d} \cap \mathcal{M}$ (see [12, Lemma 2]). \square

Remark 1. If T is a finitely presented 1-tilting module, then the dual module T^d is a 1-cotilting module over the endomorphism ring of T . Hence, in this case, the category $\mathcal{E} = {}_\perp T^d$ is zero, so \mathcal{M} coincides with $\text{Mod-}S$ and $(\mathcal{X}, \mathcal{Y})$ is the cotilting torsion pair associated to the 1-cotilting module T^d . Thus, we recover both Brenner and Butler's Theorem for the case of Artin algebras and the Colby-Fuller Tilting theorem over an arbitrary ring. So, Theorem 4.5 can be viewed

as the generalization to the case of infinitely generated 1-tilting modules of the Brenner-Butler and Colby-Fuller Theorems.

The categories $\text{Ker}(- \otimes_S T)$ and $\text{Ker}(\text{Tor}_1^S(-, T))$ are not Serre subcategories of $\text{Mod-}S$ in general. Thus, we cannot perform the corresponding quotient categories in Gabriel's sense. However, we can localize the category $\text{Mod-}S$ at a suitable multiplicative system, as we are going to explain.

In the next proposition we use the terminology as in Gabriel and Zisman's book [14].

Proposition 4.6. *Let T_R be a good 1-tilting module and let $(\mathcal{T}, \mathcal{F})$ be the associated torsion pair in $\text{Mod-}R$. Let Σ be the system of morphisms $u \in \text{Mod-}S$ such that $u \otimes_S 1_T$ is invertible in $\text{Mod-}R$. Then the following hold:*

- (1) Σ admits a calculus of left fractions.
- (2) There is an equivalence $\rho: \text{Mod-}S[\Sigma^{-1}] \rightarrow \mathcal{T}$ such that $\rho \circ q = - \otimes_S T$, where $q: \text{Mod-}S \rightarrow \text{Mod-}S[\Sigma^{-1}]$ is the canonical localization functor.
- (3) There is an equivalence between $\text{Mod-}S[\Sigma^{-1}]$ and the category $\mathcal{Y} = \mathcal{F}_{T^d} \cap \mathcal{M}$.

Proof. (1) Note that $N \otimes_S T \in \mathcal{T}$ for every right S -module N and that \mathcal{T} is a full subcategory of $\text{Mod-}R$. Hence the functor $H = \text{Hom}_R(T, -): \mathcal{T} \rightarrow \text{Mod-}S$ is right adjoint to the functor $G = - \otimes_S T: \text{Mod-}S \rightarrow \mathcal{T}$. By Proposition 3.4 (1), the counit adjunction $\phi: GH \rightarrow 1_{\mathcal{T}}$ is invertible, and, by [14, Proposition 1.3], H is a fully faithful functor. Hence Σ admits a calculus of left fractions by [14, 2.5 (b)].

(2) This follows by Proposition 3.4 (1) and by [14, Proposition 1.3].

(3) Combine (2) with Theorem 4.5(1). □

Remark 2. We couldn't get an analogous result for the pair of functors $\text{Ext}_R^1(T, -)$ and $\text{Tor}_1^S(-, T)$ because they are not an adjoint pair in general.

Moreover, we don't know whether the category of fractions $\text{Mod-}S[\Sigma^{-1}]$, considered in Proposition 4.6, is the quotient of $\text{Mod-}S$ modulo a suitable subcategory.

The above remark indicates that a better understanding of the whole situation can be obtained in the setting of derived categories.

5. DERIVED EQUIVALENCE

Before stating the main result of this section we recall some notions and facts about derived categories which will be used later on.

Let $\mathcal{D}(R)$ and $\mathcal{D}(S)$ be the unbounded derived categories of $\text{Mod-}R$ and $\text{Mod-}S$, respectively. The following hold:

- (Bökstedt and Neeman [4] or Spaltenstein [20]) For every complex $M \in \mathcal{D}(R)$ there is a quasi isomorphism $M \rightarrow I$, where I is a complex with injective terms. I is also denoted by $\mathbf{i}M$ and called a K -injective or fibrant resolution of M .

Symmetrically, for every complex $M \in \mathcal{D}(R)$ there is a quasi isomorphism $P \rightarrow M$, where P is a complex with projective terms. P is also denoted by $\mathbf{p}M$ and called a K -projective or cofibrant resolution of M .

- ([19, Theorem 3.2 (b)] and Bökstedt and Neeman [4]) Every additive functor F defined on the module category $\text{Mod-}R$ admits a total right derived functor $\mathbb{R}F$ and a total left derived functor $\mathbb{L}F$ defined on $\mathcal{D}(R)$.

Moreover, if M^\cdot is a complex in $\mathcal{D}(R)$, then $\mathbb{R}F(M^\cdot) = F(\underline{\mathbf{i}}M^\cdot)$ and $\mathbb{L}F(M^\cdot) = F(\underline{\mathbf{p}}M^\cdot)$. (We also denote by F the functor induced on the homotopy category.)

- ([19, Theorem 3.2 (c)]) If T is an S - R -bimodule, then the adjoint pair (G, H) of functors given by

$$H = \text{Hom}_R(T, -): \text{Mod-}R \rightleftarrows \text{Mod-}S: G = - \otimes_S T$$

induces an adjoint pair of total derived functors

$$\mathbb{R}H = \mathbb{R}\text{Hom}_R(T, -): \mathcal{D}(R) \rightleftarrows \mathcal{D}(S): \mathbb{L}G = - \overset{\mathbb{L}}{\otimes}_S T.$$

Theorem 5.1. *Let T_R be a good 1-tilting module with endomorphism ring S . The following hold:*

- (1) *The counit adjunction morphism*

$$\eta: \mathbb{L}G \circ \mathbb{R}H \rightarrow \text{Id}_{\mathcal{D}(R)}$$

is invertible.

- (2) *The functor $\mathbb{R}H: \mathcal{D}(R) \rightarrow \mathcal{D}(S)$ is fully faithful.*
- (3) *There is a triangle equivalence $\Theta: \mathcal{D}(S)[\Sigma^{-1}] \rightarrow \mathcal{D}(R)$ such that $\mathbb{L}G = \Theta \circ q$, where q is the canonical quotient functor $q: \mathcal{D}(S) \rightarrow \mathcal{D}(S)[\Sigma^{-1}]$.*
- (4) *If Σ is the system of morphisms $u \in \mathcal{D}(S)$ such that $\mathbb{L}Gu$ is invertible in $\mathcal{D}(R)$, then Σ admits a calculus of left fractions and the category $\mathcal{D}(S)[\Sigma^{-1}]$ coincides with the quotient category $\mathcal{D}(S)$ modulo the full triangulated subcategory $\text{Ker}(\mathbb{L}G)$ of the objects annihilated by the functor $\mathbb{L}G$.*

We first prove condition (1) of Theorem 5.1 by a lemma.

Lemma 5.2. *In the assumptions of Theorem 5.1, the counit adjunction morphism*

$$\eta: \mathbb{L}G \circ \mathbb{R}H \rightarrow \text{Id}_{\mathcal{D}(R)}$$

is invertible.

Proof. Let M^\cdot be a complex in $\mathcal{D}(R)$ and consider a K -injective resolution $\underline{\mathbf{i}}M^\cdot$ of M^\cdot . We have:

$$(1) \quad \mathbb{R}H(M^\cdot) = \mathbb{R}\text{Hom}_R(T, -)(M^\cdot) = H(\underline{\mathbf{i}}M^\cdot).$$

Let $C^\cdot = H(\underline{\mathbf{i}}M^\cdot)$. C^\cdot is a complex of right S -modules and

$$\mathbb{L}G(C^\cdot) = \mathbb{L}(- \otimes_S T)(C^\cdot) = G(\underline{\mathbf{p}}C^\cdot),$$

where $\underline{\mathbf{p}}C^\cdot$ is a K -projective resolution of C^\cdot as a complex in $\mathcal{D}(S)$.

Consider the complex $T^\cdot: 0 \rightarrow {}_S T \rightarrow 0$ concentrated in degree 0. A K -projective resolution $\underline{\mathbf{p}}T^\cdot$ of T^\cdot in $\mathcal{D}(S)$ is the complex $0 \rightarrow I \xrightarrow{\delta} S \rightarrow 0$ (from the exact sequence (b) in Lemma 3.3).

From the quasi isomorphism $\underline{\mathbf{p}}T^\cdot \rightarrow T^\cdot$ and $\underline{\mathbf{p}}C^\cdot \rightarrow C^\cdot$ we get the chain of quasi isomorphisms:

$$G(\underline{\mathbf{p}}C^\cdot) = \underline{\mathbf{p}}C^\cdot \otimes_S T \leftarrow \underline{\mathbf{p}}C^\cdot \otimes_S \underline{\mathbf{p}}T^\cdot \rightarrow C^\cdot \otimes_S \underline{\mathbf{p}}T^\cdot.$$

Thus, $\mathbb{L}G(C^\cdot) = C^\cdot \otimes_S \underline{\mathbf{p}}T^\cdot$ and this gives

$$\mathbb{L}G(C^\cdot) = C^\cdot \overset{\mathbb{L}}{\otimes}_S T = \text{Cone}(1 \otimes \delta).$$

From the exact sequence (b) in Lemma 3.3 we obtain the exact sequence of complexes of right R -modules:

$$\mathrm{Tor}_1^S(C^\cdot, T) \rightarrow C^\cdot \otimes_S I \rightarrow C^\cdot \otimes_S S \rightarrow C^\cdot \otimes_S T \rightarrow 0.$$

Now recalling that C^\cdot is the complex $\mathbb{R}H(M^\cdot) = H(\mathbf{i}M^\cdot)$, Proposition 3.4 (2) yields that the complex $\mathrm{Tor}_1^S(C^\cdot, T)$ has zero terms; hence we have the short exact sequence of complexes of right R -modules:

$$(2) \quad 0 \rightarrow C^\cdot \otimes_S I \rightarrow C^\cdot \otimes_S S \rightarrow C^\cdot \otimes_S T \rightarrow 0,$$

From (2) we obtain the long exact sequence in cohomology:

$$(*) \quad \begin{aligned} \dots &\rightarrow H^{n+1}(C^\cdot \otimes_S T) \rightarrow H^n(C^\cdot \otimes_S I) \\ &\rightarrow H^n(C^\cdot \otimes_S S) \rightarrow H^n(C^\cdot \otimes_S T) \rightarrow \dots \end{aligned}$$

We also have the exact sequence of complexes of right R -modules:

$$0 \rightarrow C^\cdot \otimes S \rightarrow \mathrm{Cone}(1 \otimes \delta) \rightarrow (C^\cdot \otimes I)[1] \rightarrow 0$$

from which we get the long exact sequence

$$(**) \quad \begin{aligned} \dots &\rightarrow H^{n+1}(\mathrm{Cone}(1 \otimes \delta)) \rightarrow H^n(C^\cdot \otimes_S I) \rightarrow H^n(C^\cdot \otimes_S S) \\ &\rightarrow H^n(\mathrm{Cone}(1 \otimes \delta)) \rightarrow C^\cdot \otimes_S I \rightarrow \dots \end{aligned}$$

Now comparing (*) with (**) we conclude that, for every $n \in \mathbb{N}$,

$$H^n(C^\cdot \otimes_S T) \cong H^n(\mathrm{Cone}(1 \otimes \delta)) \cong H^n(C^\cdot \overset{\mathbb{L}}{\otimes}_S T).$$

Hence,

$$\mathbb{L}G(C^\cdot) = C^\cdot \overset{\mathbb{L}}{\otimes}_S T \text{ is quasi-isomorphic to } C^\cdot \otimes_S T.$$

Letting $I = \mathbf{i}M^\cdot$, we have $C^\cdot = \mathrm{Hom}_R(T, I)$ and we have also the commutative diagram:

$$\begin{array}{ccccccc} \dots & \longrightarrow & \mathrm{Hom}_R(T, I^n) \otimes_S T & \longrightarrow & \mathrm{Hom}_R(T, I^{n+1}) \otimes_S T & \longrightarrow & \dots \\ & & \nu \downarrow & & \downarrow \nu & & \\ \dots & \longrightarrow & I^n & \longrightarrow & I^{n+1} & \longrightarrow & \dots \end{array}$$

where the vertical maps are canonical isomorphisms by Proposition 3.4 (1), since I is a complex of injective right R -modules, hence belonging to the tilting class T^\perp .

Hence, $\mathrm{Hom}_R(T, I) \otimes_S T$ and I are canonically isomorphic as complexes of R -modules, so we have:

$$\mathbb{L}G(\mathbb{R}H(M^\cdot)) = H(\mathbf{i}M^\cdot) \overset{\mathbb{L}}{\otimes}_S T \cong \mathbf{i}M^\cdot \cong M^\cdot. \quad \square$$

Proof of Theorem 5.1. Condition (1) is proved by Lemma 5.2, and the equivalence of (1) with the other conditions follows essentially by applying [14, Proposition 1.3].

To complete the proof we add only a few comments.

The equivalence $\Theta: \mathcal{D}(S)[\Sigma^{-1}] \rightarrow \mathcal{D}(R)$, guaranteed by [14, Proposition 1.3], is a triangle equivalence, since $\mathbb{L}G$ is a triangle functor and q is the canonical localization functor, so that the triangles in $\mathcal{D}(S)[\Sigma^{-1}]$ are images of triangles in $\mathcal{D}(S)$.

The functor $\mathbb{L}G = - \overset{\mathbb{L}}{\otimes}_S T$ is a triangle functor; hence $\text{Ker}(\mathbb{L}G)$ is a full triangulated subcategory of $\mathcal{D}(S)$. It is well known that the quotient category $\mathcal{D}(S)/\text{Ker}(\mathbb{L}G)$ is the localization of $\mathcal{D}(S)$ at the multiplicative system Σ consisting of the morphisms $u \in \mathcal{D}(S)$ such that there exists a triangle

$$K \rightarrow M \xrightarrow{u} N \rightarrow K[1],$$

where $K \in \text{Ker}(\mathbb{L}G)$ and $M, N \in \mathcal{D}(S)$. Thus Σ coincides with the system of morphisms $u \in \mathcal{D}(S)$ such that $\mathbb{L}G(u)$ is invertible in $\mathcal{D}(R)$. \square

Added in proof. In a forthcoming paper by F. Mantese, A. Tonolo and the author we will generalize the results to the case of an n -tilting module.

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