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FRACTIONAL CAUCHY TRANSFORMS, MULTIPLIERS AND CESÀRO OPERATORS

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ABSTRACT. Let B_n denote the unit ball in $\mathbb{C}^n$, $n \geq 1$. Given an $\alpha > 0$, let $\mathcal{K}_\alpha(n)$ denote the class of functions defined for $z \in B_n$ by integrating the kernel $(1 - \langle z, \zeta \rangle)^{-\alpha}$ against a complex Borel measure on the sphere $\{\zeta \in \mathbb{C}^n : |\zeta| = 1\}$. We study properties of the holomorphic functions g such that $fg \in \mathcal{K}_\alpha(n)$ for all $f \in \mathcal{K}_\alpha(n)$. Also, we investigate extended Cesàro operators on $\mathcal{K}_\alpha(n)$.

1. Introduction

For $n \geq 1$, put $B_n = \{z \in \mathbb{C}^n : |z| < 1\}$. Let $\mathcal{M}(n)$ denote the space of complex-valued Borel measures defined on the sphere ∂B_n.

Let $\alpha > 0$. Given a measure $\mu \in \mathcal{M}(n)$, its fractional Cauchy transform of order α is defined by the formula

$$K_\alpha[\mu](z) = \int_{\partial B_n} \frac{1}{(1 - \langle z, \zeta \rangle)^\alpha} \, d\mu(\zeta), \quad z \in B_n.$$

Here and in what follows we use the principal branch of the logarithm. Put

$$\mathcal{K}_\alpha(n) = \{K_\alpha[\mu] : \mu \in \mathcal{M}(n)\}.$$

Let $\mathcal{H}ol(B_n)$ denote the space of holomorphic functions in the ball B_n. A function $g \in \mathcal{H}ol(B_n)$ is called a (pointwise) multiplier for $\mathcal{K}_\alpha(n)$ provided that $fg \in \mathcal{K}_\alpha(n)$ for every $f \in \mathcal{K}_\alpha(n)$. Let $\mathfrak{M}_\alpha(n)$ denote the set of all multipliers for $\mathcal{K}_\alpha(n)$.

Standard arguments show that $\mathcal{K}_\alpha(n)$, $\alpha > 0$, is a Banach space with respect to the norm defined by

$$\|f\|_{\mathcal{K}_\alpha(n)} = \inf \left\{ \|\mu\|_{\mathcal{M}(n)} : f = K_\alpha[\mu] \right\}, \quad f \in \mathcal{K}_\alpha(n).$$

The space $\mathfrak{M}_\alpha(n)$, $\alpha > 0$, is a Banach algebra with the natural norm defined by

$$\|g\|_{\mathfrak{M}_\alpha(n)} = \sup \left\{ \|fg\|_{\mathcal{K}_\alpha(n)} : \|f\|_{\mathcal{K}_\alpha(n)} \leq 1 \right\}, \quad g \in \mathfrak{M}_\alpha(n).$$

The classical spaces $\mathcal{K}_1(1)$ and $\mathfrak{M}_1(1)$ are investigated in monograph [4]. Various properties of the spaces $\mathcal{K}_\alpha(1)$ and $\mathfrak{M}_\alpha(1)$ are collected in monograph [10]. For

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$n \in \mathbb{N}$, certain properties of $\mathcal{K}_\alpha(n)$ are obtained in [8]. The present work focuses on the study of $\mathfrak{M}_\alpha(n)$, $n \in \mathbb{N}$.

Necessary conditions. The main properties of $\mathfrak{M}_\alpha(1)$, $\alpha > 0$, were obtained in [16, 15, 9, 12, 10]. As far as the author is aware, only basic properties of the spaces $\mathfrak{M}_\alpha(n)$ are known for arbitrary $n \in \mathbb{N}$. Namely, assume that $g \in \mathfrak{M}_\alpha(n)$, $\alpha > 0$, $n \in \mathbb{N}$. Then, as shown in [8, Proposition 7.3], we have

$$\sup_{\xi \in \partial B_n} \int_0^1 |\mathcal{R} g(r\xi)|\, dr < +\infty,$$

where the radial derivative $\mathcal{R} g$ is defined by the identity

$$\mathcal{R} g(z) = \sum_{j=1}^n z_j \frac{\partial g}{\partial z_j}(z), \quad z \in B_n.$$

In particular, $\mathfrak{M}_\alpha(n) \subset H^\infty(B_n)$, where $H^\infty(B_n)$ denotes the space of bounded holomorphic functions in the ball. Also, the embedding $\mathfrak{M}_\alpha(n) \subset \mathcal{K}_\alpha(n)$ holds.

On the other hand, let ν_n denote the normalized Lebesgue measure on the ball B_n. The following theorem is formulated in [12].

Theorem 1.1 (F. Nazarov, unpublished)**.** *Assume that* $0 < \alpha < 1$*,* $g \in \mathfrak{M}_\alpha(1)$ *and* $\alpha < \beta \le 1$*. Then*

$$\sup_{\zeta \in \partial B_1} \int_{B_1} \frac{|g'(z)|(1-|z|)^{\beta-1}}{|1 - z\overline{\zeta}|^\beta}\, d\nu_1(z) < \infty.$$

Put $R = \mathcal{R} + I$. In this paper, we prove the following result for arbitrary $n \in \mathbb{N}$.

Theorem 1.2. *Suppose that* $n \in \mathbb{N}$*,* $M \in \{1, \dots, n\}$*,* $\alpha > n - M$ *and* $g \in \mathfrak{M}_\alpha(n)$*. Then*

$$\sup_{\zeta \in \partial B_n} \int_{B_n} \frac{|R^k g(z)|(1-|z|)^{\alpha+M-n-1}}{|1 - \langle z, \zeta\rangle|^{\alpha+M-k}} \left(\log \frac{e}{1-|z|}\right)^{-1-\varepsilon} d\nu_n(z) < \infty \tag{1.1}$$

for $k = 1, \dots, M$ *and any* $\varepsilon > 0$*.*

The proof of Theorem 1.2 uses the following fact.

Proposition 1.3 ([8, Proposition 2.5])**.** *Let* $n \in \mathbb{N}$ *and let* $0 < \beta < \alpha$*. Then* $\mathfrak{M}_\beta(n) \subset \mathfrak{M}_\alpha(n)$*.*

Sufficient conditions. Some explicit conditions sufficient for the property $g \in \mathfrak{M}_\alpha(n)$, $n \in \mathbb{N}$, are given in [7]. In the present paper, we apply a different method, so we obtain the following improvement of [7, Theorem 4.1].

Theorem 1.4. *Let* $n \in \mathbb{N}$*. Suppose that* $M \in \{1, \dots, n\}$*,* $n > \alpha > n - M$*,* $g \in \mathfrak{M}_n(n)$ *and*

$$\sup_{\zeta \in \partial B_n} \int_{B_n} \frac{|R^k g(z)|(1-|z|)^{\alpha+M-n-1}}{|1 - \langle z, \zeta\rangle|^{\alpha+M-k}}\, d\nu_n(z) < \infty, \quad k = 1, \dots, M. \tag{1.2}$$

Then $g \in \mathfrak{M}_\alpha(n)$*.*

Note that the sufficient condition in Theorem 1.4 is not too far from the necessary one in Theorem 1.2. Indeed, dropping the logarithmic term in (1.1), we obtain (1.2).

Organization of the paper. Some definitions and auxiliary results are collected in Section 2. Theorem 1.2 is proved in Section 3. Theorem 1.4 and other sufficient conditions are obtained in Section 4. Taking into account Proposition 1.3, it is natural to ask whether $\mathfrak{M}_\beta(n) \neq \mathfrak{M}_\alpha(n)$ for $\beta \neq \alpha$. A partial answer to this question is given in Section 5. Finally, extended Cesàro operators on $\mathcal{K}_\alpha(n)$ are investigated in Section 6.

2. Preliminaries

For $t \in \mathbb{R}$, define an operator $R^t : \mathcal{H}ol(B_n) \to \mathcal{H}ol(B_n)$ as follows. If

$$f(z) = \sum_{k=0}^{\infty} f_k(z), \quad z \in B_n,$$

is the homogeneous expansion of $f \in \mathcal{H}ol(B_n)$, then

$$R^t f(z) = \sum_{k=0}^{\infty} (k+1)^t f_k(z), \quad z \in B_n.$$

In particular, $R^1 = R = \mathcal{R} + I$. Also, note that

$$R(fg) = f \cdot Rg + \mathcal{R}f \cdot g \quad \text{for all } f, g \in \mathcal{H}ol(B_n). \tag{2.1}$$

Theorem 2.1 ([8, Corollaries 5.6 and 5.7])**.** *Suppose that* $n \in \mathbb{N}$, $f \in \mathcal{H}ol(B_n)$ *and* $\alpha > 0$. *Then* $f \in \mathcal{K}_\alpha(n)$ *if and only if* $\mathcal{R}f \in \mathcal{K}_{\alpha+1}(n)$ *if and only if* $Rf \in \mathcal{K}_{\alpha+1}(n)$.

Let $q, p > 0$. Put

$$L^p_q(B_n) = L^p(B_n, (1-|z|)^{q-1}\, d\nu_n(z)).$$

Let $s \geq 0$. By definition, the Bergman–Sobolev space $A^p_{q,s}(B_n)$ consists of those $f \in \mathcal{H}ol(B_n)$ for which $R^s f \in L^p_q(B_n)$.

Theorem 2.2 ([8, Propositions 5.2 and 5.3])**.** *Let* $n \in \mathbb{N}$. *Suppose that* $j \in \{0, 1, \ldots, n\}$ *and* $\alpha > n - j$. *Then* $A^1_{\alpha-n+j,j}(B_n) \subset \mathcal{K}_\alpha(n)$.

Proposition 2.3. *Let* $n \in \mathbb{N}$. *Suppose that* $j \in \{0, 1, \ldots, n\}$, $\alpha > n - j$ *and* $f \in \mathcal{K}_\alpha(n)$. *Then*

$$\int_{B_n} |R^j f(z)|(1-|z|)^{\alpha+j-n-1} \left(\log \frac{e}{1-|z|}\right)^{-1-\varepsilon} d\nu_n(z) < \infty$$

for all $\varepsilon > 0$.

Proof. By Theorem 2.1, $R^j f \in \mathcal{K}_{\alpha+j}(n)$; that is,

$$R^j f(z) = \int_{\partial B_n} \frac{1}{(1-\langle z, \zeta\rangle)^{\alpha+j}}\, d\mu(\zeta), \quad z \in B_n,$$

for some $\mu \in \mathcal{M}(n)$. Let σ_n denote the normalized Lebesgue measure on the sphere ∂B_n. Then we obtain

$$\int_{\partial B_n} |R^j f(r\xi)|\, d\sigma_n(\xi) \leq \int_{\partial B_n} \int_{\partial B_n} \frac{1}{|1 - r\langle \xi, \zeta\rangle|^{\alpha+j}}\, d\sigma_n(\xi)\, d|\mu|(\zeta)$$

for $0 \leq r < 1$. We have $\alpha + j > n$; hence,

$$\int_{\partial B_n} |R^j f(r\xi)|\, d\sigma_n(\xi) \leq C\|\mu\|_{\mathcal{M}(n)}(1-r)^{n-\alpha-j}$$

by [14, Proposition 1.4.10]. Thus, integrating in polar coordinates, we obtain

$$\begin{aligned}\int_{B_n} |R^j f(z)|(1-|z|)^{\alpha+j-n-1} \left(\log\frac{e}{1-|z|}\right)^{-1-\varepsilon} d\nu_n(z) \\ \le C\|\mu\| \int_0^1 (1-r)^{-1} \left(\log\frac{e}{1-r}\right)^{-1-\varepsilon} dr < \infty\end{aligned}$$

for all $\varepsilon > 0$. □

3. Proof of Theorem 1.2

Lemma 3.1. *Suppose that $n \in \mathbb{N}$, $\beta > 0$ and $g \in \mathfrak{M}_\beta(n)$. Then*

$$fR^m g \in \mathcal{K}_{\beta+j+m}(n) \text{ for all } f \in \mathcal{K}_{\beta+j}(n),\ j = 0,1,\dots,\ m = 0,1,\dots. \tag{3.1}$$

Proof. We argue by induction on m. By Proposition 1.3, we have $\mathfrak{M}_\beta(n) \subset \mathfrak{M}_{\beta+j}(n)$, $j = 0,1,\dots$. Hence, (3.1) holds for $m = 0$.

Now, assume that (3.1) holds for some $m \ge 0$. Let $f \in \mathcal{K}_{\beta+j}(n)$. For the induction step we use identity (2.1), which guarantees that $fR^{m+1}g = R(fR^m g) - \mathcal{R}f \cdot R^m g$.

On the one hand, the induction hypothesis and Theorem 2.1 imply $R(fR^m g) \in \mathcal{K}_{\beta+j+m+1}(n)$. On the other hand, $\mathcal{R}f \in \mathcal{K}_{\beta+j+1}(n)$ by Theorem 2.1. Hence, $\mathcal{R}f \cdot R^m g \in \mathcal{K}_{\beta+j+m+1}(n)$ by the induction hypothesis.

In sum, (3.1) holds for $m+1$. The induction step is complete. □

Proof of Theorem 1.2. Let $k \in \{1,\dots,M\}$. We have $\alpha + M - k \ge \alpha$; therefore, $g \in \mathfrak{M}_{\alpha+M-k}(n)$ by Proposition 1.3. Applying Lemma 3.1 with $j = 0$, $m = k$ and $\beta = \alpha + M - k$, we obtain

$$fR^k g \in \mathcal{K}_{\alpha+M}(n) \text{ for all } f \in \mathcal{K}_{\alpha+M-k}(n). \tag{3.2}$$

Fix an $\varepsilon > 0$. Given $h \in \mathcal{K}_{\alpha+M}(n)$, we have

$$\int_{B_n} |h(z)|(1-|z|)^{\alpha+M-n-1} \left(\log\frac{e}{1-|z|}\right)^{-1-\varepsilon} d\nu_n(z) < \infty$$

by Proposition 2.3 with $j = 0$. So, (3.2) and the closed graph theorem guarantee that

$$\begin{aligned}\int_{B_n} |f(z)||R^k g(z)|(1-|z|)^{\alpha+M-n-1} \left(\log\frac{e}{1-|z|}\right)^{-1-\varepsilon} d\nu_n(z) \\ \le C\|f\|_{\mathcal{K}_{\alpha+M-k}(n)}\end{aligned}$$

for all $f \in \mathcal{K}_{\alpha+M-k}(n)$. We remark that

$$\left\| \frac{1}{(1-\langle\cdot,\zeta\rangle)^{\alpha+M-k}} \right\|_{\mathcal{K}_{\alpha+M-k}(n)} = 1 \quad \text{for all } \zeta \in \partial B_n$$

by the definition of $\mathcal{K}_{\alpha+M-k}(n)$. Hence,

$$\sup_{\zeta\in\partial B_n} \int_{B_n} \frac{|R^k g(z)|(1-|z|)^{\alpha+M-n-1}}{|1-\langle z,\zeta\rangle|^{\alpha+M-k}} \left(\log\frac{e}{1-|z|}\right)^{-1-\varepsilon} d\nu_n(z) < \infty.$$

The proof is finished. □

Recall that $\mathfrak{M}_\alpha(n) \subset H^\infty(B_n)$ for all $\alpha > 0$, $n \in \mathbb{N}$. So, condition (1.1) provides no new information when $\alpha + M > n + k$ and $\varepsilon > 0$. Indeed, if $g \in H^\infty(B_n)$, then

$$|R^k g(z)| \le C(1-|z|)^{-k}, \quad z \in B_n.$$

Since $\alpha + M - k > n$, we have

$$\int_{\partial B_n} \frac{d\sigma_n(\xi)}{|1 - r\langle \xi, \zeta\rangle|^{\alpha+M-k}} \le C(1-r)^{k+n-\alpha-M}, \quad 0 \le r < 1,\ \xi \in \partial B_n,$$

by [14, Proposition 1.4.10]. So, integrating in polar coordinates, we obtain

$$\begin{aligned}\sup_{\zeta\in\partial B_n} \int_{B_n} & \frac{|R^k g(z)|(1-|z|)^{\alpha+M-n-1}}{|1-\langle z,\zeta\rangle|^{\alpha+M-k}} \left(\log \frac{e}{1-|z|}\right)^{-1-\varepsilon} d\nu_n(z) \\ &\le C \int_0^1 (1-r)^{-1} \left(\log\frac{e}{1-r}\right)^{-1-\varepsilon} dr \\ &< \infty\end{aligned}$$

for any $\varepsilon > 0$.

4. Sufficient conditions

Lemma 4.1. *Suppose that $n \in \mathbb{N}$, $g \in \mathcal{H}ol(B_n)$ and $0 < \alpha \le \beta$. Then the following properties are equivalent:*

(i) *$fg \in \mathcal{K}_\beta(n)$ for all $f \in \mathcal{K}_\alpha(n)$.*
(ii) *For all $\zeta \in \partial B_n$,*

$$\left\| \frac{g(z)}{(1-\langle z,\zeta\rangle)^\alpha} \right\|_{\mathcal{K}_\beta(n)} \le \Sigma < \infty.$$

To prove Lemma 4.1, it suffices to repeat mutatis mutandis the arguments used in the proofs of [16, Theorem 1] and [9, Lemma 2.1] for $n = 1$ and $\alpha = \beta$.

Proof of Theorem 1.4. Fix a function $f \in \mathcal{K}_\alpha(n)$.

Let $k \in \{1, \dots, M\}$. Theorem 2.2 and the closed graph theorem guarantee that $\|\cdot\|_{\mathcal{K}_{\alpha+M}(n)} \le C\|\cdot\|_{A^1_{\alpha+M-n,0}(B_n)}$. Thus,

$$\begin{aligned}\sup_{\zeta\in\partial B_n} & \left\| \frac{R^k g(z)}{(1-\langle z,\zeta\rangle)^{\alpha+M-k}} \right\|_{\mathcal{K}_{\alpha+M}(n)} \\ &\le C \sup_{\zeta\in\partial B_n} \left\| \frac{R^k g(z)}{(1-\langle z,\zeta\rangle)^{\alpha+M-k}} \right\|_{A^1_{\alpha+M-n,0}(B_n)} < \infty\end{aligned}$$

by (1.2). Therefore, $FR^k g \in \mathcal{K}_{\alpha+M}(n)$ for all $F \in \mathcal{K}_{\alpha+M-k}(n)$ by Lemma 4.1. Also, we have $\mathcal{R}^{M-k} f \in \mathcal{K}_{\alpha+M-k}(n)$ by Theorem 2.1. Hence,

$$\mathcal{R}^{M-k} f \cdot R^k g \in \mathcal{K}_{\alpha+M}(n), \quad k = 1, \dots, M. \tag{4.1}$$

We have $\mathcal{R}^M f \in \mathcal{K}_{\alpha+M}(n)$ by Theorem 2.1 and $g \in \mathfrak{M}_n(n) \subset \mathfrak{M}_{\alpha+M}(n)$ by Proposition 1.3. Thus,

$$\mathcal{R}^M f \cdot g \in \mathcal{K}_{\alpha+M}(n). \tag{4.2}$$

Identity (2.1) and properties (4.1) and (4.2) imply that $R^M(fg) \in \mathcal{K}_{\alpha+M}(n)$. Therefore, $fg \in \mathcal{K}_\alpha(n)$ by Theorem 2.1. So, by the definition, $g \in \mathfrak{M}_\alpha(n)$. □

For $n = 1$, it is known that the hypothesis $g \in \mathfrak{M}_1(1)$ in Theorem 1.4 can be replaced by $g \in H^\infty(B_1)$ (see [12]). Nevertheless, Theorem 1.4 yields explicit sufficient conditions which will be used in Section 5.

For $\delta > 0$, the holomorphic Lipschitz space $\Lambda_\delta(B_n)$ consists of those $f \in \mathcal{H}ol(B_n)$ for which

$$|R^j f(z)| \le C(1-|z|)^{\delta - j}, \quad z \in B_n,$$

where j is the least integer such that $j > \delta$.

Corollary 4.2 (cf. [7, Corollary 5.1])**.** *Let $n \in \mathbb{N}$. Suppose that $0 < \alpha < n$ and $g \in \Lambda_\beta(B_n)$ for some $\beta > n - \alpha$. Then $g \in \mathfrak{M}_\alpha(n)$.*

Proof. In order to apply Theorem 1.4 let us first prove that $g \in \mathfrak{M}_n(n)$. We have

$$|Rg(z)| \le C(1-|z|)^{\delta - 1}, \quad z \in B_n,$$

for some $\delta \in (0, 1/2)$. Note that

$$g(z) = \int_0^1 Rg(tz)\, dt, \quad z \in B_n.$$

Hence,

$$|\mathcal{R}g(z)| \le C(1-|z|)^{\delta - 1}, \quad z \in B_n. \tag{4.3}$$

Consider the real Lipschitz space $\Lambda^{\mathbb{R}}_\delta(\overline{B}_n)$ which consists of those $f : \overline{B}_n \to \mathbb{C}$ for which

$$|f(z) - f(w)| \le C|z-w|^\delta \quad \text{for all } z, w \in \overline{B}_n.$$

By (4.3) and [14, Theorem 6.4.10], there exists $G \in \Lambda^{\mathbb{R}}_\delta(\overline{B}_n)$ such that $G(z) = g(z)$ for all $z \in B_n$. So, if $n = 1$, then $g \in \mathfrak{M}_1(1)$ by [10, Corollary 7.5]. If $n \ge 2$, then $g = K_n[G\,|_{\partial B_n}] \in \mathfrak{M}_n(n)$ by [8, Corollary 7.6].

Without loss of generality, we may assume that $\beta \in (0, n) \setminus \mathbb{N}$. Let j be the least integer such that $j > \beta$. Remark that $j \in \{1, \dots, n\}$.

By the definition of $\Lambda_\beta(B_n)$, we have

$$|R^j g(z)| \le C(1-|z|)^{\beta - j}, \quad z \in B_n.$$

Therefore, using the inequalities $\alpha + \beta > n$ and $\beta > 0$, we obtain

$$\begin{aligned} &\sup_{\zeta \in \partial B_n} \int_{B_n} \frac{|R^j g(z)|(1-|z|)^{\alpha + j - n - 1}}{|1 - \langle z, \zeta\rangle|^\alpha}\, d\nu_n(z) \\ &\quad \le C \sup_{\zeta \in \partial B_n} \int_{B_n} \frac{(1-|z|)^{\alpha + \beta - n - 1}}{|1 - \langle z, \zeta\rangle|^\alpha}\, d\nu_n(z) \\ &\quad < \infty \end{aligned}$$

by [14, Proposition 1.4.10]. Note that $\alpha > n - \beta > n - j$. So, if $j = 1$, then $g \in \mathfrak{M}_\alpha(n)$ by Theorem 1.4 with $M = 1$.

Now, suppose that $j \in \{2, \dots, n\}$ and $k \in \{1, \dots, j-1\}$. We have $k \le \beta$ and $\beta \notin \mathbb{N}$; hence, $k < \beta$. So, $g \in \Lambda_\beta(B_n)$ implies $R^k g \in H^\infty(B_n)$. Therefore, using the inequalities $\alpha + j - n > 0$ and $k > 0$, we obtain

$$\begin{aligned}&\sup_{\zeta\in\partial B_n}\int_{B_n}\frac{|R^k g(z)|(1-|z|)^{\alpha+j-n-1}}{|1-\langle z,\zeta\rangle|^{\alpha+j-k}}\,d\nu_n(z)\\ &\quad\le C\sup_{\zeta\in\partial B_n}\int_{B_n}\frac{(1-|z|)^{\alpha+j-n-1}}{|1-\langle z,\zeta\rangle|^{\alpha+j-k}}\,d\nu_n(z)\\ &\quad<\infty\end{aligned}$$

by [14, Proposition 1.4.10]. So, $g \in \mathfrak{M}_\alpha(n)$ by Theorem 1.4 with $M = j$. □

Other sufficient conditions can be expressed in terms of the Hardy–Sobolev spaces $H^p_s(B_n)$. Given $p > 0$ and $s > 0$, the space $H^p_s(B_n)$ consists of those $f \in \mathcal{H}ol(B_n)$ for which $R^s f$ is in the classical Hardy space $H^p(B_n)$. The following result improves [7, Corollary 5.4].

Proposition 4.3. *Let $n \ge 2$. Assume that $j \in \{1, \dots, n-1\}$ and $g \in H^p_j(B_n)$ for some $p > n/j$. Then $g \in \mathfrak{M}_{n-j}(n)$.*

Proof. Fix a function $f \in \mathcal{K}_{n-j}(n)$. By [3, Theorem 5.13], there exist $p_k > n/k$ such that $g \in H^{p_k}_k(B_n)$, $k = j, \dots, 1$.

Property $R^1 g \in H^{p_1}(B_n)$ implies that

$$|R^1 g(z)| \le (1-|z|)^{-n/p_1}, \quad z \in B_n.$$

Since $n < p_1$, Corollary 4.2 guarantees that $g \in \mathfrak{M}_n(n)$. Note that $\mathcal{R}^j f \in \mathcal{K}_n(n)$ by Theorem 2.1; therefore,

$$\mathcal{R}^j f \cdot g \in \mathcal{K}_n(n). \tag{4.4}$$

Now, assume that $k \in \{1, \dots, j\}$. By Theorem 2.1 and [8, Corollary 3.3(i)], we obtain $\mathcal{R}^{j-k} f \in \mathcal{K}_{n-k}(n) \subset H^q(B_n)$ for all $q \in (0, n/(n-k))$. Recall that $R^k g \in H^{p_k}(B_n)$ for some $p_k > n/k$; hence,

$$\mathcal{R}^{j-k} f \cdot R^k g \in H^1(B_n) \tag{4.5}$$

by Hölder's inequality.

Note that $H^1(B_n) \subset \mathcal{K}_n(n)$ by [8, Corollary 3.3(ii)]. Therefore, identity (2.1) and properties (4.4) and (4.5) guarantee that $R^j(fg) \in \mathcal{K}_n(n)$. Thus, $fg \in \mathcal{K}_{n-j}(n)$ by Theorem 2.1. So, $g \in \mathfrak{M}_{n-j}(n)$, as required. □

5. Non-coincidence of multiplier families

Proposition 5.1. *Suppose that $n \in \mathbb{N}$, $\beta < n$ and $0 < \beta < \alpha$. Then $\mathfrak{M}_\beta(n)$ is a proper subset of $\mathfrak{M}_\alpha(n)$.*

Proof. Recall that $\mathfrak{M}_\beta(n) \subset \mathfrak{M}_{\widetilde{\beta}}(n)$ for all $\widetilde{\beta} > \beta$. So, without loss of generality, we may assume that $n - j < \beta < \alpha < n - j + 1$ for some $j \in \{1, \dots, n\}$.

Fix γ such that $\beta < \gamma < \alpha$.

We have $n - j - \gamma < 0$. Hence, [6, Lemma 1.2] provides a number $M = M(n) \in \mathbb{N}$ and functions $h_m \in \mathcal{H}ol(B_n)$, $m = 0, 1, \dots, M$, such that

$$|h_m(z)| \le C(1-|z|)^{n-j-\gamma}, \quad z \in B_n,\ m = 0, 1, \dots, M, \tag{5.1}$$

$$\sum_{m=0}^{M} |h_m(z)| \ge (1-|z|)^{n-j-\gamma}, \quad z \in B_n. \tag{5.2}$$

By (5.2), there exists $m_0 \in \{0, 1, \dots, M\}$ such that

$$\int_{B_n} |h_{m_0}(z)|(1-|z|)^{\gamma+j-n-1}\, d\nu_n(z) = \infty. \tag{5.3}$$

Put $g = R^{-j} h_{m_0}$. Note that j is the least integer such that $j > n - \gamma > 0$, so, $g \in \Lambda_{n-\gamma}(B_n)$ by (5.1). Since $n-\gamma > n-\alpha$, we obtain $g \in \mathfrak{M}_\alpha(n)$ by Corollary 4.2.

Recall that $\mathfrak{M}_\beta(n) \subset \mathcal{K}_\beta(n)$. Next, we have $\gamma > \beta > n-j$; thus, $\mathcal{K}_\beta(n) \subset A^1_{\gamma+j-n,j}(B_n)$ by [8, Proposition 5.3(ii)]. Hence, $g \notin \mathfrak{M}_\beta(n)$ by (5.3). □

Note that a different proof of Proposition 5.1 is known for $n = 1$ (see [10, Proposition 7.26]).

6. Extended Cesàro operators

Given $g \in \mathcal{H}ol(B_n)$, the extended Cesàro operator J_g is defined by the identity

$$J_g f(z) = \int_0^1 f(tz)\mathcal{R}g(tz)\,\frac{dt}{t}, \quad f \in \mathcal{H}ol(B_n),\ z \in B_n.$$

For $n = 1$, the studies of the operator J_g were initiated by Pommerenke [13]; see also [1, 2]. For arbitrary $n \in \mathbb{N}$, the operator J_g was introduced by Hu [11].

Direct calculations show that

$$\mathcal{R}(J_g f)(z) = f(z)\mathcal{R}g(z), \quad f, g \in \mathcal{H}ol(B_n),\ z \in B_n. \tag{6.1}$$

Proposition 6.1. *Assume that $n \in \mathbb{N}$, $g \in \mathcal{H}ol(B_n)$ and $\alpha > 0$. Then the following properties are equivalent:*

(i) *$J_g f \in \mathcal{K}_\alpha(n)$ for all $f \in \mathcal{K}_\alpha(n)$.*
(ii) *$f\mathcal{R}g \in \mathcal{K}_{\alpha+1}(n)$ for all $f \in \mathcal{K}_\alpha(n)$.*
(iii) *For all $\zeta \in \partial B_n$,*

$$\left\| \frac{\mathcal{R}g(z)}{(1-\langle z,\zeta\rangle)^\alpha} \right\|_{\mathcal{K}_{\alpha+1}(n)} \le \Sigma < \infty.$$

Proof. Given $f \in \mathcal{K}_\alpha(n)$, identity (6.1) and Theorem 2.1 guarantee that $J_g f \in \mathcal{K}_\alpha(n)$ if and only if $f\mathcal{R}g = \mathcal{R}(J_g f) \in \mathcal{K}_{\alpha+1}(n)$. So, properties (i) and (ii) are equivalent. It remains to remark that (ii)⇔(iii) by Lemma 4.1. □

For $n \in \mathbb{N}$, denote by $\mathcal{K}_0(n)$ the family of $f \in \mathcal{H}ol(B_n)$ such that

$$f(z) - f(0) = \int_{\partial B_n} \log \frac{1}{1-\langle z,\zeta\rangle}\, d\mu(\zeta), \quad z \in B_n,$$

for some $\mu \in \mathcal{M}(n)$. Given $\alpha > 0$, put

$$\mathfrak{C}_\alpha(n) = \{g \in \mathcal{H}ol(B_n) :\ J_g \text{ is bounded on } \mathcal{K}_\alpha(n)\}.$$

Proposition 6.2. *Assume that $n \in \mathbb{N}$ and $\alpha > 0$. Then $\mathcal{K}_0(n) \subset \mathfrak{C}_\alpha(n)$.*

Proof. Suppose that $g \in \mathcal{K}_0(n)$ and $f \in \mathcal{K}_\alpha(n)$. Elementary calculations show that $\mathcal{R}g \in \mathcal{K}_1(n)$. Thus, $f\mathcal{R}g \in \mathcal{K}_\alpha(n)\cdot\mathcal{K}_1(n) \subset \mathcal{K}_{\alpha+1}(n)$ by [8, Corollary 2.3]. It remains to apply Proposition 6.1 and the closed graph theorem. □

For $g \in \mathcal{H}ol(B_1)$, it is more natural to consider the following operator:

$$\mathcal{C}_g f(z) = \frac{1}{z} J_g f(z) = \frac{1}{z}\int_0^z f(w)g'(w)\, dw, \quad f \in \mathcal{H}ol(B_1),\ z \in B_1.$$

Indeed, if $g(z) = \log \frac{1}{1-z}$ for $z \in B_1$, then $\mathcal{C}_g$ coincides with the classical Cesàro operator $\mathcal{C} : \mathcal{H}ol(B_1) \to \mathcal{H}ol(B_1)$, which is defined as follows. If $f(z) = \sum_{j=0}^{\infty} a_j z^j \in \mathcal{H}ol(B_1)$, then

$$\mathcal{C}f(z) = \sum_{j=0}^{\infty} \left(\frac{1}{j+1} \sum_{k=0}^{j} a_k \right) z^j, \quad z \in B_1.$$

It is known that $\mathcal{C}$ is a bounded operator on various spaces of holomorphic functions. We will need the following auxiliary result.

Proposition 6.3. *Assume that $f \in \mathcal{H}ol(B_1)$, $\alpha > 0$ and $zf(z) \in \mathcal{K}_\alpha(1)$. Then $f \in \mathcal{K}_\alpha(1)$.*

Proof. For $z \in B_1$, put $h(z) = zf(z)$. Since $h \in \mathcal{K}_\alpha(1)$, we have $h' \in \mathcal{K}_{\alpha+1}(1)$ by [10, Theorem 2.8]. Note that $Rf(z) = zf'(z) + f(z) = h'(z)$. So, $f \in \mathcal{K}_\alpha(1)$ by Theorem 2.1. □

Corollary 6.4. *Let $g \in \mathcal{H}ol(B_1)$ and $\alpha > 0$. Then $\mathcal{C}_g$ is bounded on $\mathcal{K}_\alpha(1)$ if and only if $g \in \mathfrak{C}_\alpha(1)$. In particular, $\mathcal{C}_g$ is bounded on $\mathcal{K}_\alpha(1)$ for any $g \in \mathcal{K}_0(1)$.*

Proof. Recall that $z \in \mathfrak{M}_\alpha(1)$. So, it remains to apply Propositions 6.3 and 6.2. □

The following result was obtained in [5] for $\alpha \geq 1$.

Corollary 6.5. *The classical Cesàro operator $\mathcal{C}$ is bounded on $\mathcal{K}_\alpha(1)$ for all $\alpha > 0$.*

Proof. Clearly, $\log \frac{1}{1-z} \in \mathcal{K}_0(1)$, so, Corollary 6.4 applies. □

In the final part of this paper, we prove that $\mathfrak{C}_\beta(n)$ is a proper subset of $\mathfrak{C}_\alpha(n)$ when $0 < \beta < \alpha$ and $\beta < n$ (cf. Proposition 5.1).

For $n \in \mathbb{N}$ and $0 < \beta \leq \gamma$, put

$$\mathfrak{M}_{\beta,\gamma}(n) = \{g \in \mathcal{H}ol(B_n) : \ fg \in \mathcal{K}_\gamma(n) \text{ for all } f \in \mathcal{K}_\beta(n)\}.$$

We will need the following generalization of Proposition 1.3.

Lemma 6.6. *Assume that $n \in \mathbb{N}$, $\gamma \geq \beta > 0$ and $\alpha > \beta$. Then $\mathfrak{M}_{\beta,\gamma}(n) \subset \mathfrak{M}_{\alpha,\alpha+\gamma-\beta}(n)$.*

Proof. Let $g \in \mathfrak{M}_{\beta,\gamma}(n)$. By Lemma 4.1, there exist measures $\mu_\zeta \in \mathcal{M}(n)$, $\zeta \in \partial B_n$, and a constant $\Sigma > 0$ such that $\|\mu_\zeta\| \leq \Sigma$ and

$$\frac{g(z)}{(1-\langle z,\zeta\rangle)^\beta} = \int_{\partial B_n} \frac{1}{(1-\langle z,\xi\rangle)^\gamma}\, d\mu_\zeta(\xi), \quad z \in B_n.$$

So, we have

$$\frac{g(z)}{(1-\langle z,\zeta\rangle)^\alpha} = \int_{\partial B_n} \frac{1}{(1-\langle z,\zeta\rangle)^{\alpha-\beta}} \frac{1}{(1-\langle z,\xi\rangle)^\gamma}\, d\mu_\zeta(\xi), \quad z \in B_n,\ \zeta \in \partial B_n.$$

Given $\zeta, \xi \in \partial B_n$, property (2.1) from [8] yields a probability measure $\rho_{\zeta,\xi} \in \mathcal{M}(n)$ such that

$$\frac{1}{(1-\langle z,\zeta\rangle)^{\alpha-\beta}} \frac{1}{(1-\langle z,\xi\rangle)^\gamma} = \int_{\partial B_n} \frac{1}{(1-\langle z,\eta\rangle)^{\gamma+\alpha-\beta}}\, d\rho_{\zeta,\xi}(\eta), \quad z \in B_n.$$

Therefore,

$$\frac{g(z)}{(1-\langle z,\zeta\rangle)^\alpha} = \int_{\partial B_n} \int_{\partial B_n} \frac{1}{(1-\langle z,\eta\rangle)^{\gamma+\alpha-\beta}}\, d\rho_{\zeta,\xi}(\eta)\, d\mu_\zeta(\xi), \quad z \in B_n.$$

Fix a point $\zeta \in \partial B_n$ and put $\mu = \mu_\zeta$. Given $\xi \in \partial B_n$, let δ_ξ denote the Dirac measure at ξ. Applying the Banach–Alaoglu theorem, select measures $\mu_k = \sum_{j=1}^{J(k)} a_{j,k}\delta_{\xi_{j,k}}$, $a_{j,k} \in \mathbb{C}$, $\xi_{j,k} \in \partial B_n$, such that $\|\mu_k\| \le \Sigma$ and the sequence $\{\mu_k\}$ converges to μ in the weak* topology. Put $\lambda_k = \sum_{j=1}^{J(k)} a_{j,k}\rho_{\zeta,\xi_{j,k}}$. Note that $\|\lambda_k\| \le \Sigma$, so $\{\lambda_k\}$ has a subsequence which converges in the weak* topology to, say, $\lambda = \lambda_\zeta \in \mathcal{M}(n)$. So, we obtain

$$\frac{g(z)}{(1-\langle z,\zeta\rangle)^\alpha} = \int_{\partial B_n} \frac{1}{(1-\langle z,\eta\rangle)^{\alpha+\gamma-\beta}}\, d\lambda_\zeta(\eta), \quad z \in B_n,$$

where $\|\lambda_\zeta\| \le \Sigma$. Now, Lemma 4.1 guarantees that $g \in \mathfrak{M}_{\alpha,\alpha+\gamma-\beta}(n)$. □

Corollary 6.7. *Assume that $n \in \mathbb{N}$ and $\alpha > \beta > 0$. Then $\mathfrak{C}_\beta(n) \subset \mathfrak{C}_\alpha(n)$.*

Proof. We apply Proposition 6.1 and Lemma 6.6 with $\gamma = \beta + 1$. □

Lemma 6.8. *Assume that $n \in \mathbb{N}$ and $\alpha > 0$. Then $\mathfrak{M}_\alpha(n) = \mathfrak{M}_{\alpha+1}(n) \cap \mathfrak{C}_\alpha(n)$.*

Proof. Assume that $g \in \mathfrak{M}_{\alpha+1}(n) \cap \mathfrak{C}_\alpha(n)$ and $f \in \mathcal{K}_\alpha(n)$. By Theorem 2.1, $\mathcal{R}f \in \mathcal{K}_{\alpha+1}(n)$; hence, $\mathcal{R}f \cdot g \in \mathcal{K}_{\alpha+1}(n)$. Proposition 6.1 guarantees that $f\mathcal{R}g \in \mathcal{K}_{\alpha+1}(n)$. In sum, we have $\mathcal{R}(fg) = \mathcal{R}f \cdot g + f\mathcal{R}g \in \mathcal{K}_{\alpha+1}(n)$; thus, $fg \in \mathcal{K}_\alpha(n)$. So, $g \in \mathfrak{M}_\alpha(n)$.

Now, assume that $g \in \mathfrak{M}_\alpha(n)$ and $f \in \mathcal{K}_\alpha(n)$. We have $g \in \mathfrak{M}_\alpha(n) \subset \mathfrak{M}_{\alpha+1}(n)$ and $\mathcal{R}f \in \mathcal{K}_{\alpha+1}(n)$; therefore, $\mathcal{R}f \cdot g \in \mathcal{K}_{\alpha+1}(n)$. Also, $fg \in \mathcal{K}_\alpha(n)$; hence, $\mathcal{R}f \cdot g + f\mathcal{R}g \in \mathcal{K}_{\alpha+1}(n)$ by Theorem 2.1. Thus, $f\mathcal{R}g \in \mathcal{K}_{\alpha+1}(n)$ for all $f \in \mathcal{K}_\alpha(n)$. By Proposition 6.1, $g \in \mathfrak{C}_\alpha(n)$. So, $g \in \mathfrak{M}_{\alpha+1}(n) \cap \mathfrak{C}_\alpha(n)$. □

Proposition 6.9. *Assume that $n \in \mathbb{N}$, $\beta < n$ and $0 < \beta < \alpha$. Then $\mathfrak{C}_\beta(n)$ is a proper subset of $\mathfrak{C}_\alpha(n)$.*

Proof. By Corollary 6.7, we may assume that $\alpha < \beta + 1$. By Proposition 5.1, there exists a function $g \in \mathfrak{M}_\alpha(n) \setminus \mathfrak{M}_\beta(n)$. On the one hand, $g \in \mathfrak{M}_\alpha(n)$, so $g \in \mathfrak{C}_\alpha(n)$ by Lemma 6.8. On the other hand, $g \in \mathfrak{M}_\alpha(n) \subset \mathfrak{M}_{\beta+1}(n)$ and $g \notin \mathfrak{M}_\beta(n)$. Hence, $g \notin \mathfrak{C}_\beta(n)$ by Lemma 6.8. □

By [10, Proposition 7.27], $\mathfrak{M}_\alpha(1) \setminus \mathfrak{M}_1(1) \neq \varnothing$ for any $\alpha > 1$. Thus, we also have the following result.

Proposition 6.10. *Assume that $\alpha > 1$. Then $\mathfrak{C}_1(1)$ is a proper subset of $\mathfrak{C}_\alpha(1)$.*

Finally, let $g \in \mathfrak{M}_2(1) \setminus \mathfrak{M}_1(1)$. Then $g \in H^\infty(B_1)$; moreover,

$$\sup_{\xi \in \partial B_1} \int_0^1 |g'(r\xi)|\, dr < +\infty.$$

Nevertheless, $g \notin \mathfrak{C}_1(1)$ by Lemma 6.8.

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References

1. A. Aleman and A. G. Siskakis, *An integral operator on H^p*, Complex Variables Theory Appl. **28** (1995), no. 2, 149–158. MR1700079 (2000d:47050)
2. A. Aleman and A. G. Siskakis, *Integration operators on Bergman spaces*, Indiana Univ. Math. J. **46** (1997), no. 2, 337–356. MR1481594 (99b:47039)
3. F. Beatrous and J. Burbea, *Holomorphic Sobolev spaces on the ball*, Dissertationes Math. (Rozprawy Mat.) **276** (1989), 60 pp. MR1010151 (90k:32010)
4. J. A. Cima, A. L. Matheson, and W. T. Ross, *The Cauchy transform*, Mathematical Surveys and Monographs, vol. 125, American Mathematical Society, Providence, RI, 2006. MR2215991 (2006m:30003)
5. J. A. Cima and A. G. Siskakis, *Cauchy transforms and Cesàro averaging operators*, Acta Sci. Math. (Szeged) **65** (1999), no. 3-4, 505–513. MR1737267 (2000m:47043)
6. E. Doubtsov, *Growth spaces on circular domains: composition operators and Carleson measures*, C. R. Math. Acad. Sci. Paris **347** (2009), no. 11-12, 609–612.
7. E. Doubtsov, *Multipliers of fractional Cauchy transforms*, Integral Equations and Operator Theory **64** (2009), no. 2, 177–192. MR2516277
8. E. S. Dubtsov, *Families of fractional Cauchy transforms in the ball*, Algebra i Analiz **21** (2009) (Russian), to appear.
9. R. A. Hibschweiler and T. H. MacGregor, *Multipliers of families of Cauchy–Stieltjes transforms*, Trans. Amer. Math. Soc. **331** (1992), no. 1, 377–394. MR1120775 (93g:30057)
10. R. A. Hibschweiler and T. H. MacGregor, *Fractional Cauchy transforms*, Chapman & Hall/CRC Monographs and Surveys in Pure and Applied Mathematics, vol. 136, Chapman & Hall/CRC, Boca Raton, FL, 2006. MR2189294 (2007b:30001)
11. Z. Hu, *Extended Cesàro operators on mixed norm spaces*, Proc. Amer. Math. Soc. **131** (2003), no. 7, 2171–2179. MR1963765 (2004a:47030)
12. D. Luo and T. MacGregor, *Multipliers of fractional Cauchy transforms and smoothness conditions*, Canad. J. Math. **50** (1998), no. 3, 595–604. MR1629835 (99d:30040)
13. Ch. Pommerenke, *Schlichte Funktionen und analytische Funktionen von beschränkter mittlerer Oszillation*, Comment. Math. Helv. **52** (1977), no. 4, 591–602. MR0454017 (56:12268)
14. W. Rudin, *Function theory in the unit ball of $\mathbf{C}^n$*, Grundlehren der Mathematischen Wissenschaften [Fundamental Principles of Mathematical Science], vol. 241, Springer-Verlag, New York, 1980. MR601594 (82i:32002)
15. S. A. Vinogradov, *Properties of multipliers of integrals of Cauchy–Stieltjes type, and some problems of factorization of analytic functions*, Mathematical programming and related questions (Proc. Seventh Winter School, Drogobych, 1974), Theory of functions and functional analysis (Russian), Central Èkonom.-Mat. Inst. Akad. Nauk SSSR, Moscow, 1976, pp. 5–39; English transl.: Amer. Math. Soc. Transl. (2) **115** (1980), 1–32. MR0586560 (58:28518)
16. S. A. Vinogradov, M. G. Goluzina, and V. P. Havin, *Multipliers and divisors of Cauchy–Stieltjes type integrals*, Zap. Naučn. Sem. Leningrad. Otdel. Mat. Inst. Steklov. (LOMI) **19** (1970), 55–78 (Russian); English transl.: Seminars in Math., V. A. Steklov Math. Inst., Leningrad **19** (1972), 29–42. MR0291471 (45:562)

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