

A NOTE ON THE ABELIANIZATIONS OF FINITE-INDEX SUBGROUPS OF THE MAPPING CLASS GROUP

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ABSTRACT. For some $g \geq 3$, let Γ be a finite index subgroup of the mapping class group of a genus g surface (possibly with boundary components and punctures). An old conjecture of Ivanov says that the abelianization of Γ should be finite. In the paper we prove two theorems supporting this conjecture. For the first, let T_x denote the Dehn twist about a simple closed curve x . For some $n \geq 1$, we have $T_x^n \in \Gamma$. We prove that T_x^n is torsion in the abelianization of Γ . Our second result shows that the abelianization of Γ is finite if Γ contains a “large chunk” (in a certain technical sense) of the Johnson kernel, that is, the subgroup of the mapping class group generated by twists about separating curves.

1. INTRODUCTION

Let $\Sigma_{g,b}^p$ be an oriented genus g surface with b boundary components and p punctures and let $\text{Mod}(\Sigma_{g,b}^p)$ be its *mapping class group*, that is, the group of isotopy classes of orientation-preserving diffeomorphisms of $\Sigma_{g,b}^p$ that fix the boundary components and punctures pointwise (we will omit b or p when they are zero). A long-standing conjecture of Ivanov (see [6] for a recent discussion) says that for $g \geq 3$, the group $\text{Mod}(\Sigma_{g,b}^p)$ does not virtually surject onto \mathbb{Z} . In other words, if Γ is a finite-index subgroup of $\text{Mod}(\Sigma_{g,b}^p)$, then $H_1(\Gamma; \mathbb{R}) = 0$.

The goal of this paper is to offer some evidence for this conjecture. If G is a group and $g \in G$, then we will denote by $[g]_G$ the corresponding element of $H_1(G; \mathbb{R})$. Also, for a simple closed curve γ on $\Sigma_{g,b}^p$, we will denote by T_γ the corresponding right Dehn twist. Observe that if Γ is any finite-index subgroup of $\text{Mod}_{g,b}^p$, then $T_\gamma^n \in \Gamma$ for some $n \geq 1$. Our first result is the following.

Theorem A (Powers of twists vanish). *For some $g \geq 3$, let $\Gamma < \text{Mod}(\Sigma_{g,b}^p)$ satisfy $[\text{Mod}(\Sigma_{g,b}^p) : \Gamma] < \infty$ and let γ be a simple closed curve on $\Sigma_{g,b}^p$. Pick $n \geq 1$ such that $T_\gamma^n \in \Gamma$. Then $[T_\gamma^n]_\Gamma = 0$.*

Remark. After this paper was written, Bridson informed us that in unpublished work he had proven a result about mapping class group actions on $\text{CAT}(0)$ spaces that implies Theorem A. Bridson’s work will appear in [3].

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We use this to verify Ivanov's conjecture for a class of examples. For a long time, the only positive evidence for Ivanov's conjecture was a result of Hain [5] that says that it holds for all finite-index subgroups containing the *Torelli group* $\mathcal{I}_{g,b}^p$, that is, the kernel of the action of $\text{Mod}(\Sigma_{g,b}^p)$ on $H_1(\Sigma_g; \mathbb{Z})$ induced by filling in all the punctures and boundary components. The group $\mathcal{I}_{g,b}^p$ contains the *Johnson kernel* $\mathcal{K}_{g,b}^p$, which is the subgroup generated by Dehn twists about separating curves. A result of Johnson [7] says that $\mathcal{K}_{g,b}^p$ is an infinite-index subgroup of $\mathcal{I}_{g,b}^p$.

For a subgroup Γ of $\text{Mod}(\Sigma_{g,b}^p)$, denote by $K(\Gamma)$ the subgroup of $\Gamma \cap \mathcal{K}_{g,b}^p$ generated by the set

$$\{T_\gamma^n \mid \gamma \text{ a separating curve, } n \in \mathbb{Z}, \text{ and } T_\gamma^n \in \Gamma\}.$$

If $\mathcal{K}_{g,b}^p < \Gamma$, then $K(\Gamma) = \Gamma \cap \mathcal{K}_{g,b}^p$, but the converse does not hold. Our second result is the following.

Theorem B (Subgroups containing large pieces of Johnson kernel). *For some $g \geq 3$, let $\Gamma < \text{Mod}(\Sigma_{g,b}^p)$ satisfy $[\text{Mod}(\Sigma_{g,b}^p) : \Gamma] < \infty$. Assume that $[\Gamma \cap \mathcal{K}_{g,b}^n : K(\Gamma)] < \infty$. Then $H_1(\Gamma; \mathbb{R}) = 0$.*

As a corollary, we obtain the following result, which was recently proven by Boggi [2] via a difficult algebro-geometric argument under the assumption $b = p = 0$.

Corollary C (Subgroups containing Johnson kernel). *For some $g \geq 3$, let $\Gamma < \text{Mod}(\Sigma_{g,b}^p)$ satisfy $[\text{Mod}(\Sigma_{g,b}^p) : \Gamma] < \infty$. Assume that $\mathcal{K}_{g,b}^n < \Gamma$. Then $H_1(\Gamma; \mathbb{R}) = 0$.*

Remark. McCarthy [11] proved that Ivanov's conjecture fails in the case $g = 2$.

2. NOTATION AND BASIC FACTS ABOUT GROUP HOMOLOGY

If M is a G -module, then M_G will denote the *coinvariants* of the action, that is, the quotient of M by the submodule generated by the set $\{x - g(x) \mid x \in M, g \in G\}$. This appears in the 5-term exact sequence [4, Corollary VII.6.4], which asserts the following. If

$$1 \longrightarrow K \longrightarrow G \longrightarrow Q \longrightarrow 1$$

is a short exact sequence of groups, then for any ring R , there is an exact sequence

$$H_2(G; R) \longrightarrow H_2(Q; R) \longrightarrow (H_1(K; R))_Q \longrightarrow H_1(G; R) \longrightarrow H_1(Q; R) \longrightarrow 0.$$

If $G_2 < G_1$ are groups satisfying $[G_1 : G_2] < \infty$ and R is a ring, then for all k there exists a *transfer map* of the form $t : H_k(G_1; R) \rightarrow H_k(G_2; R)$ (see, e.g., [4, Chapter III.9]). The key property of t (see [4, Proposition III.9.5]) is that if $i : H_k(G_2; R) \rightarrow H_k(G_1; R)$ is the map induced by the inclusion, then $i \circ t : H_k(G_1; R) \rightarrow H_k(G_1; R)$ is multiplication by $[G_1 : G_2]$. In particular, if $R = \mathbb{R}$, then we obtain a right inverse $\frac{1}{[G_1 : G_2]}t$ to i . This yields the following standard lemma.

Lemma 2.1. *Let $G_2 < G_1$ be groups satisfying $[G_1 : G_2] < \infty$. For all k , the map $H_k(G_2; \mathbb{R}) \rightarrow H_k(G_1; \mathbb{R})$ is surjective.*

3. PROOF OF THEOREM A

Let $n \geq 1$ be the smallest integer such that $T_\gamma^n \in \Gamma$.

We first claim that there exists a subsurface $S \hookrightarrow \Sigma_{g,b}^p$ whose genus is at least 2 with the following property. Let $i : \text{Mod}(S) \rightarrow \text{Mod}(\Sigma_{g,b}^p)$ be the induced map (“extend by the identity”). Then there exists some boundary component β of S such that $i(T_\beta) = T_\gamma$. There are two cases. If γ is nonseparating, then let S be the complement of a regular neighborhood of γ . Observe that $S \cong \Sigma_{g-1,b+2}^p$, so the genus of S is at least 2. If instead γ is separating, then let S be the component of $\Sigma_{g,b}^p$ cut along γ whose genus is maximal. Since $g \geq 3$, this subsurface must have genus at least 2. The claim follows.

Define $\Gamma' = i^{-1}(\Gamma)$. We have $T_\beta^n \in \Gamma'$, and it is enough to show that $[T_\beta^n]_{\Gamma'} = 0$. Let \bar{S} be the result of gluing a punctured disc to β and let $\bar{\Gamma}'$ be the image of Γ' in $\text{Mod}(\bar{S})$. There is a diagram of central extensions

$$\begin{array}{ccccccc} 1 & \longrightarrow & \mathbb{Z} & \longrightarrow & \Gamma' & \longrightarrow & \bar{\Gamma}' & \longrightarrow & 1 \\ & & \downarrow \times n & & \downarrow & & \downarrow & & \\ 1 & \longrightarrow & \mathbb{Z} & \longrightarrow & \text{Mod}(S) & \longrightarrow & \text{Mod}(\bar{S}) & \longrightarrow & 1 \end{array}$$

with $\mathbb{Z} < \text{Mod}(S)$ and $\mathbb{Z} < \Gamma'$ generated by T_β and T_β^n , respectively. The last 4 terms of the corresponding diagram of 5-term exact sequences are

$$\begin{array}{ccccccc} H_2(\bar{\Gamma}'; \mathbb{R}) & \xrightarrow{f_1} & \mathbb{R} & \longrightarrow & H_1(\Gamma'; \mathbb{R}) & \longrightarrow & H_1(\bar{\Gamma}'; \mathbb{R}) & \longrightarrow & 0 \\ \downarrow f_2 & & \downarrow \cong & & \downarrow & & \downarrow & & \\ H_2(\text{Mod}(\bar{S}); \mathbb{R}) & \xrightarrow{f_3} & \mathbb{R} & \longrightarrow & H_1(\text{Mod}(S); \mathbb{R}) & \longrightarrow & H_1(\text{Mod}(\bar{S}); \mathbb{R}) & \longrightarrow & 0. \end{array}$$

We remark that there are no nontrivial coinvariants in these sequences since our extensions are central. We must show that f_1 is a surjection. Since S has genus at least 2, we have $H_1(\text{Mod}(S); \mathbb{R}) = 0$ (see, e.g., [10]), so f_3 is a surjection. Since $[\text{Mod}(\bar{S}) : \bar{\Gamma}'] < \infty$, Lemma 2.1 implies that f_2 is a surjection, so f_1 is a surjection, as desired.

4. PROOF OF THEOREM B

4.1. **Two facts about $\text{Sp}_{2g}(\mathbb{Z})$.** We will need two standard facts about finite-index subgroups Γ of $\text{Sp}_{2g}(\mathbb{Z})$, both of which follow from the fact that Γ is a lattice in $\text{Sp}_{2g}(\mathbb{R})$.

For the first, since $\text{Sp}_{2g}(\mathbb{R})$ is a connected simple Lie group with finite center and real rank g , the group Γ has Kazhdan’s property (T) when $g \geq 2$ (see, e.g., [13, Theorem 7.1.4]). One standard property of groups with property (T) is that they have no nontrivial homomorphisms to \mathbb{R} (see, e.g., [13, Theorem 7.1.7]). Combining these facts, we obtain the following theorem.

Theorem 4.1. *For some $g \geq 2$, let $\Gamma < \text{Sp}_{2g}(\mathbb{Z})$ satisfy $[\text{Sp}_{2g}(\mathbb{Z}) : \Gamma] < \infty$. Then $H_1(\Gamma; \mathbb{R}) = 0$.*

For the second, since $\text{Sp}_{2g}(\mathbb{R})$ is a connected noncompact simple real algebraic group, we can apply the Borel density theorem (see, e.g., [13, Theorem 3.2.5]) to deduce that Γ is Zariski dense in $\text{Sp}_{2g}(\mathbb{R})$. This implies that any finite-dimensional

nontrivial irreducible $\mathrm{Sp}_{2g}(\mathbb{R})$ -representation V must also be an irreducible Γ -representation; indeed, if V' was a nontrivial proper Γ -submodule of V , then the subgroup of $\mathrm{Sp}_{2g}(\mathbb{R})$ preserving V' would be a proper subvariety of $\mathrm{Sp}_{2g}(\mathbb{R})$ containing Γ . Recall that the ring of coinvariants V_Γ of V under Γ is the quotient V/K , where $K = \langle x - g(x) \mid x \in V, g \in \Gamma \rangle$. Since $K \neq 0$, we can apply Schur's lemma to deduce that $K = V$, i.e. that $V_\Gamma = 0$. We record this fact as the following theorem.

Theorem 4.2. *For some $g \geq 1$, let $\Gamma < \mathrm{Sp}_{2g}(\mathbb{Z})$ satisfy $[\mathrm{Sp}_{2g}(\mathbb{Z}) : \Gamma] < \infty$ and let V be a nontrivial irreducible $\mathrm{Sp}_{2g}(\mathbb{R})$ -representation. Then $V_\Gamma = 0$.*

4.2. Two preliminary lemmas. We will need two lemmas. The first is the following, which slightly generalizes a theorem of Johnson [8].

Lemma 4.3. *For $g \geq 3$, we have $\mathcal{I}_{g,b}^p/\mathcal{K}_{g,b}^p \cong (\wedge^3 H)/H \oplus H^{b+p}$, where $H = H_1(\Sigma_g; \mathbb{Z})$.*

Proof. Since $\mathcal{K}_{g,b}^p$ contains all twists about boundary curves, we can assume that $b = 0$.

Building on work of Johnson [9], Hain [5] proved that

$$H_1(\mathcal{I}_g^p; \mathbb{R}) \cong (\wedge^3 H_{\mathbb{R}})/H_{\mathbb{R}} \oplus H_{\mathbb{R}}^p,$$

where $H_{\mathbb{R}} = H_1(\Sigma_g; \mathbb{R})$. Also, Johnson [9, Lemma 2] proved that for $x \in \mathcal{K}_g^p$, we have $[x]_{\mathcal{I}_g^p} = 0$. (Johnson only considered the case where $p = 0$, but his argument works in general.) It follows that

$$(1) \quad H_1(\mathcal{I}_g^p/\mathcal{K}_g^p; \mathbb{R}) \cong (\wedge^3 H_{\mathbb{R}})/H_{\mathbb{R}} \oplus H_{\mathbb{R}}^p.$$

We will prove the lemma by induction on p . The base case $p = 0$ is a theorem of Johnson [8]. Assume now that $p > 0$ and that the lemma is true for all smaller p . Fixing a puncture $*$ of Σ_g^p , work of Birman [1] and Johnson [9] gives an exact sequence

$$1 \longrightarrow \pi_1(\Sigma_g^{p-1}, *) \longrightarrow \mathcal{I}_g^p \longrightarrow \mathcal{I}_g^{p-1} \longrightarrow 1,$$

where the map $\mathcal{I}_g^p \rightarrow \mathcal{I}_g^{p-1}$ comes from “forgetting the puncture $*$ ”. Quotienting out by \mathcal{K}_g^p , we obtain an exact sequence

$$1 \longrightarrow \pi_1(\Sigma_g^{p-1}, *) / (\pi_1(\Sigma_g^{p-1}, *) \cap \mathcal{K}_g^p) \longrightarrow \mathcal{I}_g^p/\mathcal{K}_g^p \longrightarrow \mathcal{I}_g^{p-1}/\mathcal{K}_g^{p-1} \longrightarrow 1.$$

By induction, we have

$$\mathcal{I}_g^{p-1}/\mathcal{K}_g^{p-1} \cong (\wedge^3 H)/H \oplus H^{p-1}.$$

Set $A = \pi_1(\Sigma_g^{p-1}, *) / (\pi_1(\Sigma_g^{p-1}, *) \cap \mathcal{K}_g^p)$. We will prove that A is a quotient of H . We will then be able to conclude that $\mathcal{I}_g^{p-1}/\mathcal{K}_g^{p-1}$ acts trivially on A , so $\mathcal{I}_g^p/\mathcal{K}_g^p$ is the abelian group

$$(\wedge^3 H)/H \oplus H^{p-1} \oplus A.$$

Using (1), a simple dimension count will then imply that A cannot be a proper quotient of H , and the lemma will follow.

The element of \mathcal{I}_g^p corresponding to $\delta \in \pi_1(\Sigma_g^{p-1}, *)$ “drags” $*$ around δ . As shown in Figure 1.a–b, a simple closed curve $\gamma \in \pi_1(\Sigma_g^{p-1}, *)$ corresponds to $T_{\gamma_1} T_{\gamma_2}^{-1} \in \mathcal{I}_g^p$, where γ_1 and γ_2 are the boundary components of a regular neighborhood of γ . In particular, if γ is a simple closed separating curve, then as shown in Figure 1.c–d, the corresponding element of \mathcal{I}_g^p is a product of separating twists. Since $[\pi_1(\Sigma_g^{p-1}, *), \pi_1(\Sigma_g^{p-1}, *)]$ is generated by simple closed separating curves (see,

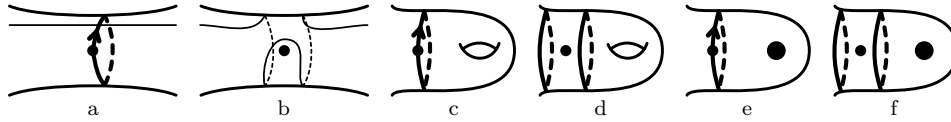


FIGURE 1. a–f. Curves needed for proof of Lemma 4.3

e.g., [12, Lemma A.1]), we deduce that $[\pi_1(\Sigma_g^{p-1}, *), \pi_1(\Sigma_g^{p-1}, *)] \subset \pi_1(\Sigma_g^{p-1}, *) \cap \mathcal{K}_g^p$. Thus $A = \pi_1(\Sigma_g^{p-1}, *) / (\pi_1(\Sigma_g^{p-1}, *) \cap \mathcal{K}_g^p)$ is a quotient of $H_1(\Sigma_g^{p-1}; \mathbb{Z})$. Finally, as shown in Figure 1.e–f, all simple closed curves that are homotopic into punctures are also contained in $\pi_1(\Sigma_g^{p-1}, *) \cap \mathcal{K}_g^p$, so we conclude that A is a quotient of $H = H_1(\Sigma_g; \mathbb{Z})$, as desired. \square

For the second lemma, define $Q_{g,b}^p = \text{Mod}_{g,b}^p / \mathcal{K}_{g,b}^p$.

Lemma 4.4. *For some $g \geq 3$, let $Q' < Q_{g,b}^p$ satisfy $[Q_{g,b}^p : Q'] < \infty$. Then $H_1(Q'; \mathbb{R}) = 0$.*

Proof. Restricting the short exact sequence

$$1 \longrightarrow \mathcal{I}_{g,b}^p / \mathcal{K}_{g,b}^p \longrightarrow Q_{g,b}^p \longrightarrow \text{Sp}_{2g}(\mathbb{Z}) \longrightarrow 1$$

to Q' , we obtain a short exact sequence

$$1 \longrightarrow B \longrightarrow Q' \longrightarrow \overline{Q}' \longrightarrow 1,$$

where B and \overline{Q}' are finite-index subgroups of $\mathcal{I}_{g,b}^p / \mathcal{K}_{g,b}^p$ and $\text{Sp}_{2g}(\mathbb{Z})$, respectively. The last 3 terms of the associated 5-term exact sequence are

$$(H_1(B; \mathbb{R}))_{\overline{Q}'} \longrightarrow H_1(Q'; \mathbb{R}) \longrightarrow H_1(\overline{Q}'; \mathbb{R}) \longrightarrow 0.$$

By Theorem 4.1, we have $H_1(\overline{Q}'; \mathbb{R}) = 0$. Letting $H = H_1(\Sigma_g; \mathbb{Z})$, Lemma 4.3 says that

$$\mathcal{I}_{g,b}^p / \mathcal{K}_{g,b}^p \cong (\wedge^3 H) / H \oplus H^{b+p}.$$

Since B is a finite-index subgroup of $\mathcal{I}_{g,b}^p / \mathcal{K}_{g,b}^p$, we get that B is itself abelian and

$$H_1(B; \mathbb{R}) \cong B \otimes \mathbb{R} \cong (\mathcal{I}_{g,b}^p / \mathcal{K}_{g,b}^p) \otimes \mathbb{R} \cong (\wedge^3 H_{\mathbb{R}}) / H_{\mathbb{R}} \oplus H_{\mathbb{R}}^{b+p},$$

where $H_{\mathbb{R}} = H_1(\Sigma_g; \mathbb{R})$. Both $(\wedge^3 H_{\mathbb{R}}) / H_{\mathbb{R}}$ and $H_{\mathbb{R}}$ are nontrivial finite-dimensional irreducible representations of $\text{Sp}_{2g}(\mathbb{R})$, so Theorem 4.2 implies that $(H_1(B; \mathbb{R}))_{\overline{Q}'} = 0$, and we are done. \square

4.3. The proof of Theorem B. The last 3 terms of the 5-term exact sequence associated to the short exact sequence

$$1 \longrightarrow \Gamma \cap \mathcal{K}_{g,b}^p \longrightarrow \Gamma \longrightarrow \Gamma / (\Gamma \cap \mathcal{K}_{g,b}^p) \longrightarrow 1$$

are

$$(H_1(\Gamma \cap \mathcal{K}_{g,b}^p; \mathbb{R}))_{\Gamma / (\Gamma \cap \mathcal{K}_{g,b}^p)} \xrightarrow{i} H_1(\Gamma; \mathbb{R}) \longrightarrow H_1(\Gamma / (\Gamma \cap \mathcal{K}_{g,b}^p); \mathbb{R}) \longrightarrow 0.$$

By assumption, $[\Gamma \cap \mathcal{K}_{g,b}^p : K(\Gamma)] < \infty$, so Lemma 2.1 implies that the map $H_1(K(\Gamma); \mathbb{R}) \rightarrow H_1(\Gamma \cap \mathcal{K}_{g,b}^p; \mathbb{R})$ is surjective. Since $K(\Gamma)$ is generated by powers of twists, Theorem A allows us to deduce that $i = 0$. Also, $\Gamma / (\Gamma \cap \mathcal{K}_{g,b}^p)$ is a finite-index subgroup of $Q_{g,b}^p$, so Lemma 4.4 implies that $H_1(\Gamma / (\Gamma \cap \mathcal{K}_{g,b}^p); \mathbb{R}) = 0$, and we are done.

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