

## $C^k$ -QUASI-ISOMETRY SETS ARE PRE-COMPACT

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ABSTRACT. Let  $M$  be a closed smooth manifold. By an argument formally similar to one used in constructing the Levi-Civita connection, it is shown that  $C^k$ -quasi-isometry sets in  $DIFF^{k+1}(M)$  are  $C^{k+1}$ -bounded, where  $0 \leq k < \infty$ . This implies, using the Arzela-Ascoli theorem, that such sets are pre-compact in  $DIFF^k(M)$ .

Let  $M$  be a closed  $C^{k+1}$ -manifold,  $1 \leq k < \infty$ . Given a  $C^k$ -Riemannian metric  $g$  on  $M$  we know that the space of  $C^{k+1}$ -diffeomorphisms that preserve  $g$  (i.e. the isometries of  $(M, g)$ ) with the  $C^0$ -topology is compact [8]. In our study of the space of negatively curved metrics ([3], [4], [5], [6]) we have had to deal with the behavior of diffeomorphisms that “almost” preserve a Riemannian metric “up to order  $k$ ”. Even though it has been mentioned sparsely in the literature that “quasi-isometries are pre-compact” (see for instance section 12.55 of Besse’s book [1, p. 253]) we have not found a precise presentation (definitions, statements and proofs) that includes the  $C^k$  cases. We believe it is useful to have such a presentation, and in this short paper we aim to give one.

First we introduce some notation and definitions. The space of  $C^k$ -Riemannian metrics on  $M$ , with the  $C^k$ -topology, will be denoted by  $\mathcal{MET}^{C^k}(M)$ , or simply by  $\mathcal{MET}(M)$ . The spaces of  $C^k$ -self maps and  $C^k$ -self-diffeomorphisms of  $M$ , with the  $C^k$ -topology, will be denoted by  $C^k(M, M)$  and  $DIFF^k(M, M)$ . Recall that  $\mathcal{MET}(M)$  is an open set in the normed vector space of all  $C^k$ -symmetric two-tensors on  $M$ . Hence the notion of a set  $A \subset \mathcal{MET}(M)$  being  $C^k$ -bounded is well-defined. In fact it is equivalent to saying the set of all matrix entries of the chart representative (with respect to a fixed finite set of charts) of all elements of  $A$  is bounded up to order  $k$ . Similarly, the notion of  $F \subset C^k(M, M)$  being  $C^k$ -bounded is well-defined (see item 2 below).

*Remarks.* (1) A technical point here. The fixed set of charts  $\mathcal{A} = \{(U, \phi)\}$  mentioned above has to have the following property. Every chart  $(U, \phi)$  can be extended to a chart  $(V, \phi)$ , with  $\bar{U} \subset V$ ,  $\bar{U}$  compact. In what follows we assume that a finite set of charts with this property has been given.

(2)  $F \subset C^k(M, M)$  is  $C^k$ -bounded if the set of representatives of all  $f : U_1 \cap f^{-1}(U_2) \rightarrow U_2$ ,  $f \in F$ ,  $(U_1, \phi_1), (U_2, \phi_2) \in \mathcal{A}$ , is bounded up to order  $k$ .

(3) The definition of  $C^k$ -boundedness (in both cases) does not depend on the choice of the set of charts.

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We say that  $A \subset \mathcal{MET}(M)$  has *volume forms away from zero* if the set of positive numbers

$$\left\{ \det(a_{ij}^\alpha)(x), a \in A, \alpha = (U, \phi) \in \mathcal{A}, x \in \phi(U) \right\}$$

is bounded away from zero, where  $(a_{ij}^\alpha)$  is the local coordinate expression of  $a \in A$  with respect to  $\alpha$ . Note that this implies that the set of entries of all  $(a_{ij}^\alpha(x))^{-1}$  is bounded, provided  $A$  is  $C^0$ -bounded. We say that  $A \subset \mathcal{MET}(M)$  is  $C^k$ -well-bounded if it is  $C^k$ -bounded and has volume forms away from zero.

Let  $g, h \in \mathcal{MET}^{C^k}(M)$ . A set  $F \subset \text{DIFF}^{k+1}(M)$  is a  $C^k$ -quasi-isometry set with respect to  $g, h$  if the sets  $\{f^*h, f \in F\}$  and  $\{f_*g, f \in F\}$  are  $C^k$ -well-bounded. More generally, a set  $F \subset \text{DIFF}^{k+1}(M)$  is a  $C^k$ -quasi-isometry set if there are  $C^k$ -well-bounded sets  $A, B \subset \mathcal{MET}^{C^k}(M)$ , such that for every  $f \in F$  there is  $a \in A$  with  $f_*a \in B$ . Clearly  $F$  is a  $C^k$ -quasi-isometry set if it is a  $C^k$ -quasi-isometry set with respect to some  $g$  and  $h$ . It is a consequence of Theorem A below that the first definition is independent of the metrics  $g$  and  $h$  and is equivalent to the second definition.

*Remark.* Let  $F$  be a  $C^k$ -quasi-isometry set. Then any subset of  $F$  is a  $C^k$ -quasi-isometry set. Also  $F^{-1} = \{f^{-1}, f \in F\}$  is a  $C^k$ -quasi-isometry set.

Here is our main theorem.

**Theorem A.**  $C^k$ -quasi-isometry sets are pre-compact in  $\text{DIFF}^k(M)$ ,  $k \geq 0$ .

If  $F$  is  $C^{k+1}$ -bounded, then it is also  $C^k$ -bounded and, since the set of  $k$ -derivatives of the elements of  $f \in F$  is bounded, these elements are Lipschitz (with the same constant); hence this set is equicontinuous. The Arselà-Ascoli theorem implies then the following:

**Lemma.** Let  $F \subset C^{k+1}(M, M)$  be  $C^{k+1}$ -bounded. Then  $F$  is pre-compact in  $C^k(M, M)$ .

Using the lemma above, the proof of Theorem A is reduced to proving:

**Theorem B.**  $C^k$ -quasi-isometry sets are  $C^{k+1}$ -bounded.

*Remarks.* (1) Theorem A implies that the definition of  $C^k$ -quasi-isometry set with respect to two metrics  $g, h$  does not depend on  $g, h$ .

(2) Rigorously, Theorem B implies that  $C^k$ -quasi-isometry sets are pre-compact in  $C^k(M, M)$ . But, using the remark before the statement of Theorem A, it is straightforward to check that they are actually pre-compact in  $\text{DIFF}^k(M)$ .

The proof of Theorem B implies (and, essentially reduces to) the following interesting fact. Let  $f$  be a diffeomorphism between open sets of  $\mathbb{R}^n$ . Let  $Df(x) = \left(\frac{\partial f_i}{\partial x_j}(x)\right)$  be its Jacobian matrix at  $x$ . Assume: (i) the entries of  $Df^T(x)Df(x)$  are bounded up to order  $k$ , and (ii)  $\det Df(x)$  is bounded away from zero. Then the entries of  $Df(x)$  are bounded up to order  $k$ . Note that it is important to assume that the map  $x \mapsto Df(x)$  has for images Jacobian matrices; otherwise the result fails: let  $x \mapsto C(x) \in O(n)$  be any wild function, but we have that  $C^T(x)C(x)$  is constant.

*Proof of Theorem B.* Let  $F \subset \text{DIFF}^{k+1}(M)$  be a  $C^k$ -quasi-isometry set. Let  $A$  and  $B$  be  $C^k$ -well-bounded sets such that for every  $f \in F$ , there is  $a \in A$  with

$f_*a \in B$ . In what follows the local representatives of  $f \in F$ ,  $a \in A$  and  $b \in B$  will be denoted by the same letters. Also, for  $f \in F$ ,  $Df(x) = \left(\frac{\partial f_i}{\partial x_j}(x)\right)$  denotes the Jacobian matrix of (a local representative of)  $f$ . Then, for  $f \in F$  there are  $a \in A$ ,  $b \in B$  such that:

$$(1) \quad Df(x)^T a(x) Df(x) = b(x)$$

for every  $x \in \phi(U)$ ,  $(U, \phi) \in \mathcal{A}$ . We now proceed to verify Theorem B by induction on  $k$  starting with the special case  $k = 0$ . Let  $c(x)$  be the (unique and smoothly defined) square root of  $a(x)$ . Then

$$\left(c(x) Df(x)\right)^T \left(c(x) Df(x)\right) = b(x).$$

Since the diagonal entries of this matrix are the norm of the columns of  $c(x)Df(x)$  and the entries of  $b(x)$  are bounded, we have that the entries of  $c(x) Df(x)$  are also bounded. But the entries of  $c^{-1}(x)$  are bounded since  $\det c(x) = \sqrt{\det a(x)}$  is bounded away from zero. We therefore conclude that the entries of  $Df(x)$  are bounded, thus proving the theorem when  $k = 0$ . Interchanging the sets  $A$  and  $B$  in the above argument, we obtain also that the  $\|Df^{-1}\|$ 's are bounded as well.

Proceeding with the induction we assume  $k > 0$  and that all derivatives of order  $\leq k$  of all  $f$ 's are bounded. We prove that all derivatives of order  $k + 1$  are also bounded by contradiction. Suppose  $F$  is not  $C^{k+1}$ -bounded. Then there are sequences  $(f_m)$  in  $F$  and  $(x_m)$  in  $M$  such that some  $k + 1$  derivative of  $f_m$ , at  $x_m$ , becomes large as  $m \rightarrow \infty$ . Since  $M$  is compact and  $(f_m)$  is pre-compact in  $C^0(M, M)$  we can assume that both sequences converge. It follows that there are two charts,  $\psi : V' \rightarrow V \subset \mathbb{R}^n$ ,  $\psi' : W' \rightarrow W \subset \mathbb{R}^n$ , such that  $f_m(V) \subset W$  and some  $k + 1$  derivative of  $f_m : V \rightarrow W$  becomes large as  $m \rightarrow \infty$ . (The charts  $\psi, \psi'$  are not necessarily in  $\mathcal{A}$  but can be extended to charts in  $\mathcal{A}$ .) We have reduced our problem to  $\mathbb{R}^n$ .

For a  $j$ -multi-index  $\alpha = (i_1, \dots, i_j)$ ,  $1 \leq i_\ell \leq n$  (i.e.  $\alpha \in \{1, 2, \dots, n\}^j$ ), we write  $\partial x_\alpha = \partial x_{i_1} \cdots \partial x_{i_j}$ . If  $\alpha$  and  $\beta$  are multi-indices,  $\alpha\beta$  denotes the "concatenation" multi-index. Also a 1-multi-index  $(i)$  will be denoted just as  $i$ . Let  $\alpha$  be a  $k$ -multi-index. Applying  $\frac{\partial^k}{\partial x_\alpha}$  to equation (1) above we get:

$$\left[ Df^T a \left( \frac{\partial^k}{\partial x_\alpha}(Df) \right) + \left( \frac{\partial^k}{\partial x_\alpha}(Df) \right)^T a Df \right] + d = \frac{\partial^k}{\partial x_\alpha} b,$$

where  $d$  is a sum of products of derivatives of order  $\leq k$  of  $a$  and  $f$ . Hence the expression inside the brackets is bounded. Note that the columns of  $Df$  are  $f_i = \frac{\partial}{\partial x_i} f$ , that is,  $Df = [f_1 \cdots f_n]$ . To simplify our notation, for a multi-index  $\beta$  we write  $f_\beta = \frac{\partial^k}{\partial x_\beta} f$ . Using this notation, the equations above tell us that the following set of expressions is uniformly bounded:

$$(2) \quad f_i^T a f_{\alpha j} + f_j^T a f_{\alpha i}$$

for all  $1 \leq i, j \leq n$  and every  $\alpha \in \{1, \dots, n\}^j$ . Note that the second term in (2) is obtained from the first term by interchanging  $i$  and  $j$ .

*Claim.* The fact that the expressions in (2) are bounded implies that the following expressions are bounded for every  $\beta \in \{1, \dots, n\}^{k+1}$ :

$$(Df)^T a f_\beta.$$

Since the  $Df^{-1}$ 's and  $a^{-1}$ 's are bounded, we conclude that the expressions  $f_\beta = \frac{\partial^{k+1}}{\partial x^\beta} f$  are bounded. This contradicts our assumption that "some  $k + 1$  derivative of  $f_m : V \rightarrow W$  becomes large as  $m \rightarrow \infty$ ", and hence proves Theorem B once we verify the claim.

*Proof of the claim.* To simplify our notation we denote the expression  $f_i^T a f_\beta$  as simply  $[i, \beta]$ . Note that the  $i$ -th entry of the column vector  $Df^T a f_\beta$  is  $[i, \beta]$ . Therefore to prove the claim we need to show that  $[i, \beta]$  is bounded, for every  $i$  and every  $(k + 1)$ -multi-index  $\beta$ . Note that our argument above proving Theorem B for  $k = 0$  also proved the claim for  $k = 0$ . Hence we may assume that  $k > 0$ . Now, with the bracket notation introduced above, the bounded expression in (2) can be rewritten as:

$$(2') \quad [i, \alpha j] + [j, \alpha i]$$

That is, the expression in (2') is bounded for every  $i, j$  and  $k$ -multi-index  $\alpha$ . Since  $k + 1 \geq 2$ , we can write  $\beta = \gamma s t$  for some  $(k - 1)$ -multi-index  $\gamma$  and  $1 \leq s, t \leq n$ . By cyclically permuting  $i, s, t$  and applying the fact that (2') is bounded, we obtain that the following three expressions are bounded:

$$\begin{aligned} [i, \gamma s t] + [t, \gamma s i] \\ [s, \gamma t i] + [i, \gamma t s] \\ [t, \gamma i s] + [s, \gamma i t]. \end{aligned}$$

Notice that only three numbers occur in these three sums, namely:

$$\begin{aligned} A &= [i, \gamma s t] = [i, \gamma t s] \\ B &= [s, \gamma t i] = [s, \gamma i t] \\ C &= [t, \gamma i s] = [t, \gamma s i]. \end{aligned}$$

Therefore the above three bounded terms become:

$$\begin{aligned} A + C \\ A + B \\ B + C. \end{aligned}$$

Consequently all three terms  $A, B$ , and  $C$  are bounded. This completes the proof of the claim because  $A = [i, \beta]$ .

*Remark.* The above proof is formally similar to that given for the Fundamental Lemma of Riemannian Geometry, i.e. for the construction of the Levi-Civita connection (see, for instance, pp. 48-49 of [7]).

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