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CIRCLE IMMERSIONS THAT CAN BE DIVIDED INTO TWO ARC EMBEDDINGS

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Abstract. We give a complete characterization of a circle immersion that can be divided into two arc embeddings in terms of its chord diagram.

1. Introduction

Let $\mathbb{S}^1$ be the unit circle. Let X be a set and $f : \mathbb{S}^1 \to X$ a map. Let n be a natural number greater than one. Suppose that there are n subspaces $I_1, \cdots, I_n$ of $\mathbb{S}^1$ with the following properties:

(1) Each I_i is homeomorphic to a closed interval.
(2) $\mathbb{S}^1 = I_1 \cup \cdots \cup I_n$.
(3) The restriction map $f|_{I_i} : I_i \to X$ is injective for each i.

Then we say that f can be divided into n arc embeddings. We define the *arc number* of f, denoted by $\mathrm{arc}(f)$, to be the smallest such n except for the case that f itself is injective. If f itself is injective, then we define $\mathrm{arc}(f) = 1$. If f cannot be divided into n arc embeddings for any natural number n, then we define $\mathrm{arc}(f) = \infty$.

Note that if f can be divided into n arc embeddings, then there exist n subspaces $I_1, \cdots, I_n$ of $\mathbb{S}^1$ with (1), (2) and (3) above together with the following additional condition.

(4) $I_i \cap I_j = \partial I_i \cap \partial I_j$ for each i and j with $1 \leq i < j \leq n$.

Namely, we may assume that $\mathbb{S}^1$ is covered by mutually interior disjoint n simple arcs $I_1, \cdots, I_n$.

Let $\mathrm{S}(f) = \{x \in \mathbb{S}^1 | f^{-1}(f(x)) \text{ is not a singleton}\}$ and $\mathrm{s}(f) = f(\mathrm{S}(f))$. We say that a map $f : \mathbb{S}^1 \to X$ has *finite multiplicity* if $\mathrm{S}(f)$ is a finite subset of $\mathbb{S}^1$. From now on we restrict our attention to maps that have finite multiplicity. The purpose of this paper is to give a characterization of a map $f : \mathbb{S}^1 \to X$ with $\mathrm{arc}(f) = 2$. By $|Y|$ we denote the cardinality of a set Y. Let $m(f)$ be the maximum of $|f^{-1}(y)|$, where y varies over all points of X. It is clear that $\mathrm{arc}(f) \geq m(f)$. Thus we further restrict our attention to a map $f : \mathbb{S}^1 \to X$ whose multiple points are only finitely many double points; namely f has finite multiplicity and $m(f) \leq 2$. Then we have $|\mathrm{S}(f)| = 2m$ for some non-negative integer m. Then the *crossing number* of f, denoted by $c(f)$, is defined to be m.

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Let m be a natural number. An *m-chord diagram* on $\mathbb{S}^1$ is a pair $\mathcal{C} = (P, \varphi)$, where P is a subset of $\mathbb{S}^1$ that contains exactly $2m$ points and φ is a fixed point free involution on P. A *chord* c of $\mathcal{C}$ is an unordered pair of points $(x, \varphi(x)) = (\varphi(x), x)$, where x is a point in P. Let $\sim_{\mathcal{C}}$ be the equivalence relation on $\mathbb{S}^1$ generated by $x \sim_{\mathcal{C}} \varphi(x)$ for every $x \in P$. Let $\mathbb{S}^1/\sim_{\mathcal{C}}$ be the quotient set and $f_{\mathcal{C}} : \mathbb{S}^1 \to \mathbb{S}^1/\sim_{\mathcal{C}}$ the quotient map. We call $f_{\mathcal{C}}$ the associated map of $\mathcal{C}$. Then the arc number of $\mathcal{C}$, denoted by $\mathrm{arc}(\mathcal{C})$, is defined to be the arc number of $f_{\mathcal{C}}$. Two m-chord diagrams $\mathcal{C}_1 = (P_1, \varphi_1)$ and $\mathcal{C}_2 = (P_2, \varphi_2)$ are *equivalent* if there is an orientation-preserving self-homeomorphism h of $\mathbb{S}^1$ such that $h(P_1) = P_2$ and $h \circ \varphi_1 = \varphi_2 \circ h$. From now on we consider m-chord diagrams up to this equivalence relation. In the following we sometimes express an m-chord diagram $\mathcal{C} = (P, \varphi)$ by m line segments in the plane $\mathbb{R}^2$ where $\mathbb{S}^1 \subset \mathbb{R}^2$, P is the set of the end points of these line segments and x and $\varphi(x)$ are joined by a line segment for each $x \in P$. Thus a line segment expresses a chord, and from now on we do not distinguish them. See for example Figure 1.1.

Let $f : \mathbb{S}^1 \to X$ be a map whose multiple points are only finitely many double points. By $\mathcal{C}(f)$ we denote the $c(f)$-chord diagram $(\mathrm{S}(f), \varphi_f)$, where $\varphi_f : \mathrm{S}(f) \to \mathrm{S}(f)$ is the fixed point free involution with $f|_{\mathrm{S}(f)} \circ \varphi_f = f|_{\mathrm{S}(f)}$. We call $\mathcal{C}(f)$ the associated chord diagram of f. Then it is clear that $\mathrm{arc}(f) = \mathrm{arc}(\mathcal{C}(f))$.

A chord diagram $\mathcal{D} = (Q, \psi)$ is called a *sub-chord diagram* of a chord diagram $\mathcal{C} = (P, \varphi)$ if Q is a subset of P and ψ is the restriction of φ on Q. Then it is clear that $\mathrm{arc}(\mathcal{D}) \leq \mathrm{arc}(\mathcal{C})$. We call $\mathcal{D} = (Q, \psi)$ a *proper sub-chord diagram* of $\mathcal{C} = (P, \varphi)$ if $\mathcal{D}$ is a sub-chord diagram of $\mathcal{C}$ and Q is a proper subset of P.

Let n be a natural number. Let $\mathcal{C}_{2n+1}$ be a $(2n+1)$-chord diagram as illustrated in Figure 1.1. Formally $\mathcal{C}_{2n+1}$ is defined as follows. Let k be a natural number greater than two. Let R_k be a regular k-gon inscribed in $\mathbb{S}^1$ and let $v_{k;1}, \cdots, v_{k;k}$ be the vertices of R_k that are arranged in this order on $\mathbb{S}^1$ along the counterclockwise orientation of $\mathbb{S}^1$; namely, $v_{k;i}$ and $v_{k;i+1}$ are adjacent in R_k for each i, where the indices are considered modulo k. Let j be a natural number less than $\frac{k}{2}$. Let $c(k; i, j)$ be the chord joining $v_{k;i}$ and $v_{k;i+j}$ for each $i \in \{1, \cdots, k\}$. Then $\mathcal{C}_{2n+1}$ is the chord diagram represented by chords $c(4n+2; 2i-1, 2n-1)$ with $i \in \{1, \cdots, 2n+1\}$. We will show that $\mathrm{arc}(\mathcal{C}_{2n+1}) = 3$ but $\mathrm{arc}(\mathcal{D}) = 2$ for any proper sub-chord diagram $\mathcal{D}$ of $\mathcal{C}_{2n+1}$. Moreover we have the following theorem.

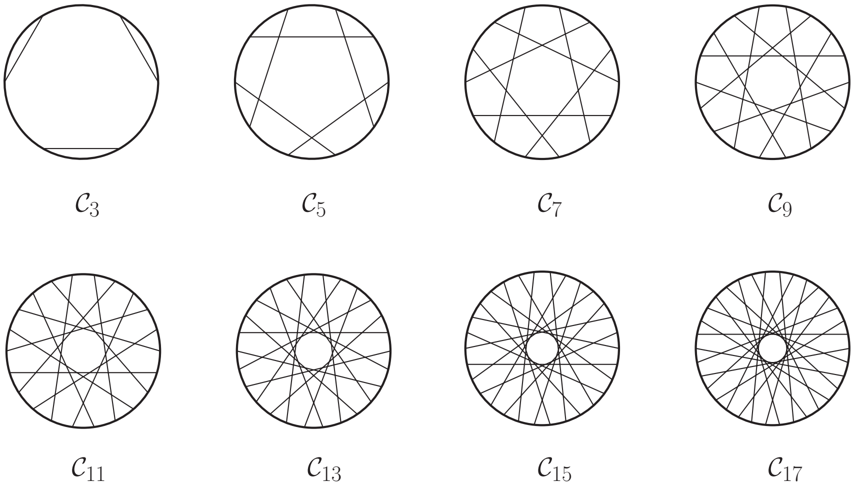
Theorem 1-1. *Let m be a natural number and $\mathcal{C}$ an m-chord diagram on $\mathbb{S}^1$. Then $\mathrm{arc}(\mathcal{C}) = 2$ if and only if no sub-chord diagram of $\mathcal{C}$ is equivalent to the chord diagram $\mathcal{C}_{2n+1}$ for any natural number n.*

We will give the proof of Theorem 1-1 in Section 2. We note here that after putting the first version of this paper on arXiv, T. Hagge generalized Theorem 1-1 in [2]. His theorem characterizes the chord diagrams with arc number k for each natural number k.

The motive for this paper was the result in [3] that every knot has a diagram which can be divided into two simple arcs. This result was rediscovered by Ozawa [4] and Shinjo [5]. See also [1]. Then it is natural to ask what plane closed curve can be divided into two simple arcs. Theorem 1-1 gives an answer to this question. However we still have a question whether or not we actually need all of $\mathcal{C}_3, \mathcal{C}_5, \cdots$. The following proposition answers this question and says that we actually need all of them.

Proposition 1-2. *For each natural number* n *there exists a smooth immersion* $f_{2n+1} : \mathbb{S}^1 \to \mathbb{R}^2$ *with* $\operatorname{arc}(f_{2n+1}) = 3$ *that has only finitely many transversal double points such that the associated chord diagram* $\mathcal{C}(f_{2n+1})$ *of* f_{2n+1} *has a sub-chord diagram which is equivalent to* $\mathcal{C}_{2n+1}$ *but has no sub-chord diagram which is equivalent to* $\mathcal{C}_{2m+1}$ *for any* $m < n$*.*

We will give the proof of Proposition 1-2 in Section 3.



$\mathcal{C}_3$ $\mathcal{C}_5$ $\mathcal{C}_7$ $\mathcal{C}_9$

$\mathcal{C}_{11}$ $\mathcal{C}_{13}$ $\mathcal{C}_{15}$ $\mathcal{C}_{17}$

FIGURE 1.1

2. PROOF OF THEOREM 1-1

First we check that $\operatorname{arc}(\mathcal{C}_{2n+1}) = 3$. Let $\mathcal{C} = (P, \varphi)$ be an m-chord diagram with $\operatorname{arc}(\mathcal{C}) = 2$. A pair of points $p, q \in \mathbb{S}^1 \setminus P$ is called a *cutting pair* of $\mathcal{C}$ if x and $\varphi(x)$ belong to the different components of $\mathbb{S}^1 \setminus \{p, q\}$ for each $x \in P$. Then p and q are "antipodal"; namely, each component of $\mathbb{S}^1 \setminus \{p, q\}$ contains exactly m points of P. Thus we can check whether or not a given m-chord diagram has arc number 2 by examining m pairs of antipodal points of it. Then by the symmetry of $\mathcal{C}_{2n+1}$ we immediately have the inequality $\operatorname{arc}(\mathcal{C}_{2n+1}) > 2$. Then it is easily seen that $\operatorname{arc}(\mathcal{C}_{2n+1}) = 3$ and $\operatorname{arc}(\mathcal{D}) = 2$ for any proper sub-chord diagram $\mathcal{D}$ of $\mathcal{C}_{2n+1}$. The 'only if part' of the proof of Theorem 1-1 immediately follows. The 'if part' immediately follows from the following proposition.

Proposition 2-1. *Let* $\mathcal{C}$ *be a chord diagram on* $\mathbb{S}^1$ *that satisfies the following condition* $(*)$*:*

$(*)$ $\operatorname{arc}(\mathcal{C}) \geq 3$ *and* $\operatorname{arc}(\mathcal{D}) = 2$ *for any proper sub-chord diagram* $\mathcal{D}$ *of* $\mathcal{C}$*.*
Then there is a natural number n *such that* $\mathcal{C}$ *is equivalent to* $\mathcal{C}_{2n+1}$*.*

Note that deleting a chord will decrease the arc number by at most one. Therefore, if $\mathcal{C}$ is a chord diagram on $\mathbb{S}^1$ that satisfies the condition $(*)$, then $\operatorname{arc}(\mathcal{C}) = 3$. For the proof of Proposition 2-1 we prepare the following lemmas. Let $\mathcal{C} = (P, \varphi)$ be a chord diagram and $c = (x, \varphi(x))$ a chord of $\mathcal{C}$. Let α and β be the components of $\mathbb{S}^1 \setminus \{x, \varphi(x)\}$. We may suppose without loss of generality that $|\alpha \cap P| \leq |\beta \cap P|$.

Then the *length* of c in $\mathcal{C}$, denoted by $l(c) = l(c, \mathcal{C})$, is defined to be $|\alpha \cap P| + 1$. By $\mathcal{C} \setminus c$, we denote the chord diagram $(P \setminus \{x, \varphi(x)\}, \varphi|_{P\setminus\{x,\varphi(x)\}})$. Let p, q, x and y be four mutually distinct points on $\mathbb{S}^1$. We say that the pair of points p and q separates the pair of points x and y if each component of $\mathbb{S}^1 \setminus \{p, q\}$ contains exactly one of x and y. Note that the pair of points p and q separates the pair of points x and y if and only if the chord joining p and q intersects the chord joining x and y.

Lemma 2-2. *Let $\mathcal{C} = (P, \varphi)$ be a chord diagram on $\mathbb{S}^1$ that satisfies the condition $(*)$. Let $c = (x, \varphi(x))$ be a chord of $\mathcal{C}$. Let p and q be a cutting pair of $\mathcal{C} \setminus c$. Then p and q do not separate x and $\varphi(x)$.*

Proof. If p and q separate x and $\varphi(x)$, then p and q form a cutting pair of $\mathcal{C}$ itself. Then it follows that $\mathrm{arc}(\mathcal{C}) = 2$. This is a contradiction. □

Lemma 2-3. *Let $\mathcal{C} = (P, \varphi)$ be an m-chord diagram on $\mathbb{S}^1$ that satisfies the condition $(*)$. Let $c = (x, \varphi(x))$ be a chord of $\mathcal{C}$. Then $l(c, \mathcal{C}) \leq m - 2$.*

Proof. Since $|P| = 2m$ we have $1 \leq l(c, \mathcal{C}) \leq m$ for any chord c. First we examine the case $l(c, \mathcal{C}) = m$. In this case each component of $\mathbb{S}^1 \setminus \{x, \varphi(x)\}$ contains exactly $m - 1$ elements of P. Let p and q be a cutting pair of $\mathcal{C} \setminus c$. Then Lemma 2-2 implies that p and q do not separate x and $\varphi(x)$. Note that each component of $\mathbb{S}^1 \setminus \{p, q\}$ also contains exactly $m - 1$ elements of $P \setminus \{x, \varphi(x)\}$. Then it follows that p and q are next to x and $\varphi(x)$ or $\varphi(x)$ and x, respectively. We may suppose without loss of generality that p and q are next to x and $\varphi(x)$, respectively. Let p' be a point on $\mathbb{S}^1$ that is next to x such that p' and p separate x and $\varphi(x)$. Then p' and q would be a cutting pair of $\mathcal{C}$. This is a contradiction. Next we examine the case $l(c, \mathcal{C}) = m - 1$. In this case one component of $\mathbb{S}^1 \setminus \{x, \varphi(x)\}$ contains exactly $m - 2$ elements of P, and the other component contains exactly m elements of P. Let p and q be a cutting pair of $\mathcal{C} \setminus c$. Then Lemma 2-2 implies that p and q do not separate x and $\varphi(x)$. Note that each component of $\mathbb{S}^1 \setminus \{p, q\}$ contains exactly $m - 1$ elements of $P \setminus \{x, \varphi(x)\}$. Then it follows that one of p and q, say p, is next to x or $\varphi(x)$, say x. Let p' be a point on $\mathbb{S}^1$ that is next to x such that p' and p separate x and $\varphi(x)$. Then p' and q would be a cutting pair of $\mathcal{C}$. This is a contradiction. Thus we have $l(c, \mathcal{C}) \leq m - 2$. □

Lemma 2-4. *Let $\mathcal{C} = (P, \varphi)$ be an m-chord diagram on $\mathbb{S}^1$ that satisfies the condition $(*)$. Let $c = (x, \varphi(x))$ be a chord of $\mathcal{C}$. Then $l(c, \mathcal{C}) \geq m - 2$.*

Proof. Suppose that there is a chord $c = (x, \varphi(x))$ of $\mathcal{C}$ with $l(c, \mathcal{C}) \leq m - 3$. Let $\mathcal{D} = (Q, \varphi|_Q)$ be the sub-chord diagram of $\mathcal{C}$ obtained by deleting all the chords that have at least one end point on the component of $\mathbb{S}^1 \setminus \{x, \varphi(x)\}$ that has exactly $l(c, \mathcal{C}) - 1$ points of P. Then $l(c, \mathcal{D}) = 1$. Let n be the number of chords of $\mathcal{D}$. Let A (resp. B) be the point in Q such that each component of $\mathbb{S}^1 \setminus \{x, A\}$ (resp. $\mathbb{S}^1 \setminus \{\varphi(x), B\}$) contains $n - 1$ points of Q. Note that $n \geq m - (l(c, \mathcal{C}) - 1) \geq m - (m - 3 - 1) = 4$. Therefore $\mathcal{D}$ has at least 4 chords. Then there is a chord $d = (y, \varphi(y))$ of $\mathcal{D}$ such that $\{y, \varphi(y)\}$ and $\{x, \varphi(x), A, B\}$ are mutually disjoint. Suppose that y and $\varphi(y)$ do not separate x and A. In this case there must be a chord $e = (z, \varphi(z))$ of $\mathcal{D}$ such that $\{z, \varphi(z)\}$ and $\{x, \varphi(x), y, \varphi(y)\}$ are mutually disjoint and z and $\varphi(z)$ do not separate y and $\varphi(y)$. Then the chords c, d and e form a sub-chord diagram of $\mathcal{C}$ that is equivalent to $\mathcal{C}_3$. This is a contradiction. Suppose that y and $\varphi(y)$ separate x and A. Let p and q be a cutting pair of $\mathcal{C} \setminus d$. Then we have by Lemma 2-2 that p and q do not separate y and $\varphi(y)$. Note that p

and q also become a cutting pair of $\mathcal{D} \setminus d$ and they separate x and $\varphi(x)$. Then the component of $\mathbb{S}^1 \setminus \{p, q\}$ that contains both A and B has more points of $Q \setminus \{y, \varphi(y)\}$ than the other. This is a contradiction. $\square$

Thus we have shown the following lemma.

Lemma 2-5. *Let $\mathcal{C} = (P, \varphi)$ be an m-chord diagram on $\mathbb{S}^1$ that satisfies the condition $(*)$. Then $\mathcal{C}$ satisfies the following condition $(\star)$.*

$(\star)$ $l(c, \mathcal{C}) = m - 2$ for every chord c of $\mathcal{C}$.

Proposition 2-6. *Let $\mathcal{C} = (P, \varphi)$ be an m-chord diagram on $\mathbb{S}^1$ that satisfies the condition $(\star)$. If m is even, then m is divisible by 4 and $\operatorname{arc}(\mathcal{C}) = 2$. If m is odd, then $\mathcal{C}$ is equivalent to $\mathcal{C}_m$.*

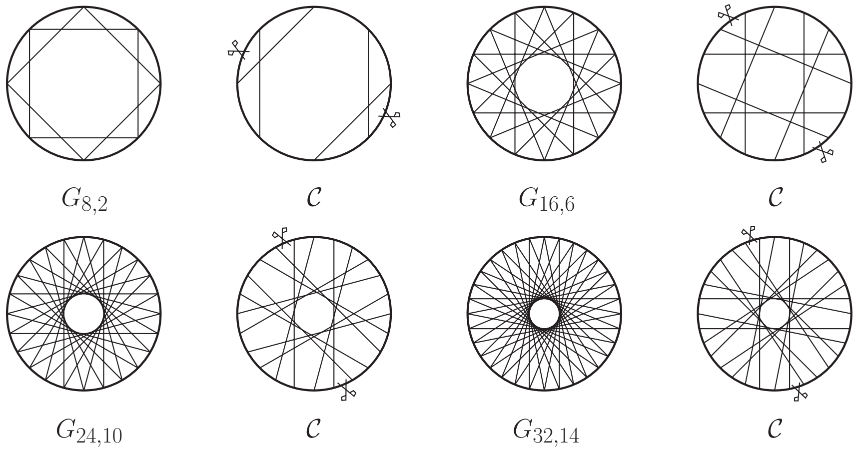
Proof. Recall that R_{2m} is a regular $(2m)$-gon inscribed in $\mathbb{S}^1$ and $v_{2m;1}, \cdots, v_{2m;2m}$ are the vertices of R_{2m} lying in this order. Let $G_{2m,m-2}$ be the graph whose vertices are $v_{2m;1}, \cdots, v_{2m;2m}$ and whose edges are the chords $c(2m; i, m-2)$ joining the vertices $v_{2m;i}$ and $v_{2m;i+m-2}$ where $i \in \{1, \cdots, 2m\}$. By calculating the greatest common divisor $(2m, m-2) = (2m - 2(m-2), m-2) = (4, m-2)$ we have the isomorphism type of the graph $G_{2m,m-2}$ as follows.

(1) If m is a multiple of 4, then $(4, m-2) = 2$ and therefore $G_{2m,m-2}$ is isomorphic to a disjoint union of two m-cycles.

(2) If m is congruent to 2 modulo 4, then $(4, m-2) = 4$ and therefore $G_{2m,m-2}$ is isomorphic to a disjoint union of four $\frac{m}{2}$-cycles.

(3) If m is odd, then $(4, m-2) = 1$ and therefore $G_{2m,m-2}$ is isomorphic to a $2m$-cycle.

Note that in each case $\mathcal{C}$ must be a perfect matching of the graph $G_{2m,m-2}$. In (1), up to symmetry, $\mathcal{C}$ is as illustrated in Figure 2.1. Then we have $\operatorname{arc}(\mathcal{C}) = 2$. In (2), $G_{2m,m-2}$ has no perfect matchings because an $\frac{m}{2}$-cycle is an odd cycle. In (3), $\mathcal{C}$ is equivalent to $\mathcal{C}_m$. This completes the proof. $\square$


$G_{8,2}$ $\mathcal{C}$ $G_{16,6}$ $\mathcal{C}$

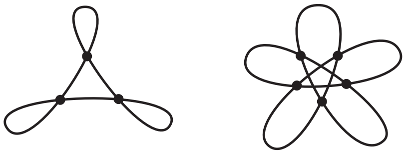
$G_{24,10}$ $\mathcal{C}$ $G_{32,14}$ $\mathcal{C}$

FIGURE 2.1

Proof of Proposition 2-1. By Lemma 2-5 and Proposition 2-6 we have the result. $\square$

3. Examples of plane curves

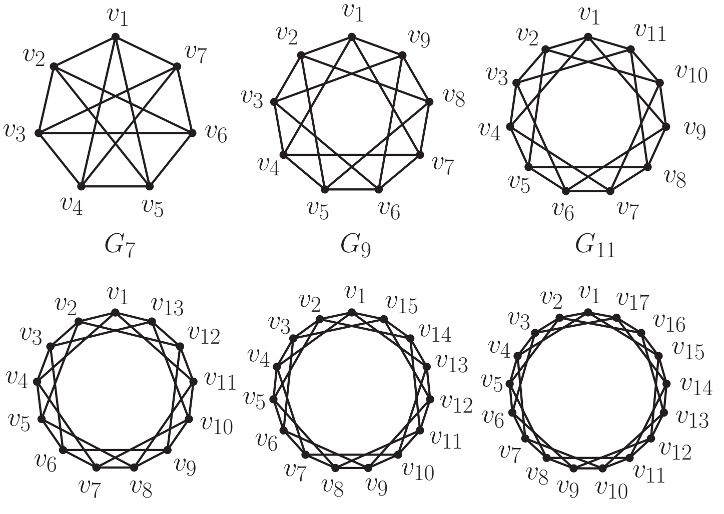
Let $\mathcal{C} = (P, \varphi)$ be a chord diagram and let $c = (x, \varphi(x))$ and $d = (y, \varphi(y))$ be two chords of $\mathcal{C}$. We say that c and d are *parallel* if the pair of points x and $\varphi(x)$ does not separate the pair of points y and $\varphi(y)$. We say that two distinct points x and y in P are *next to each other* if there is a component of $\mathbb{S}^1 \setminus \{x, y\}$ that is disjoint from P. We say that c and d are *close to each other* if x and y are next to each other and $\varphi(x)$ and $\varphi(y)$ are next to each other, or x and $\varphi(y)$ are next to each other and $\varphi(x)$ and y are next to each other.



$f_3(\mathbb{S}^1)$ $f_5(\mathbb{S}^1)$

Figure 3.1

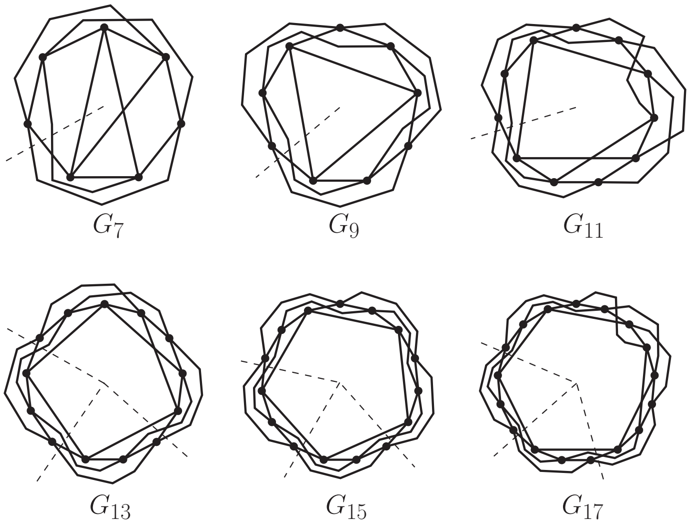
Proof of Proposition 1-2. The cases $n = 1, 2$ are shown in Figure 3.1. We consider the case $n \geq 3$. Let $G_{2n+1} = \mathbb{S}^1/\sim_{\mathcal{C}_{2n+1}}$ be the 4-regular graph obtained from $\mathbb{S}^1$ by identifying the end points of each chord of $\mathcal{C}_{2n+1}$. It is easy to observe that G_{2n+1} is isomorphic to a graph obtained from a $(2n+1)$-cycle Γ_{2n+1} on vertices $v_1, \cdots, v_{2n+1}$ lying in this order by adding edges joining v_i and v_{i+3} for each i, so that along the counterclockwise orientation of $\mathbb{S}^1$ the vertices of G_{2n+1} appear as $v_i, v_{i+1}, v_{i+1-3}, v_{i+1-3+1}, v_{i+1-3+1-3}, \cdots$. Here the indices of the vertices are listed modulo $2n+1$. See Figure 3.2. We may suppose that Figure 3.2 illustrates not



G_7 G_9 G_{11}

G_{13} G_{15} G_{17}

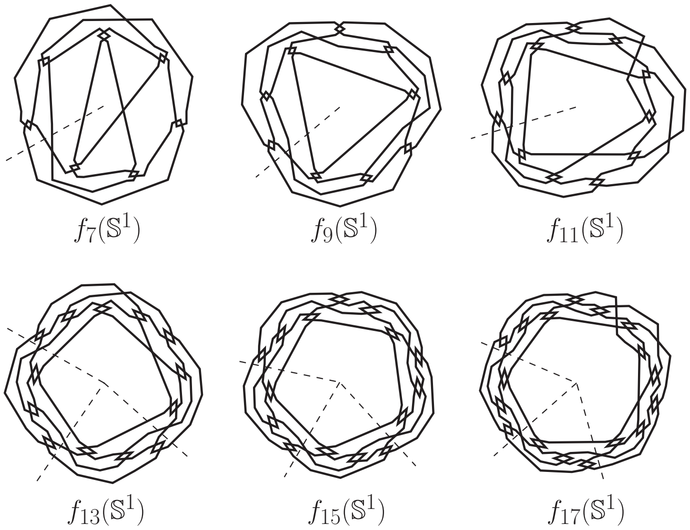
Figure 3.2



G_7 G_9 G_{11}

G_{13} G_{15} G_{17}

FIGURE 3.3



$f_7(\mathbb{S}^1)$ $f_9(\mathbb{S}^1)$ $f_{11}(\mathbb{S}^1)$

$f_{13}(\mathbb{S}^1)$ $f_{15}(\mathbb{S}^1)$ $f_{17}(\mathbb{S}^1)$

FIGURE 3.4

only abstract graphs but graphs immersed in $\mathbb{R}^2$. Then we deform them on $\mathbb{R}^2$ as illustrated in Figure 3.3. Note that they are classified into three types by $2n+1$ modulo 6; namely, G_{2n+1+6} is obtained from G_{2n+1} by cutting open G_{2n+1} along the dotted line and inserting two pieces of a pattern as illustrated in Figure 3.3. We modify this G_{2n+1} and have the image $f_{2n+1}(\mathbb{S}^1)$ as illustrated in Figure 3.4. Note that each vertex of G_{2n+1} is replaced by two transversal double points. We call them a *twin pair*. The chords corresponding to them are also called a twin pair. By choosing any one of them for each twin pair we have a sub-chord diagram of $\mathcal{C}(f_{2n+1})$ that is equivalent to $\mathcal{C}_{2n+1}$ by the construction. Observe that each

$f_{2n+1}(\mathbb{S}^1)$ is made of $(2n+1)$-times repetitions of "one step forward and three steps back" along the $(2n+1)$-cycle Γ_{2n+1} and it totally goes around Γ_{2n+1} twice. Here "one step forward" corresponds to an edge of G_{2n+1} joining v_i and v_{i+1} and "three steps back" corresponds to an edge of G_{2n+1} joining v_{i+1} and v_{i+1-3}. It has no local double points, and each double point comes from a part and another part that is one lap behind. Therefore we have $\mathrm{arc}(f_{2n+1}) = 3$.

Now we will check that no sub-chord diagram of $\mathcal{C}(f_{2n+1})$ is equivalent to $\mathcal{C}_{2m+1}$ for any $m < n$. Note that two chords in a twin pair are close to each other in $\mathcal{C}(f_{2n+1})$. Let $\mathcal{D}(f_{2n+1})$ be a sub-chord diagram of $\mathcal{C}(f_{2n+1})$ obtained from $\mathcal{C}(f_{2n+1})$ by deleting one of two chords for each twin pair in $\mathcal{C}(f_{2n+1})$. Since no two chords in $\mathcal{C}_{2m+1}$ are close to each other it is sufficient to check that no sub-chord diagram of $\mathcal{D}(f_{2n+1})$ is equivalent to $\mathcal{C}_{2m+1}$ for any $m < n$. Suppose that $\mathcal{E}$ is a sub-chord diagram of $\mathcal{D}(f_{2n+1})$ that is equivalent to $\mathcal{C}_{2m+1}$ for some $m < n$. Since no proper sub-chord diagram of $\mathcal{C}_{2n+1}$ is equivalent to $\mathcal{C}_{2m+1}$, there is a chord c of $\mathcal{E}$ that does not belong to any twin pair of $\mathcal{C}(f_{2n+1})$; namely, c corresponds to a transversal double point of f_n that comes from a double point of $G_{2n+1} \subset \mathbb{R}^2$ in Figure 3.3. Observe that for each chord $d = (x, \varphi(x))$ of $\mathcal{C}_{2m+1}$ there exist exactly two chords $g = (y, \varphi(y))$ and $h = (z, \varphi(z))$ of $\mathcal{C}_{2m+1}$ that are parallel to d such that all of $y, \varphi(y), z$ and $\varphi(z)$ are contained in the same component of $\mathbb{S}^1 \setminus \{x, \varphi(x)\}$.

Therefore c must have such two chords in $\mathcal{D}(f_{2n+1})$. By the "one step forward and three steps back" structure of $f_{2n+1}(\mathbb{S}^1)$ mentioned above the double points corresponding to such chords must lie in a small neighbourhood of the double point corresponding to c. Then we can check that there are no such two chords for c in $\mathcal{D}(f_{2n+1})$ except that the case $2n+1$ is congruent to 5 modulo 6 and c is one of the three chords of $\mathcal{C}(f_{2n+1})$ that come from the three double points on the same edge of $G_{2n+1} \subset \mathbb{R}^2$ in Figure 3.3.

For this exceptional case we further observe that the chords g and h above intersect unless $m = 1$ and the end point of g (resp. h) that is next to x or $\varphi(x)$ in $\mathcal{C}_{2m+1}$ is not next to any end point of h (resp. g) in $\mathcal{C}_{2m+1}$.

However in this exceptional case we can check that there are no such two chords for c in $\mathcal{D}(f_{2n+1})$. This is a contradiction. □

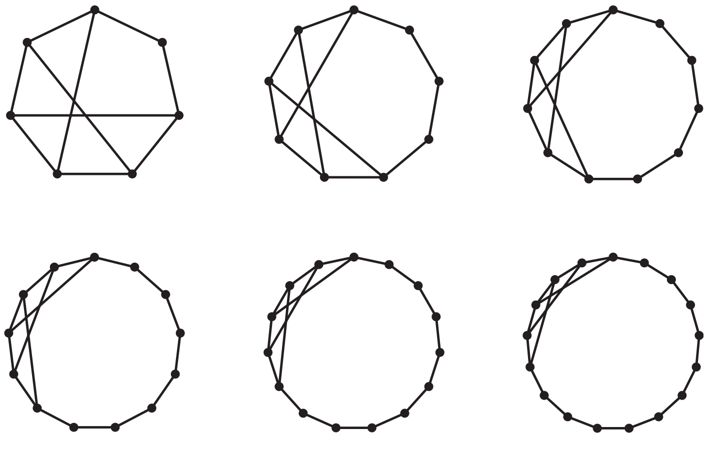


FIGURE 3.5

Remark 3-1. It is easy to see that the graph $G_{2n+1} = \mathbb{S}^1 / \sim_{\mathcal{C}_{2n+1}}$ is a non-planar graph for $n \geq 2$. In fact G_5 is the complete graph K_5, and G_{2n+1} with $n \geq 3$ contains a subgraph that is a subdivision of the complete bipartite graph $K_{3,3}$. See Figure 3.5. Therefore for $n \geq 2$ there is no smooth immersion $f : \mathbb{S}^1 \to \mathbb{R}^2$ that has only finitely many transversal double points whose associated chord diagram $\mathcal{C}(f)$ itself is equivalent to $\mathcal{C}_{2n+1}$.

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