

MOTIVES OF HYPERSURFACES OF VERY SMALL DEGREE

ANDRE CHATZISTAMATIOU

(Communicated by Ted Chinburg)

ABSTRACT. We study the Chow motive (with rational coefficients) of a hypersurface X in the projective space by using the variety $F(X)$ of l -dimensional planes contained in X . If the degree of X is sufficiently small, we show that the primitive part of the motive of X is the tensor product of a direct summand in the motive of a suitable complete intersection in $F(X)$ and the l -th twist $\mathbb{Q}(-l)$ of the Lefschetz motive.

INTRODUCTION

Let X be a smooth hypersurface of degree d in the projective space \mathbb{P}_k^n over a field k . In this paper we study the Chow motive (with rational coefficients) of X provided that d is sufficiently small.

Roitman has shown that the Chow group of zero-dimensional cycles of degree 0 is a torsion group if $d \leq n$ [4]. For higher dimensional cycles it is known [1, Theorem 4.6] that the Chow groups satisfy

$$\mathrm{CH}_{l'}(X) \otimes \mathbb{Q} = \mathrm{CH}_{l'}(\mathbb{P}^n) \otimes \mathbb{Q} = \mathbb{Q} \quad \text{for } 0 \leq l' \leq l-1$$

if $n \geq \binom{d+l-1}{l}$ and $d \geq 3$. The identity also holds if X is covered by l -dimensional planes [5, Theorem 9.28], or more generally if X is a hyperplane section of a hypersurface Y which is covered by l -dimensional planes [3].

Results on triviality of Chow groups give rise to a decomposition of the motive $h(X)$ associated with X . In our case, we get

$$h(X) \cong M_D \otimes \mathbb{Q}(-l) \oplus \bigoplus_{i=0}^{l-1} \mathbb{Q}(-i),$$

where $\mathbb{Q}(-1)$ is the Lefschetz motive and M_D is a direct summand of the motive $h(D)$ of some variety D . Our purpose is to describe M_D .

In order to state the theorem we need the following notation. An l -dimensional plane E in \mathbb{P}^n is called an osculating plane if the intersection $E \cap X$ is an $(l-1)$ -dimensional plane or if E is contained in X . We say that X has sufficiently many osculating planes if there exists an osculating plane through every closed point of X (the planes may be defined over a field extension of k).

Received by the editors January 18, 2008, and, in revised form, April 27, 2009.

2000 *Mathematics Subject Classification*. Primary 14-XX.

The author was supported by a fellowship within the Post-Doc program of the Deutsche Forschungsgemeinschaft (DFG).

Theorem (see Theorem 2.5). *Let n, d, l be numbers such that a general hypersurface of degree d in the projective space \mathbb{P}^n has sufficiently many osculating l -dimensional planes. Let $X \subset \mathbb{P}^n$ be a smooth hypersurface of degree d such that the Fano variety $F_l(X)$ of l -dimensional planes contained in X is smooth and has the expected dimension. Furthermore, let $HF_l(X) \subset F_l(X)$ be a smooth complete intersection of hyperplanes (in the Plücker embedding) with $\dim HF_l(X) = n - 2l - 1$. Then there is an isomorphism in the category of Chow motives with rational coefficients:*

$$h(X) \cong M_{HF_l(X)} \otimes \mathbb{Q}(-l) \oplus \bigoplus_{i=0}^{n-1} \mathbb{Q}(-i),$$

where $M_{HF_l(X)}$ is a direct summand in the motive of $HF_l(X)$.

The conditions on n, d, l hold if $n \geq \binom{l+d-1}{l} + l - 1$. For a finite field k there may be no smooth complete intersection $HF_l(X)$ of hyperplanes over k . In this case the variety $HF_l(X)$ exists over a suitable finite field extension of k .

Let us sketch the idea of the proof. We consider the family of planes

$$\Xi = \{(x, E) \in X \times HF_l(X) \mid x \in E\} \subset X \times HF_l(X)$$

over $HF_l(X)$. The cycle Ξ defines a correspondence $\phi_1 : HF_l(X) \otimes \mathbb{Q}(-l) \rightarrow X$, resp. $\phi_2 : X \rightarrow HF_l(X) \otimes \mathbb{Q}(-l)$. The composite $\phi_1 \circ \phi_2$ is the cycle

$$Z_X = \{(x, y) \in X \times X \mid x, y \in E, E \in HF_l(X)\}$$

in $\text{CH}^{n-1}(X \times X)$. The most important step is to show that

$$(0.0.1) \quad Z_X = \iota^*(a) + m \cdot \Delta_X,$$

for some $a \in \text{CH}^{n-1}(\mathbb{P}^n \times \mathbb{P}^n)$, some nonzero integer m , the inclusion $\iota : X^2 \hookrightarrow (\mathbb{P}^n)^2$ and the diagonal Δ_X . In order to prove (0.0.1) we introduce the doubled incidence variety of degree d hypersurfaces together with two points:

$$\Sigma = \{(x, y, Y) \in (\mathbb{P}^n)^2 \times \mathbb{P}(\text{Sym}^d(k^{n+1})) \mid x, y \in Y\}.$$

The projection $p : \Sigma \rightarrow (\mathbb{P}^n)^2$ is a projective bundle over $(\mathbb{P}^n)^2 - \Delta_{\mathbb{P}^n}$ and $\Delta_{\mathbb{P}^n}$, so that $\text{CH}^{n-1}(\Sigma)$ can be calculated by using the projective bundle formula and the localization sequence. The cycle $p^{-1}(\Delta_{\mathbb{P}^n}) \in \text{CH}^{n-1}(\Sigma)$ maps to the diagonal Δ_X by the pullback map j^* of the inclusion $j : X \times X \rightarrow \Sigma$. One defines a cycle $Z \in \text{CH}^{n-1}(\Sigma)$ with $j^*(Z) = Z_X$, and by comparing Z with $p^{-1}(\Delta_{\mathbb{P}^n})$ we obtain (0.0.1) after applying j^* .

1. CYCLES IN THE DOUBLED INCIDENCE VARIETY

1.1. Let k be a field and let $\mathbb{P}(\text{Sym}^d(k^{n+1}))$ denote the hypersurfaces of degree d in the projective space \mathbb{P}_k^n . We denote by Σ the doubled incidence variety

$$\Sigma = \{(x, y, X) \in (\mathbb{P}^n)^2 \times \mathbb{P}(\text{Sym}^d(k^{n+1})) \mid x, y \in X\}.$$

Letting $p : \Sigma \rightarrow (\mathbb{P}^n)^2$ be the projection, we define $\Sigma_1 := p^{-1}(\Delta_{\mathbb{P}^n})$ and $\Sigma_0 := \Sigma - \Sigma_1$. In the diagram

$$\begin{array}{ccccc} \Sigma_0 & \longrightarrow & \Sigma & \longleftarrow & \Sigma_1 \\ \downarrow & & \downarrow p & & \downarrow \\ (\mathbb{P}^n)^2 - \Delta_{\mathbb{P}^n} & \longrightarrow & (\mathbb{P}^n)^2 & \longleftarrow & \Delta_{\mathbb{P}^n} \end{array}$$

the varieties Σ_0 , resp. Σ_1 , are projective bundles with fiber dimension $N - 2$, resp. $N - 1$, where $N = \dim \mathbb{P}(\mathrm{Sym}^d(k^{n+1}))$. It is easy to see that $\mathrm{Sing}(\Sigma) = \{(x, x, X) \mid x \in \mathrm{Sing}(X)\}$. The singular locus is a projective bundle over $\Delta_{\mathbb{P}^n}$ with fiber dimension $N - 1 - n$.

1.2. There is an exact sequence

$$(1.2.1) \quad \mathrm{CH}^0(\Sigma_1) \rightarrow \mathrm{CH}^{n-1}(\Sigma) \rightarrow \mathrm{CH}^{n-1}(\Sigma_0) \rightarrow 0$$

from the localization sequence of Chow groups (CH^i denotes the group of i codimensional cycles modulo rational equivalence). Moreover, there is a natural splitting defined as follows.

Let $\phi : \Sigma \rightarrow \mathbb{P}(\mathrm{Sym}^d(k^{n+1}))$ be the projection and set $c = \phi^*(c_1(\mathcal{O}(1)))$. For the other projection $p : \Sigma \rightarrow (\mathbb{P}^n)^2$ we may define p^* to be the composite $\epsilon^* \circ \mathrm{pr}^*$, where $\epsilon : \Sigma \rightarrow (\mathbb{P}^n)^2 \times \mathbb{P}(\mathrm{Sym}^d(k^{n+1}))$ is the regular embedding and pr is the projection to $(\mathbb{P}^n)^2$. From the projective bundle formula we see that

$$\bigoplus_{i=0}^{n-1} c^{n-1-i} \cdot p^* \mathrm{CH}^i((\mathbb{P}^n)^2) \subset \mathrm{CH}^{n-1}(\Sigma)$$

splits the sequence (1.2.1), so that every class Z in $\mathrm{CH}^{n-1}(\Sigma)$ can be written as

$$Z = \sum_{i=0}^{n-1} c^{n-1-i} \cdot p^*(a_i) + m \cdot [\Sigma_1]$$

with $a_i \in \mathrm{CH}^i((\mathbb{P}^n)^2)$ and m is an integer.

1.3. Let \mathcal{H} be a smooth, connected projective k -scheme and $\Xi \subset \mathcal{H} \times \mathbb{P}^n$ be a family of κ -dimensional subschemes of \mathbb{P}^n , flat over \mathcal{H} . We assume that $\kappa \geq 1$ and that the sheaf $\mathrm{pr}_{1*}(\mathcal{O}_\Xi \otimes \mathrm{pr}_2^* \mathcal{O}_{\mathbb{P}^n}(d))$ on \mathcal{H} is locally free. Furthermore, we assume that the map

$$(1.3.1) \quad \mathrm{Sym}^d((k^{n+1})^\vee) \otimes \mathcal{O}_{\mathcal{H}} = \mathrm{pr}_{1*} \mathrm{pr}_2^* \mathcal{O}_{\mathbb{P}^n}(d) \rightarrow \mathrm{pr}_{1*}(\mathcal{O}_\Xi \otimes \mathrm{pr}_2^* \mathcal{O}_{\mathbb{P}^n}(d)),$$

induced by $\mathcal{O}_{\mathcal{H} \times \mathbb{P}^n} \rightarrow \mathcal{O}_\Xi$ and $H^0(\mathbb{P}^n, \mathcal{O}_{\mathbb{P}^n}(d)) = \mathrm{Sym}^d((k^{n+1})^\vee)$, is surjective.

We denote by E the kernel of (1.3.1). It is convenient to write QE for the sheaf $\mathrm{pr}_{1*}(\mathcal{O}_\Xi \otimes \mathrm{pr}_2^* \mathcal{O}_{\mathbb{P}^n}(d))$. In the following commutative diagram we fix the notation for the various maps:

$$\begin{array}{ccccc} \Xi \times_{\mathcal{H}} \Xi \times_{\mathcal{H}} \mathbb{P}_{\mathcal{H}}(E^\vee) & \xrightarrow{f_\Sigma} & \Sigma & \xrightarrow{p} & (\mathbb{P}^n)^2 \\ \downarrow f_{\mathbb{P}_{\mathcal{H}}(E^\vee)} & & \downarrow \phi & & \\ \mathbb{P}_{\mathcal{H}}(E^\vee) & \xrightarrow{\psi} & \mathbb{P}(\mathrm{Sym}^d(k^{n+1})) & & \\ \downarrow f_{\mathcal{H}} & & & & \\ \mathcal{H} & & & & \end{array}$$

Here $\psi : \mathbb{P}_{\mathcal{H}}(E^\vee) \rightarrow \mathbb{P}(\mathrm{Sym}^d(k^{n+1}))$ is the composite

$$\mathbb{P}_{\mathcal{H}}(E^\vee) \xrightarrow{\tilde{\psi}} \mathcal{H} \times \mathbb{P}(\mathrm{Sym}^d((k^{n+1})^\vee)) \xrightarrow{\mathrm{pr}_2} \mathbb{P}(\mathrm{Sym}^d((k^{n+1})^\vee)),$$

and $\tilde{\psi}$ is induced by the surjective map $\text{Sym}^d((k^{n+1})^\vee) \otimes \mathcal{O}_{\mathcal{H}} \rightarrow E^\vee$. By definition, $\mathbb{P}_{\mathcal{H}}(E^\vee)$ is the incidence variety $\{(L, X) \in \mathcal{H} \times \mathbb{P}(\text{Sym}^d((k^{n+1})^\vee)) \mid L \subset X\}$. The map f_Σ is defined as follows. By using $\Xi \subset \mathcal{H} \times \mathbb{P}^n$ we get

$$\begin{array}{ccc} \Xi \times_{\mathcal{H}} \Xi \times_{\mathcal{H}} \mathbb{P}_{\mathcal{H}}(E^\vee) & \longrightarrow & (\mathcal{H} \times \mathbb{P}^n) \times_{\mathcal{H}} (\mathcal{H} \times \mathbb{P}^n) \times_{\mathcal{H}} \mathbb{P}_{\mathcal{H}}(E^\vee) \\ & \searrow \tilde{f}_\Sigma & \downarrow (\text{pr}_2, \text{pr}_2, \psi) \\ & & \mathbb{P}^n \times \mathbb{P}^n \times \mathbb{P}(\text{Sym}^d(k^{n+1})), \end{array}$$

and the image of \tilde{f}_Σ is the incidence variety $\{(x, y, X) \mid \exists L \in \mathcal{H} \ x, y \in L \subset X\}$. In particular, \tilde{f}_Σ factors through Σ , and we denote this map by f_Σ .

Defining

$$(1.3.2) \quad Z := f_{\Sigma*} f_{\mathbb{P}_{\mathcal{H}}(E^\vee)}^* f_{\mathcal{H}}^* : \text{CH}^{*-e}(\mathcal{H}) \rightarrow \text{CH}^*(\Sigma),$$

we will be mainly interested in cycles $Z(a) \in \text{CH}^{n-1}(\Sigma)$ and their pullback to $X \times X \subset \Sigma$ for a hypersurface X .

The cycle $\Xi \in \text{CH}^{n-\kappa}(\mathcal{H} \times \mathbb{P}^n)$ has a unique representation,

$$(1.3.3) \quad [\Xi] = \sum_{i=0}^{n-\kappa} \xi_{n-\kappa-i} \otimes H^i,$$

with $\xi_j \in \text{CH}^j(\mathcal{H})$ and $H = c_1(\mathcal{O}_{\mathbb{P}^n}(1))$.

Lemma 1.4. *For $a \in \text{CH}^{n-1-e}(\mathcal{H})$, let $Z(a)|_{\Sigma_0} = \sum_{i=0}^{n-1} c^{n-1-i} \cdot p^*(a_i)$ be the pullback of $Z(a)$ to Σ_0 . The classes a_i can be computed as follows:*

$$\sum_i a_i = \frac{\sum_{0 \leq s, t \leq n-\kappa} b_{s,t} H^s \otimes H^t}{(1 + d \otimes H)(1 + H \otimes d)}$$

in $\text{CH}^*((\mathbb{P}^n)^2 - \Delta_{\mathbb{P}^n})$ and where

$$b_{s,t} = \int_{\mathcal{H}} \xi_{n-\kappa-s} \cdot \xi_{n-\kappa-t} \cdot c_{\text{rk}(QE)-n+s+t-1}(QE) \cdot a.$$

Proof. On $U := (\mathbb{P}^n)^2 - \Delta_{\mathbb{P}^n}$ the evaluation morphism

$$(1.4.1) \quad \text{Sym}^d((k^{n+1})^\vee) \otimes \mathcal{O}_U \rightarrow \text{pr}_1^* \mathcal{O}_{\mathbb{P}^n}(d) \oplus \text{pr}_2^* \mathcal{O}_{\mathbb{P}^n}(d)$$

is surjective; let G be the kernel. We have $\Sigma_0 = \mathbb{P}_U(G^\vee)$ and $c_{1\Sigma_0} = c_1(\mathcal{O}_{\mathbb{P}_U(G^\vee)}(1))$, so that

$$(1.4.2) \quad p_* \left(\frac{1}{1-c} Z(a)|_{\Sigma_0} \right) = \sum_{j \geq 0} s_j(G) \cdot \sum_{i=0}^{n-1} a_i,$$

where $s_j(G)$ are the Segre classes. Since G is the kernel of (1.4.1), we see that

$$(1.4.3) \quad (1 + d \cdot \text{pr}_1^* c_1(\mathcal{O}_{\mathbb{P}^n}(1)))(1 + d \cdot \text{pr}_2^* c_1(\mathcal{O}_{\mathbb{P}^n}(1))) \sum_{j \geq 0} s_j(G) = 1.$$

Define

$$(1.4.4) \quad T_i(a) = p_*(c^{N-n-1+i} Z(a))$$

and let $j : U \subset (\mathbb{P}^n)^2$ be the open immersion. It follows from (1.4.2) and (1.4.3) that

$$(1.4.5) \quad \sum_{i=0}^{n-1} a_i = \frac{\sum_{i \geq 0} j^* T_i(a)}{(1 + d \otimes H)(1 + H \otimes d)}.$$

Let us now compute the T_i :

$$\begin{aligned} p_*(c^{N-n-1+i} Z(a)) &= p_* f_{\Sigma*}((\phi \circ f_{\Sigma})^* c_1(\mathcal{O}(1))^{N-n-1+i} \cdot (f_{\mathcal{H}} \circ f_{\mathbb{P}_{\mathcal{H}}(E^{\vee})})^*(a)) \\ &= p_* f_{\Sigma*} f_{\mathbb{P}_{\mathcal{H}}(E^{\vee})}^*(\psi^* c_1(\mathcal{O}(1))^{N-n-1+i} \cdot f_{\mathcal{H}}^*(a)). \end{aligned}$$

The map $T = p_* f_{\Sigma*} f_{\mathbb{P}_{\mathcal{H}}(E^{\vee})}^*$ is given by the correspondence $\mathbb{P}_{\mathcal{H}}(E^{\vee}) \times_{\mathcal{H}} \Xi \times_{\mathcal{H}} \Xi$ in $\mathbb{P}_{\mathcal{H}}(E^{\vee}) \times (\mathbb{P}^n)^2$, and using (1.3.3), we see that

$$[\Xi \times_{\mathcal{H}} \Xi] = \sum_{0 \leq s, t \leq n-\kappa} (\xi_{n-\kappa-s} \cdot \xi_{n-\kappa-t}) \otimes H^s \otimes H^t$$

in $\text{CH}^*(\mathcal{H} \times (\mathbb{P}^n)^2)$, so that the coefficient of $H^s \otimes H^t$ in $T_i(a)$ is

$$(1.4.6) \quad \begin{aligned} &\int_{\mathbb{P}_{\mathcal{H}}(E^{\vee})} f_{\mathcal{H}}^* \xi_{n-\kappa-s} \cdot f_{\mathcal{H}}^* \xi_{n-\kappa-t} \cdot \psi^* c_1(\mathcal{O}(1))^{N-n-1+i} \cdot f_{\mathcal{H}}^*(a) \\ &= \int_{\mathcal{H}} \xi_{n-\kappa-s} \cdot \xi_{n-\kappa-t} \cdot s_{N-n-\text{rk}(E)+i}(E) \cdot a. \end{aligned}$$

If $Z(a) \in \text{CH}^{n-1}(\Sigma)$, then $T_i(a) \in \text{CH}^i((\mathbb{P}^n)^2)$ by definition so that the coefficient of $H^s \otimes H^t$ in $T_i(a)$ vanishes if $s + t \neq i$. The identity $s_{N-n-\text{rk}(E)+i}(E) = c_{\text{rk}(QE)-n-1+i}(QE)$ completes the proof. \square

The next lemma computes the pullback of $Z(a)$ to $X \times X$ for a smooth hypersurface X . We write ι_{Σ} for the inclusion $X \times X \rightarrow \Sigma$ and $\iota_{(\mathbb{P}^n)^2}$ for the inclusion $X \times X \rightarrow (\mathbb{P}^n)^2$. Note that both inclusions are locally complete intersections; thus the pullback is well-defined. Indeed, since X is smooth the image $\iota_{\Sigma}(X \times X)$ is contained in the open smooth part of Σ (see (1.1)).

Lemma 1.5. *For $Z(a) \in \text{CH}^{n-1}(\Sigma)$ and $Z(a)|_{\Sigma_0} = \sum_{i=0}^{n-1} c^{n-1-i} \cdot p^*(a_i)$, we have*

$$\iota_{\Sigma}^* Z(a) = \iota_{(\mathbb{P}^n)^2}^*(a_{n-1}) - m \cdot \Delta_X,$$

where

$$m = d \cdot \sum_{j=\kappa-1}^{n-1} (-d)^j \int_{\mathcal{H}} \xi_{n-\kappa} \cdot \xi_{j-\kappa+1} \cdot c_{\text{rk}(QE)-2-j}(QE) \cdot a.$$

Proof. We know that

$$(1.5.1) \quad Z(a) = \sum_{i=0}^{n-1} c^{n-1-i} \cdot p^*(a_i) - m \cdot [\Sigma_1]$$

for some m . The line bundle $\iota_{\Sigma}^* \phi^* \mathcal{O}(1)$ is trivial and $\iota_{\Sigma}^* [\Sigma_1] = \Delta_X$; therefore

$$(1.5.2) \quad \iota_{\Sigma}^* Z(a) = \iota_{(\mathbb{P}^n)^2}^* a_{n-1} - m \cdot \Delta_X.$$

We claim that

$$(1.5.3) \quad \iota_{(\mathbb{P}^n)^2*} \iota_{\Sigma}^* \beta = p_*(c^N \cdot \beta)$$

for every class $\beta \in \text{CH}^*(\Sigma)$. This follows from the following fact. If $g : D \subset Y$ is a Cartier divisor on Y and L is the associated line bundle, then $g_* g^*(\beta) = c_1(L) \cdot \beta$.

Claim (1.5.3) implies $\iota_{(\mathbb{P}^n)^{2*}} \iota_{\Sigma}^* Z(a) = T_{n+1}(a)$ (see (1.4.4)). The coefficient of $H \otimes H^n$ in $T_{n+1}(a)$ is computed in (1.4.6) and vanishes for trivial reasons (if $\kappa \geq 1$). Using Lemma 1.4, we see that the coefficient γ of $1 \otimes H^{n-1}$ in a_{n-1} is

$$\gamma = \sum_{j=\kappa-1}^{n-1} (-d)^j \cdot b_{0,n-1-j}.$$

By applying $\iota_{(\mathbb{P}^n)^{2*}}$ to (1.5.2) and using $\iota_{(\mathbb{P}^n)^{2*}} \iota_{(\mathbb{P}^n)^2}^* a_{n-1} = d^2(H \otimes H) \cdot a_{n-1}$ and $\iota_{(\mathbb{P}^n)^{2*}} \Delta_X = d \cdot \sum_{i \geq 0}^{n-1} H^{i+1} \otimes H^{n-i}$, we find that $m = d \cdot \gamma$, as claimed. \square

2. MOTIVES OF HYPERSURFACES AND THEIR FANO VARIETIES

2.1. In the following we work with the Grassmannian of κ -planes $\mathcal{H} = \text{Gr}_{\kappa}$ in projective space where $\Xi \subset \text{Gr}_{\kappa} \times \mathbb{P}^n$ is the universal family. We denote by V , resp. QV , the tautological bundle $V \subset \mathcal{O}_{\text{Gr}}^{n+1}$, resp. the quotient $\mathcal{O}_{\text{Gr}}^{n+1}/V$. The family Ξ is the projective bundle $\Xi = \mathbb{P}_{\text{Gr}_{\kappa}}(V^{\vee})$, and it is easy to see that

$$[\Xi] = \sum_{i=0}^{n-\kappa} c_{n-\kappa-i}(QV) \otimes H^i$$

in $\text{CH}^{n-\kappa}(\text{Gr} \times \mathbb{P}^n)$. Furthermore we have that $QE = \text{pr}_{1*}(\mathcal{O}_{\Xi} \otimes \text{pr}_2^* \mathcal{O}_{\mathbb{P}^n}(d)) = \text{Sym}^d(V^{\vee})$.

2.2. We will be interested in cycles $Z(c_1(V^{\vee})^s) \in \text{CH}^{n-1}(\Sigma)$, for $s \geq 0$ (notation as in (1.3.2)). By counting dimensions we see that

$$(2.2.1) \quad s = \dim \Xi - \text{rk}(QE) - (n-1) = \kappa(n-\kappa) - \binom{d+\kappa}{\kappa} + \kappa + 1.$$

Let us consider the variety

$$(2.2.2) \quad \{(x, E_{\kappa-1}, E_{\kappa}, X) \in \mathbb{P}^n \times \text{Gr}_{\kappa-1} \times \text{Gr}_{\kappa} \times \mathbb{P}(\text{Sym}^d(k^{n+1})) \mid x \in E_{\kappa-1} \subset E_{\kappa}, E_{\kappa} \cap X = E_{\kappa-1} \text{ or } E_{\kappa} \subset X\}.$$

More formally, this variety is defined as follows. On $\Xi = \mathbb{P}_{\text{Gr}_{\kappa}}(V^{\vee})$ there is an exact sequence of vector bundles

$$(2.2.3) \quad 0 \rightarrow V_1^{\vee} \rightarrow V^{\vee} \otimes \mathcal{O}_{\Xi} \rightarrow \mathcal{O}_{\Xi}(1) \rightarrow 0,$$

and the points of $\mathbb{P}_{\Xi}(V_1)$ are $\{(x, E_{\kappa-1}, E_{\kappa}) \mid x \in E_{\kappa-1} \subset E_{\kappa}\}$.

Since $\mathcal{O}_{\mathbb{P}_{\Xi}(V_1)}(-1) \subset V_1^{\vee} \otimes \mathcal{O}_{\mathbb{P}_{\Xi}(V_1)} \subset V^{\vee} \otimes \mathcal{O}_{\mathbb{P}_{\Xi}(V_1)}$, we can define G to be the kernel of

$$\text{Sym}^d((k^{n+1})^{\vee}) \otimes \mathcal{O}_{\mathbb{P}_{\Xi}(V_1)} \rightarrow \text{Sym}^d(V^{\vee})/\mathcal{O}_{\mathbb{P}_{\Xi}(V_1)}(-d).$$

Then $\mathbb{P}_{\mathbb{P}_{\Xi}(V_1)}(G^{\vee})$ is the variety (2.2.2).

The following condition will imply that the diagonal Δ_X , for a hypersurface X , can be written in terms of the pullback of $Z(c_1(V^{\vee})^s)$ to $X \times X$ (i.e. $m \neq 0$ in Lemma 1.5).

(B) The following map is surjective:

$$\begin{aligned} \mathbb{P}_{\mathbb{P}_{\Xi}(V_1)}(G^{\vee}) &\rightarrow \{(x, X) \in \mathbb{P}^n \times \mathbb{P}(\text{Sym}^d(k^{n+1})) \mid x \in X\}, \\ (x, E_{\kappa-1}, E_{\kappa}, X) &\mapsto (x, X). \end{aligned}$$

By counting dimensions we see that a necessary condition for (B) is $s \geq 0$ (with s as in (2.2.1)). If $d = 2$, then $s \geq 0$ is not sufficient; the first example is $\kappa = 3$

and $n = 5$. In fact, (B) is equivalent to $n \geq 2 \cdot \kappa$ if $d = 2$, which can be checked by using the following lemma. However, we don't know any counterexamples to

$$(2.2.4) \quad (B) \Leftrightarrow s \geq 0$$

for $d > 2$. If $\kappa = 1$, then (2.2.4) is true. This also holds for

$$(n, d, \kappa) = (6, 3, 2), (8, 4, 2), (11, 5, 2), (9, 3, 3).$$

It is known [1, Lemma 1.1+Lemma 4.2] that (B) is true if $d \geq 3$ and

$$n - \kappa + 1 \geq \binom{\kappa - 1 + d}{\kappa}.$$

Lemma 2.3. *Condition (B) holds if and only if*

$$m = d \cdot \sum_{j=\kappa-1}^{n-1} (-d)^j \int_{\mathcal{H}} \xi_{n-\kappa} \cdot \xi_{j-\kappa+1} \cdot c_{\mathrm{rk}(QE)-2-j}(QE) \cdot c_1(V^\vee)^s$$

is nonzero.

Proof. From the construction of $\mathbb{P}_{\mathbb{P}_{\Xi}(V_1)}(G^\vee)$ we have the maps

$$\mathbb{P}_{\mathbb{P}_{\Xi}(V_1)}(G^\vee) \xrightarrow{f} \mathbb{P}_{\Xi}(V_1) \xrightarrow{g} \Xi \xrightarrow{h} \mathrm{Gr}_\kappa.$$

We claim that

$$(2.3.1) \quad \begin{aligned} & (h \circ g \circ f)_*(c_1(\mathcal{O}_{\Xi}(1))^{n-1} \cdot c_1(\mathcal{O}_{\mathbb{P}_{\mathbb{P}_{\Xi}(V_1)}(G^\vee)}(1))^N) \\ &= (-1)^{\kappa-1} d \cdot \sum_{j=\kappa-1}^{n-1} (-d)^j \xi_{n-\kappa} \cdot \xi_{j-\kappa+1} \cdot c_{\mathrm{rk}(QE)-2-j}(QE) \end{aligned}$$

($N = \dim \mathbb{P}(\mathrm{Sym}^d(k^{n+1}))$). Indeed,

$$\begin{aligned} g_* f_* (c_1(\mathcal{O}_{\mathbb{P}_{\mathbb{P}_{\Xi}(V_1)}(G^\vee)}(1))^N &= g_* s_{\mathrm{rk}(QE)-1}(G) = g_* c_{\mathrm{rk}(QE)-1}(QE/\mathcal{O}_{\mathbb{P}_{\Xi}(V_1)}(-d)) = \\ g_* \sum_i d^i c_{\mathrm{rk}(QE)-i-1}(QE) \cdot c_1(\mathcal{O}_{\mathbb{P}_{\Xi}(V_1)}(1))^i &= \sum_i d^i c_{\mathrm{rk}(QE)-i-1}(QE) \cdot s_{i-\kappa+1}(V_1^\vee), \end{aligned}$$

and from (2.2.3) we obtain $s_j(V_1^\vee) = s_j(V^\vee) + c_1(\mathcal{O}_{\Xi}(1)) \cdot s_{j-1}(V^\vee)$ for all j . Thus,

$$(2.3.2) \quad \begin{aligned} & (h \circ g \circ f)_*(c_1(\mathcal{O}_{\Xi}(1))^{n-1} \cdot c_1(\mathcal{O}_{\mathbb{P}_{\mathbb{P}_{\Xi}(V_1)}(G^\vee)}(1))^N) \\ &= \sum_i d^i c_{\mathrm{rk}(QE)-i-1}(QE) \cdot (s_{n-\kappa-1}(V) s_{i-\kappa+1}(V^\vee) + s_{n-\kappa}(V) s_{i-\kappa}(V^\vee)). \end{aligned}$$

On Ξ the natural morphism $QE \otimes \mathcal{O}_{\Xi}(-d) \rightarrow \mathcal{O}_{\Xi}$ is surjective, so that the top Chern class of $QE \otimes \mathcal{O}_{\Xi}(-d)$ vanishes:

$$\begin{aligned} h_* c_{\mathrm{rk}(QE)}(QE \otimes \mathcal{O}_{\Xi}(-d)) &= h_* \sum_i (-d)^i c_{\mathrm{rk}(QE)-i}(QE) \cdot c_1(\mathcal{O}_{\Xi}(1))^i \\ &= \sum_i (-d)^i c_{\mathrm{rk}(QE)-i}(QE) s_{i-\kappa}(V) = (-1)^\kappa d \cdot \sum_i d^i c_{\mathrm{rk}(QE)-i-1}(QE) s_{i+1-\kappa}(V^\vee) \end{aligned}$$

is zero and together with (2.3.2) we obtain

$$\begin{aligned} & (h \circ g \circ f)_*(c_1(\mathcal{O}_{\Xi}(1))^{n-1} \cdot c_1(\mathcal{O}_{\mathbb{P}_{\mathbb{P}_{\Xi}(V_1)}(G^\vee)}(1))^N) \\ &= d \cdot \sum_i d^i c_{\mathrm{rk}(QE)-i-2}(QE) \cdot s_{n-\kappa}(V) s_{i+1-\kappa}(V^\vee). \end{aligned}$$

Then, $(-1)^j s_j(V^\vee) = s_j(V) = c_j(QV) = \xi_j$ proves the claim.

Let π be the map in condition (B). For a general closed point (x, X) the irreducible components of $\pi^{-1}(x, X)$ map generically one to one to Gr_κ . The class $c_1(V^\vee) = c_1(\Lambda^{\kappa+1}V^\vee)$ is the class of an ample line bundle, and s is the dimension of the generic fiber of π if π is surjective. Thus, π is surjective if and only if $\int_{\mathbb{P}_{\mathbb{P}^\Xi(V_1)}(G^\vee)} c_1(V^\vee)^s \cdot [\pi^{-1}(x, X)] > 0$.

Since $\mathcal{O}_\Xi(1) = \pi^* \mathcal{O}_{\mathbb{P}^n}(1)$ and $\mathcal{O}_{\mathbb{P}_{\mathbb{P}^\Xi(V_1)}(G^\vee)}(1) = \pi^* \mathcal{O}_{\mathbb{P}(\text{Sym}^d(k^{n+1}))}(1)$ we get

$$\begin{aligned} \pi^*(z) &= c_1(\mathcal{O}_\Xi(1))^{n-1} \cdot c_1(\mathcal{O}_{\mathbb{P}_{\mathbb{P}^\Xi(V_1)}(G^\vee)}(1))^N, \\ z &= c_1(\mathcal{O}_{\mathbb{P}^n}(1))^{n-1} \cdot c_1(\mathcal{O}_{\mathbb{P}(\text{Sym}^d(k^{n+1}))}(1))^N. \end{aligned}$$

The zero-dimensional cycle z is the class of a complete intersection of $n - 1$ hyperplanes in \mathbb{P}^n and N hyperplanes in $\mathbb{P}(\text{Sym}^d(k^{n+1}))$. Therefore $z = \sum_i (x_i, X_i)$ for some sufficiently general points (x_i, X_i) , and $c_1(\mathcal{O}_\Xi(1))^{n-1} \cdot c_1(\mathcal{O}_{\mathbb{P}_{\mathbb{P}^\Xi(V_1)}(G^\vee)}(1))^N = \sum_i \pi^*(x_i, X_i)$. Now (2.3.1) implies the lemma. \square

In the following theorem we work with the category of (pure) Chow motives with rational coefficients (see [2, Chapter 16]), where $\mathbb{Q}(-1)$ denotes the Lefschetz motive.

Theorem 2.5. *Let n, d, κ be numbers satisfying (B). Let $X \subset \mathbb{P}^n$ be a smooth hypersurface of degree d such that the Fano variety $F_\kappa(X)$ of κ -dimensional planes contained in X has the expected dimension (i.e. $\dim F_\kappa(X) = (\kappa+1)(n-\kappa) - \binom{d+\kappa}{\kappa}$), and let $HF_\kappa(X) \subset F_\kappa(X)$ be a complete intersection of hyperplanes (in the Plücker embedding) with $\dim HF_\kappa(X) = n - 2\kappa - 1$. Furthermore, let $\psi : \widetilde{HF_\kappa(X)} \rightarrow HF_\kappa(X)$ be a generically finite surjective morphism which resolves the singularities of $HF_\kappa(X)$. Then there is an isomorphism in the category of Chow motives with rational coefficients*

$$(X, id_X) \cong (\widetilde{HF_\kappa(X)}, P) \otimes \mathbb{Q}(-\kappa) \oplus \bigoplus_{i=0}^{n-1} \mathbb{Q}(-i)$$

for a suitable projector P . We give an explicit formula for P in the proof.

Proof. From Lemma 1.5 and Lemma 2.3 we obtain

$$(2.5.1) \quad \Delta_X = -\frac{1}{m} i_{\Sigma}^* Z(c_1(V^\vee)^s) + \frac{1}{m} i_{(\mathbb{P}^n)^2}^*(a_{n-1}).$$

In the Cartesian diagram

$$\begin{array}{ccc} \Xi \times_{F_\kappa(X)} \Xi & \xrightarrow{f_{X \times X}} & X \times X \\ \downarrow & & \downarrow i_\Sigma \\ \Xi \times_{\text{Gr}_\kappa} \Xi \times_{\text{Gr}_\kappa} \mathbb{P}_{\text{Gr}_\kappa}(E^\vee) & \xrightarrow{f_\Sigma} & \Sigma \end{array}$$

the pullback and the pushforward commute [2, Theorem 6.2, Remark 6.2.1], because $F_\kappa(X)$ has the expected dimension. Therefore

$$(2.5.2) \quad i_{\Sigma}^* Z(c_1(V^\vee)^s) = f_{X \times X}^*(c_1(V^\vee)^s) = (f_{X \times X} \circ \eta)_*(\Xi \times_{HF_\kappa(X)} \Xi),$$

where $\eta : HF_\kappa(X) \rightarrow F_\kappa(X)$ is a complete intersection of s hyperplanes (thus the dimension is $\dim HF_\kappa(X) = n - 2\kappa - 1$). It is more convenient to write $H = HF_\kappa(X)$ and $\tilde{H} = \widetilde{HF_\kappa(X)}$.

There is a cycle $Y \in \text{CH}_{n-2\kappa-1}(\tilde{H}) \otimes \mathbb{Q}$ (i.e. Y is a rational linear combination of connected components of \tilde{H}) such that $\psi_*(Y) = [H]$.

Let $\phi_1 \in \text{Cor}(\tilde{H} \otimes \mathbb{Q}(-\kappa), X)$ (resp. $\phi_2 \in \text{Cor}(X, \tilde{H} \otimes \mathbb{Q}(-\kappa))$) be the correspondence defined by the cycle $\tilde{H} \times_H \Xi$ in $\tilde{H} \times X$ (resp. $[\Xi \times_H \tilde{H}] \cdot pr_{\tilde{H}}^*(Y)$ in $X \times \tilde{H}$, where $pr_{\tilde{H}} : X \times \tilde{H} \rightarrow \tilde{H}$ is the projection).

We consider the commutative diagram

$$\begin{array}{ccccc}
 & & \Xi \times_{\tilde{H}} \Xi & \xrightarrow{\psi'} & \Xi \times_H \Xi \\
 & \swarrow & \downarrow \cap & & \downarrow \cap \\
 \tilde{H} & & X \times \tilde{H} \times X & \xrightarrow{\text{id} \times \psi \times \text{id}} & X \times H \times X \\
 & \swarrow pr_{\tilde{H}} & & & \swarrow \\
 & & & & X \times X \\
 & & & & \nwarrow f_{X \times X} \circ \eta
 \end{array}$$

It is easy to see that $\phi_1 \circ \phi_2 = (f_{X \times X} \circ \eta \circ \psi')_*([\Xi \times_{\tilde{H}} \Xi] \cdot pr_{\tilde{H}}^*(Y))$, and $\psi'_*([\Xi \times_{\tilde{H}} \Xi] \cdot pr_{\tilde{H}}^*(Y)) = [\Xi \times_H \Xi]$ together with (2.5.2) yields

$$(2.5.3) \quad \phi_1 \circ \phi_2 = Z(c_1(V^\vee)^s).$$

If H is the class of a hyperplane in \mathbb{P}^n , then we write P_i for the pullback of $\frac{1}{d}H^{n-1-i} \otimes H^i \in \text{CH}^{n-1}(\mathbb{P}^n \times \mathbb{P}^n) \otimes \mathbb{Q}$ to $X \times X$. The correspondences P_0, \dots, P_{n-1} are idempotent and orthogonal. We may write $a_{n-1} = \frac{m}{d} \sum_{i=0}^{n-1} \beta_i H^{n-1-i} \otimes H^i$, and it follows from (2.5.1) and (2.5.3) that

$$\Delta_X - \sum_i P_i = -\frac{1}{m} \phi_1 \circ \phi_2 + \sum_{i=0}^{n-1} (\beta_i - 1) P_i.$$

Composition with P_i shows that $\frac{1}{m} \phi_1 \circ \phi_2 \circ P_i = (\beta_i - 1) P_i$ and

$$(2.5.4) \quad \Delta_X - \sum_i P_i = \phi_1 \circ \left(-\frac{1}{m} \phi_2 + \frac{1}{m} \sum_{i=0}^{n-1} \phi_2 \circ P_i \right).$$

Since $(X, P_i) \cong \mathbb{Q}(-i)$ and $(X, \Delta_X - \sum_i P_i) \cong (\tilde{H}, -\frac{1}{m} \phi_2 \circ (\Delta_X - \sum_i P_i) \circ \phi_1)$ by (2.5.4), this proves the theorem. \square

ACKNOWLEDGMENTS

I thank H. Esnault for introducing me to this subject and for her interest in my work. I am grateful to Y. André for valuable discussions. I thank the École Normale Supérieure for its hospitality.

REFERENCES

- [1] Esnault, Hélène; Levine, Marc; Viehweg, Eckart, *Chow groups of projective varieties of very small degree*, Duke Math. J. **87** (1997), 29–58. MR1440062 (98d:14002)
- [2] Fulton, William, *Intersection theory*. Second edition. Ergebnisse der Mathematik und ihrer Grenzgebiete. 3. Folge, Springer-Verlag, Berlin, 1998. MR1644323 (99d:14003)
- [3] Otwinowska, Anna, *Remarques sur les groupes de Chow des hypersurfaces de petit degré* [Remarks on Chow groups of hypersurfaces of low degree], C. R. Acad. Sci. Paris Sér. I Math. **329** (1999), 51–56. MR1703267 (2000g:14007)

- [4] Roĭtman, A. A., *Rational equivalence of zero-dimensional cycles*. (Russian), Mat. Sb. (N.S.) **89** (131) (1972), 569–585. (Translation in Math. USSR-Sb. **18** (1974), 571–588.) MR0327767 (48:6109)
- [5] Voisin, Claire, *Hodge theory and complex algebraic geometry. II*. Translated from the French by Leila Schneps. Cambridge Studies in Advanced Mathematics, 77, Cambridge University Press, Cambridge, 2003. MR1997577 (2005c:32024b)

FACHBEREICH MATHEMATIK, UNIVERSITÄT DUISBURG-ESSEN, 45117 ESSEN, GERMANY
E-mail address: a.chatzistamatiou@uni-due.de