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BOCKSTEIN THEOREM FOR NILPOTENT GROUPS

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Abstract. We extend the definition of Bockstein basis $\sigma(G)$ to nilpotent groups G. A metrizable space X is called a *Bockstein space* if $\dim_G(X) = \sup\{\dim_H(X)|H \in \sigma(G)\}$ for all Abelian groups G. The Bockstein First Theorem says that all compact spaces are Bockstein spaces.

Here are the main results of the paper:

Theorem 0.1. *Let X be a Bockstein space. If G is nilpotent, then* $\dim_G(X) \leq 1$ *if and only if* $\sup\{\dim_H(X)|H \in \sigma(G)\} \leq 1$.

Theorem 0.2. *X is a Bockstein space if and only if* $\dim_{\mathbf{Z}_{(l)}}(X) = \dim_{\hat{Z}_{(l)}}(X)$ *for all subsets l of prime numbers.*

1. Introduction

We use the Kuratowski notation $X\tau M$ in the case when every map from a closed subset of X to M can be extended over all X.

Recall that the cohomological dimension of a space X with respect to an Abelian group G is less than or equal to n, denoted by $\dim_G X \leq n$, if $\check{H}^{n+1}(X, A; G) = 0$ for all closed $A \subset X$.

Of basic importance in cohomological dimension theory is the *Bockstein basis* $\sigma(G)$ of an Abelian group G (see Definition 3.1) and the following result of Bockstein (see [12] or [4]):

Theorem 1.1 (Bockstein First Theorem)**.** *If X is a compact space, then*

$$\dim_G(X) = \sup\{\dim_H(X)|H \in \sigma(G)\}.$$

The aim of this paper is to generalize Theorem 1.1 to nilpotent groups. There are two issues to resolve first:

(1) Define cohomological dimension with respect to non-Abelian groups.
(2) Define the Bockstein basis of nilpotent groups.

The definition of $\dim_G(X)$ for G non-Abelian was first introduced by A. Dranishnikov and D. Repovš [5] as follows: By [4, Theorem 1.1], $\dim_G(X) \leq n$ is equivalent (for Abelian groups G) to $X\tau K(G, n)$, where $K(G, n)$ is an Eilenberg-MacLane

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space. One ought to use the same equivalence in the definition of $\dim_G(X)$ for non-Abelian groups. For nilpotent groups G this definition was used for the characterization of nilpotent CW complexes as absolute extensors of metric compacta (see [3]). There is no Eilenberg-MacLane space $K(G,n)$, $1 < n < \infty$, for non-Abelian groups G, so $\dim_G X \in \{0,1,\infty\}$. Since $\dim_G X = 0$ means $\dim(X) = 0$ (see Lemma 1.2), the only interesting question is if $X\tau K(G,1)$ holds.

Our definition of the Bockstein basis for nilpotent groups can be found in Definition 3.2.

The remainder of this section is devoted to elementary properties of cohomological dimension over non-Abelian groups.

Lemma 1.2. *If X is a metrizable space and $\dim_G(X) = 0$ for some group $G \neq 1$, then $\dim_H(X) = 0$ for any group H.*

Proof. We will show that for a nontrivial group G we have $\dim_G(X) = 0$ if and only if $\dim_{\mathbf{Z}_2}(X) = 0$ and $\dim_{\mathbf{Z}_2}(X) = 0$ means $\dim(X) = 0$..

Suppose $\dim_G(X) = 0$ for some $G \neq 1$. Let $i\colon \mathbf{Z}_2 \to G$ be an injection and let $r\colon G \to \mathbf{Z}_2$ be a map such that $r \circ i = id_{\mathbf{Z}_2}$. Let A be a closed subset of X and $f\colon A \to K(\mathbf{Z}_2, 0) = \mathbf{Z}_2$ a map. Then there exists an extension $f'\colon X \to G$ of $i \circ f\colon A \to G$. Then $F = r \circ f'\colon X \to \mathbf{Z}_2$ is an extension of f.

Suppose that $\dim_{\mathbf{Z}_2}(X) = 0$. Let $A \subset X$ be a closed subset and let $f\colon A \to G$ be a continuous map. For every $g \in G$ we define $A_g = f^{-1}(g)$ and $B_g = \{x \in X \mid d(x, A \setminus A_g) \leq d(x, A_g)\}$. Because $A \setminus A_g$ is a closed subset of X, B_g is a closed subset of X. Because $B_g \cap A_g = \emptyset$ we can define a continuous map $f_g\colon A_g \cup B_g \to \mathbf{Z}_2$ by $f_g(A_g) = 1$ and $f_g(B_g) = 0$. Let $F_g\colon X \to \mathbf{Z}_2$ be an extension of f_g. For every $g \in G$ we define $X_g = F_g^{-1}(1)$. Sets X_g are open and closed in X and pairwise disjoint. So we can define a continuous map $F\colon X \to G$ as

$$F(x) = \begin{cases} g & ; x \in X_g, \\ e & ; x \notin \bigcup_{g \in G} X_g. \end{cases}$$

The map F is an extension of f. □

Lemma 1.3. *Let X be a metrizable space. If $1 \to J \to G \to I \to 1$ is an exact sequence of groups and $\dim_J X \leq 1$, then $\dim_I(X) \leq 1$ if and only if $\dim_G(X) \leq 1$.*

Proof. In view of Lemma 1.2 the only interesting case is that of $\dim_J X = 1$. Use the fibration $K(J,1) \to K(G,1) \to K(I,1)$ and the fact that $X\tau K(J,1)$ to conclude $X\tau K(I,1)$ if and only if $X\tau K(G,1)$. □

If G is a group, then $\mathrm{Ab}(G)$ is its *abelianization*.

Lemma 1.4. *If X is a metrizable space, then $\dim_{Ab(G)}(X) \leq \dim_G X$ for any group G.*

Proof. In view of Lemma 1.2 the only interesting case is that of G non-Abelian and $\dim_G X = 1$. Since $X\tau K(G,1)$ one gets $X\tau K(H_1(K(G,1)),1)$ by Theorem 3.4 of [8]. As $H_1(K(G,1)) = \mathrm{Ab}(G)$, we are done. □

2. Nilpotent groups

If $A, B \subset G$ are subgroups, then the *commutator subgroup* $[A,B]$ is a group generated by all commutators $[a,b]$, $a \in A$ and $b \in B$. The *lower central series* $\{\Gamma_n(G)\}$ for a group G is defined as follows: $\Gamma_1(G) = G$, and $\Gamma_{n+1}(G) = [\Gamma_n(G), G]$.

If a group G is nilpotent, then there exists an integer c such that $\Gamma_c(G) \neq \{1\}$ but $\Gamma_{c+1}(G) = \{1\}$. The number c (denoted by $h(G)$) is called the *nilpotency class* of the nilpotent group G or its *Hirsch length*. Abelian groups are nilpotent of Hirsch length 1. By [14, Theorem 3.1], for every n there exists an epimorphism

$$\otimes^n \mathrm{Ab}(G) \to \Gamma_n(G)/\Gamma_{n+1}(G).$$

In particular, there is an epimorphism $\otimes^c \mathrm{Ab}(G) \to \Gamma_c(G)$. It follows from the definition that $\Gamma_c(G)$ is in the center of G. Therefore $1 \to \Gamma_c(G) \to G \to G/\Gamma_c(G) \to 1$ is a central extension. A short calculation shows that the epimorphism $G \to G/\Gamma_c(G)$ induces the trivial homomorphism $\Gamma_c(G) \to \Gamma_c(G/\Gamma_c(G)) = 1$. Therefore $G/\Gamma_c(G)$ is nilpotent of nilpotency class strictly less than c. This motivates the following definition.

Definition 2.1. A central extension $K \to G \to I$ of groups where G is nilpotent (or equivalently I is nilpotent), for which there exists an epimorphism $\otimes^n \mathrm{Ab}(G) \to K$ for some n, is called a *nilpotent central extension*.

Thus, for every (non-Abelian) nilpotent group G, there exists a nilpotent central extension $K \to G \to I$ such that the Hirsch length of I is less than the Hirsch length of G.

Lemma 2.2. *Let $1 \to K \to G \xrightarrow{\pi} I \to 1$ be a central extension of nilpotent groups.*

(a) *If K and I are p-divisible, then G is p-divisible.*
(b) *If the extension is a nilpotent central extension and G is p-divisible, then K and I are p-divisible.*

Proof. Suppose K and I are p-divisible. Let $g \in G$. Then $\pi(g) = i^p$ for some $i \in I$. Let $\bar{g} \in G$ be such that $\pi(\bar{g}) = i$. Then $\pi(\bar{g}^p g^{-1}) = 1$, so $\bar{g}^p g^{-1} = k^p$ for some $k \in K$. Because $k \in K \subset C(G)$, $g = (\bar{g}k^{-1})^p$, so G is p-divisible.

If G is p-divisible, then any epimorphic image of G is p-divisible. Thus both I and $\mathrm{Ab}(G)$ are p-divisible. As there is an epimorphism $\otimes^n \mathrm{Ab}(G) \to K$, K is also p-divisible. □

Lemma 2.3. *Suppose $\mathcal{P}_i$, $i = 1, 2$, are two classes of nilpotent groups such that for any nilpotent central extension $1 \to K \to G \to I \to 1$ where $h(I) < h(G)$ the following conditions hold:*

(a) *K and I belong to $\mathcal{P}_1$ if $G \in \mathcal{P}_1$,*
(b) *$G \in \mathcal{P}_2$ if $K, I \in \mathcal{P}_2$.*

If $A \in \mathcal{P}_1 \implies A \in \mathcal{P}_2$ for all Abelian groups A, then $\mathcal{P}_1 \subset \mathcal{P}_2$.

Proof. We prove the implication $G \in \mathcal{P}_1 \implies G \in \mathcal{P}_2$ by induction on the Hirsch length of G. By assumption the implication holds for an Abelian group G. Therefore suppose $1 \to K \to G \to I \to 1$ is a nilpotent central extension of groups and I is of lower Hirsch length than the Hirsch length of G. By (a) we have $K, I \in \mathcal{P}_1$. By the inductive hypothesis, $K, I \in \mathcal{P}_2$. The condition (b) yields $G \in \mathcal{P}_2$. □

Corollary 2.4. *Let A be an Abelian group and $n \in \{1, 2\}$. Consider the following statements:*

(1) $\widetilde{H}_*(G; A) = 0$,
(2) $\widetilde{H}_i(G; A) = 0$ *for all* $i \leq n$.

If (1) *is equivalent to* (2) *for all Abelian groups G, then the two statements are equivalent for all nilpotent groups G.*

Proof. Let $\mathcal{P}_1 = \mathcal{P}_1^r$ (respectively, $\mathcal{P}_2 = \mathcal{P}_2^r$) be the class of all nilpotent groups G such that $\widetilde{H}_i(G;A) = 0$ for all $i \leq n$ (respectively, $\widetilde{H}_i(G;A) = 0$ for all i) and $h(G) \leq r$. Our goal is to prove, by induction on r, that $\mathcal{P}_1^r = \mathcal{P}_2^r$. It is clearly so for $r = 1$. Assume $\mathcal{P}_1^m = \mathcal{P}_2^m$ for all $m < r$.

Suppose $1 \to K \to G \to I \to 1$ is a nilpotent central extension of groups such that $h(I) < h(G) = r$. If $G \in \mathcal{P}_j^r$, then $H_1(K;A) = 0$, which implies $H_i(G;A) \to H_i(I;A)$ is an epimorphism for $i \leq 2$, so $H_i(I;A) = 0$ for $1 \leq i \leq n$. By the inductive assumption, $H_i(I;A) = 0$ for all $i \geq 1$. If $\widetilde{H}_i(K;A)$ is the first nontrivial reduced homology group of K, then the Leray-Serre spectral sequence implies $H_{i+1}(I;A) \neq 0$, a contradiction. Thus both K and I belong to $\mathcal{P}_j^r$. Conversely, if $K, I \in \mathcal{P}_j^{r-1}$, then (by the inductive assumption) they have trivial homology with coefficients in A resulting in G having trivial homology with coefficients in A and $G \in \mathcal{P}_j^r$. Applying Lemma 2.3 one gets $\mathcal{P}_1^r = \mathcal{P}_2^r$. □

3. Bockstein basis

If G is a group, then $\mathrm{Tor}(G)$ is the subgroup generated by torsion elements of G, $\mathrm{Tor}_p(G)$ is the subgroup generated by all elements of G whose order is a power of p, $F_p(G) = G/\mathrm{Tor}_p(G)$, and $F(G) = G/\mathrm{Tor}(G)$.

The *Bockstein groups* are: rationals $\mathbf{Q}$, cyclic groups $\mathbf{Z}/p$ of p elements, p-adic circles $\mathbf{Z}/p^\infty$, and p-localizations of integers $\mathbf{Z}_{(p)} = \{\frac{m}{n} \in \mathbf{Q} \mid n \text{ is not divisible by } p\}$, where p is a prime number. Here is a classical definition of the maximal Bockstein basis of an Abelian group G:

Definition 3.1. $\sigma(G)$ is a subset of all Bockstein groups satisfying the following conditions:

(1) $\mathbf{Q} \in \sigma(G)$ if and only if $F(G) \neq 1$.
(2) $\mathbf{Z}/p^\infty \in \sigma(G)$ if and only if $\mathrm{Tor}_p(G) \neq 1$ or $F(G)$ is not divisible by p.
(3) $\mathbf{Z}/p \in \sigma(G)$ if and only if $\mathrm{Tor}_p(G)$ is not divisible by p or $F(G)$ is not divisible by p.
(4) $\mathbf{Z}_{(p)} \in \sigma(G)$ if and only if $F(G)$ is not divisible by p.

Notice that our Definition 3.1 differs from that in [4] in the sense that ours is maximal (if $\mathbf{Z}/p \in \sigma(G)$, then $\mathbf{Z}/p^\infty \in \sigma(G)$ and if $\mathbf{Z}_{(p)} \in \sigma(G)$, then $\mathbf{Z}/p \in \sigma(G)$) and the one in [4] is minimal (if $\mathbf{Z}/p \in \sigma(G)$ or $\mathbf{Z}_{(p)} \in \sigma(G)$, then $\mathbf{Z}/p^\infty \notin \sigma(G)$). From the point of view of the First Bockstein Theorem, both definitions are equivalent.

Here is a definition for nilpotent groups which is more convenient in this paper as it allows using localization of short exact sequences of nilpotent groups. Recall that a (multiplicative) group is $\bar{p}$-local iff the map $x \mapsto x^p$ is a bijection. We call a nilpotent group p-local iff it is $\bar{q}$-local for all primes $q \neq p$.

Definition 3.2. Let G be a nilpotent group. Then the Bockstein basis $\sigma(G)$ is defined as follows:

(1) $\mathbf{Q} \notin \sigma(G)$ if and only if $G = \mathrm{Tor}(G)$.
(2) $\mathbf{Z}/p^\infty \notin \sigma(G)$ if and only if G is $\bar{p}$-local.
(3) $\mathbf{Z}/p \notin \sigma(G)$ if and only if G is divisible by p.
(4) $\mathbf{Z}_{(p)} \notin \sigma(G)$ if and only if $F(G)$ is $\bar{p}$-local.

Remark 3.3. Note that according to the above definition we have

$$\mathbf{Z}_{(p)} \in \sigma(G) \Rightarrow \mathbf{Z}/p \in \sigma(G) \Rightarrow \mathbf{Z}/p^\infty \in \sigma(G).$$

Corollary 3.4. *For a nilpotent group G the following statements are equivalent:*

(1) $\mathbf{Q} \notin \sigma(G)$,
(2) $\widetilde{H}_*(G;\mathbf{Q}) = 0$, *and*
(3) $H_1(G;\mathbf{Q}) = 0$.

Proof. For Abelian groups, (2) and (3) are equivalent [1, Theorem V.6.4(ii)]. With Corollary 2.4 we extend this equivalence to all nilpotent groups.

Let $\mathcal{P}_1$ be the class of torsion nilpotent groups and let $\mathcal{P}_2$ be the class of nilpotent groups such that $H_1(G;\mathbf{Q}) = 0$ (i.e., $\mathrm{Ab}(G)$ is torsion). Use Lemma 2.3 to conclude that $\mathcal{P}_1 = \mathcal{P}_2$ and thus (1) and (3) are equivalent. □

Corollary 3.5. *For a nilpotent group G, $\mathbf{Z}/p \notin \sigma(G)$ if and only if $H_1(G;\mathbf{Z}/p) = 0$.*

Proof. Let $\mathcal{P}_1$ be the class of p-divisible nilpotent groups and let $\mathcal{P}_2$ be the class of nilpotent groups such that $H_1(G;\mathbf{Z}/p) = 0$. Use Lemmas 2.3 and 2.2 to conclude that $\mathcal{P}_1 = \mathcal{P}_2$. □

Corollary 3.6. *For a nilpotent group G the following statements are equivalent:*

(1) $\mathbf{Z}/p^\infty \notin \sigma(G)$,
(2) $\widetilde{H}_*(G;\mathbf{Z}/p^\infty) = 0$, *and*
(3) $H_1(G;\mathbf{Z}/p^\infty) = H_2(G;\mathbf{Z}/p^\infty) = 0$.

Proof. If $\mathbf{Z}/p^\infty \notin \sigma(G)$, then G is $\bar{p}$-local, so all its integral homology groups are $\bar{p}$-local and $H_i(G;\mathbf{Z}/p^\infty) = 0$ for all $i \geq 1$, which proves the implication (1) $\implies$ (2).

Notice (2) $\iff$ (3) by Corollary 2.4. Indeed, if A is Abelian and $H_1(A;\mathbf{Z}/p^\infty) = H_2(A;\mathbf{Z}/p^\infty) = 0$, then A must be $\bar{p}$-local.

Let $\mathcal{P}_1$ be the class of $\bar{p}$-local nilpotent groups and let $\mathcal{P}_2$ be the class of nilpotent groups such that $H_1(G;\mathbf{Z}/p^\infty) = H_2(G;\mathbf{Z}/p^\infty) = 0$. Use Lemma 2.3 to conclude that $\mathcal{P}_1 = \mathcal{P}_2$. □

Corollary 3.7. *For a nilpotent group G the following statements are equivalent:*

(1) $\mathbf{Z}_{(p)} \notin \sigma(G)$,
(2) $\widetilde{H}_*(F(G);\mathbf{Z}/p^\infty) = 0$, *and*
(3) $H_1(F(G);\mathbf{Z}/p^\infty) = H_2(F(G);\mathbf{Z}/p^\infty) = 0$.

It is obvious that $\mathbf{Z}/p^\infty \notin \sigma(G)$ if and only if $G \to G_{(\bar{p})}$ is an isomorphism. The following lemma characterizes $\mathbf{Z}_{(p)} \notin \sigma(G)$ via localizations.

Lemma 3.8. *For a nilpotent group G the following statements are equivalent:*

(1) $\mathbf{Z}_{(p)} \notin \sigma(G)$,
(2) $G \to G_{(\bar{p})}$ *is an epimorphism.*

Proof. If $G \to G_{(\bar{p})}$ is an epimorphism, then its kernel has trivial $\bar{p}$-localization (by exactness of the localization functor) and must be a torsion group. Therefore $F(G) = F(G_{(\bar{p})})$ is $\bar{p}$-local. If $F(G)$ is $\bar{p}$-local, then apply exactness of the localization functor to the short exact sequence $1 \to \mathrm{Tor}(G) \to G \to F(G) \to 1$ and derive that $G \to G_{(\bar{p})}$ is an epimorphism. □

Definition 3.9. The *torsion-divisible Bockstein basis* $\sigma_{TD}(G)$ of G consists of all $\mathbf{Z}/p^\infty$ belonging to $\sigma(G)$. We set $\sigma_{NTD}(G) = \sigma(G) \backslash \sigma_{TD}(G)$.

Lemma 3.10. *If $G \to I$ is an epimorphism of nilpotent groups. Then $\sigma_{NTD}(I) \subset \sigma_{NTD}(G)$.*

Proof. Suppose $\mathbf{Q} \notin \sigma(G)$. Then G is a torsion group. So I is a torsion group; hence $\mathbf{Q} \notin \sigma(I)$.

Let $\mathbf{Z}/p \notin \sigma(G)$. Then G is p-divisible and then also I is p-divisible.

Let $\mathbf{Z}_{(p)} \notin \sigma(G)$. Then $G \to G_{(\bar{p})}$ is an epimorphism. Because $\bar{p}$-localization is an exact functor, the map $G_{(\bar{p})} \to I_{(\bar{p})}$ is an epimorphism and hence $I \to I_{(\bar{p})}$ is an epimorphism. □

Lemma 3.11. *Let $1 \to K \to G \to I \to 1$ be a central extension of nilpotent groups. Then $\sigma(G) \subset \sigma(K) \cup \sigma(I)$.*

Proof. Let $\mathbf{Q} \notin \sigma(K) \cup \sigma(I)$. Then K and I are torsion groups. Hence G is torsion, so $\mathbf{Q} \notin \sigma(G)$.

Let $\mathbf{Z}/p \notin \sigma(K) \cup \sigma(I)$. Then K and I are p-divisible. By Lemma 2.2, also G is p-divisible.

Let $\mathbf{Z}/p^\infty \notin \sigma(K) \cup \sigma(I)$. Then $K \to K_{(\bar{p})}$ and $I \to I_{(\bar{p})}$ are isomorphisms. Using the Five Lemma and the fact that $\bar{p}$-localization is an exact functor, we conclude that also $G \to G_{(\bar{p})}$ is an isomorphism.

Let $\mathbf{Z}_{(p)} \notin \sigma(K) \cup \sigma(I)$. Then $K \to K_{(\bar{p})}$ and $I \to I_{(\bar{p})}$ are epimorphisms. By the Three Lemma [10, Lemma 2.8], also $G \to G_{(\bar{p})}$ is an epimorphism. □

Lemma 3.12. *Let $1 \to K \to G \to I \to 1$ be a nilpotent central extension of groups. If $\mathbf{Z}_{(p)} \notin \sigma(Ab(G))$ for some prime p, then $\mathbf{Z}_{(p)} \notin \sigma(K)$ and $\mathbf{Z}_{(p)} \notin \sigma(Ab(I))$.*

Proof. Assume $\mathbf{Z}_{(p)} \notin \sigma(\mathrm{Ab}(G))$. That means the map $F(\mathrm{Ab}(G)) \to F(\mathrm{Ab}(G))_{(\bar{p})}$ is an isomorphism. Because $\mathrm{Ab}(G) \to \mathrm{Ab}(I)$ is an epimorphism and F is a right exact functor, the map $F(\mathrm{Ab}(G)) \to F(\mathrm{Ab}(I))$ is an epimorphism. Hence the map $F(\mathrm{Ab}(I)) \to F(\mathrm{Ab}(I))_{(\bar{p})}$ is an epimorphism. Its kernel is a p-torsion group, so the kernel is trivial and the map $F(\mathrm{Ab}(I)) \to F(\mathrm{Ab}(I))_{(\bar{p})}$ is an isomorphism. That means $\mathbf{Z}_{(p)} \notin \sigma(\mathrm{Ab}(I))$.

There exists an epimorphism $\otimes^n \mathrm{Ab}(G) \to K$ for some integer n. Because $F(\mathrm{Ab}(\otimes^n G)) \to F(\mathrm{Ab}(\otimes^n G))_{(\bar{p})}$ is an isomorphism in the same way as in the previous paragraph, we can prove that $F(K) \to F(K)_{(\bar{p})}$ is an isomorphism. Hence $\mathbf{Z}_{(p)} \notin \sigma(K)$. □

Lemma 3.13. *Let G be a nilpotent group. Then $\sigma_{NTD}(G) = \sigma_{NTD}(Ab(G))$ and $\sigma(Ab(G)) \subset \sigma(G)$.*

Proof. The inclusion $\sigma_{NTD}(\mathrm{Ab}(G)) \subset \sigma_{NTD}(G)$ follows from Lemma 3.10.

Let us prove $\sigma_{NTD}(G) \subset \sigma_{NTD}(\mathrm{Ab}(G))$. Suppose $\mathbf{Q} \notin \sigma_{NTD}(\mathrm{Ab}(G))$. Because G is a torsion group if and only if $\mathrm{Ab}(G)$ is a torsion group, $\mathbf{Q} \notin \sigma_{NTD}(G)$.

Suppose $\mathbf{Z}/p \notin \sigma_{NTD}(\mathrm{Ab}(G))$. Because G is p-divisible if and only if $\mathrm{Ab}(G)$ is p-divisible, $\mathbf{Z}/p \notin \sigma_{NTD}(G)$.

Consider the class $\mathcal{P}$ of all nilpotent groups such that $\mathbf{Z}_{(p)} \notin \sigma_{NTD}(\mathrm{Ab}(G))$ implies $\mathbf{Z}_{(p)} \notin \sigma_{NTD}(G)$. $\mathcal{P}$ clearly contains all Abelian groups. To show $\mathcal{P}$ equals the class $\mathcal{N}$ of all nilpotent groups it suffices to show (see Lemma 2.3) that for any nilpotent central extension $A \to G \to G'$ such that the Hirsch length of G' is less than $h(G)$, $A, G' \in \mathcal{P}$ implies $G \in \mathcal{P}$. Assume $\mathbf{Z}_{(p)} \notin \sigma_{NTD}(\mathrm{Ab}(G))$. By Lemma 3.12, we conclude $\mathbf{Z}_{(p)} \notin \sigma_{NTD}(G')$ as $G' \in \mathcal{P}$ and $\mathbf{Z}_{(p)} \notin \sigma_{NTD}(A)$. By Lemma 3.11, $\mathbf{Z}_{(p)} \notin \sigma_{NTD}(G)$.

Let us now prove that $\sigma(\mathrm{Ab}(G)) \subset \sigma(G)$. By Lemma 3.10, $\sigma_{NTD}(\mathrm{Ab}(G)) \subset \sigma_{NTD}(G)$. Suppose $\mathbf{Z}/p^\infty \notin \sigma(G)$. Then G is uniquely p-divisible, so $H_*(G; \mathbf{Z})$ is

uniquely p-divisible. In particular $\mathrm{Ab}(G) = H_1(G;\mathbf{Z})$ is uniquely p-divisible; hence $\mathbf{Z}/p^\infty \notin \sigma(\mathrm{Ab}(G))$. □

Theorem 3.14. *If $1 \to K \to G \to I \to 1$ is a nilpotent central extension, then $\sigma(G) = \sigma(K) \cup \sigma(I)$.*

Proof. By Lemma 3.11, $\sigma(G) \subset \sigma(K) \cup \sigma(I)$.

Let us prove that $\sigma(K) \cup \sigma(I) \subset \sigma(G)$. Suppose $\mathbf{Q} \notin \sigma(G)$. Then G is a torsion group. Therefore K and I are also torsion groups, so $\mathbf{Q} \notin \sigma(K) \cup \sigma(I)$.

Suppose $\mathbf{Z}/p \notin \sigma(G)$. Then G is p-divisible. By Lemma 2.2, K and I are p-divisible, so $\mathbf{Z}/p \notin \sigma(K) \cup \sigma(I)$.

Suppose $\mathbf{Z}/p^\infty \notin \sigma(G)$. Then $G \to G_{(\bar{p})}$ is an isomorphism. Because $\bar{p}$-localization is an exact functor, the map $K \to K_{(\bar{p})}$ is a monomorphism and the map $\mathrm{Ab}(G) \to (\mathrm{Ab}(G))_{(\bar{p})}$ is an epimorphism. Because there exists an epimorphism $\otimes^n \mathrm{Ab}G \to K$, the map $K \to K_{(\bar{p})}$ is also an epimorphism; hence it is an isomorphism. By the Five Lemma, also the map $I \to I_{(\bar{p})}$ is an isomorphism, so $\mathbf{Z}/p^\infty \notin \sigma(K) \cup \sigma(I)$.

If $\mathbf{Z}_{(p)} \notin \sigma(G)$, then $\mathbf{Z}_{(p)} \notin \sigma(K)$ and $\mathbf{Z}_{(p)} \notin \sigma(I)$ by Lemmas 3.12 and 3.13. □

4. Bockstein spaces

Definition 4.1. A metrizable space X is called a *Bockstein space* if $\dim_G X = \sup\{\dim_H X \mid H \in \sigma(G)\}$ for all Abelian groups G.

Remark 4.2. In the above definition observe that $\dim_G X$ is an element of $\mathbf{N} \cup \{0, \infty\}$ and not only in $\{0, 1, \infty\}$ as in the case of non-Abelian groups G.

Dranishnikov-Repovš-Shchepin [6] showed the existence of a separable metric space X of dimension 2 such that $\dim_{\mathbf{Z}_{(p)}} X = 1$ for all primes p. Thus, X is not a Bockstein space as $\dim_{\mathbf{Z}} X = 2 > 1 = \sup\{\dim_H X \mid H \in \sigma(\mathbf{Z})\}$.

Problem 4.3. Is every metric ANR a Bockstein space?

Proposition 4.4. *Suppose $X = \bigcup_{n=1}^\infty X_n$ is metrizable and each X_n is closed in X. If all X_n are Bockstein spaces, then so is X.*

Proof. Suppose G is an Abelian group and $H \in \sigma(G)$. If $\dim_G(X) \le m$, then $\dim_G(X_n) \le m$ for all n and $\dim_H(X_n) \le m$ for all n resulting in $\dim_H(X) \le m$.

If $\dim_H(X) < m$ for all $H \in \sigma(G)$, then $\dim_H(X_n) < m$ for all n and $\dim_G(X_n) < m$ for all n resulting in $\dim_G(X) < m$. □

For a subset $l \subset \mathbb{P}$ of prime integers let $\mathbf{Z}_l = \{\frac{m}{n} \in \mathbf{Q} \mid n$ is not divisible by any $p \in l\}$ and let $\hat{\mathbf{Z}}_l$ be the group of l-adic integers.

Theorem 4.5. *A metrizable space X is a Bockstein space if and only if $\dim_{\mathbf{Z}_l} X = \dim_{\hat{\mathbf{Z}}_l} X$ for all subsets $l \subset \mathbb{P}$ of the set of prime numbers.*

Proof. Since $\sigma(\hat{\mathbf{Z}}_l) = \sigma(\mathbf{Z}_l)$ for all $l \subset \mathbb{P}$, $\dim_{\mathbf{Z}_l} X = \dim_{\hat{\mathbf{Z}}_l} X$ holds for any Bockstein space X.

Assume $\dim_{\mathbf{Z}_l} X = \dim_{\hat{\mathbf{Z}}_l} X$ for all subsets $l \subset \mathbb{P}$ of the set of prime numbers. Suppose G is a torsion-free Abelian group G. If $\mathbf{Z}_{(p)} \in \sigma(G)$, then Theorem B(d) of [7] says that $\dim_{\hat{\mathbf{Z}}_{(p)}} X \leq \dim_G X$. Therefore $\dim_G X \geq \sup\{\dim_H X \mid H \in \sigma(G)\}$. Suppose $\sup\{\dim_H X \mid H \in \sigma(G)\} = n$ and consider $l = \{p \mid p \cdot G \neq G\}$. Theorem B(f) of [7] says that $\dim_G(X) \leq \dim_{\mathbf{Z}_l} X$. Since $\sigma(G) = \sigma(\mathbf{Z}_l)$, $\dim_G X \leq \dim_{\mathbf{Z}_l} X = \sup\{\dim_H X \mid H \in \sigma(\mathbf{Z}_l)\} = \sup\{\dim_H X \mid H \in \sigma(G)\} = n$. That proves $\dim_G X = \sup\{\dim_H X \mid H \in \sigma(G)\}$ for all torsion-free Abelian groups. The same equality holds for all torsion Abelian groups by Theorem B(a) of [7]. In the case of arbitrary Abelian groups G, as $\sigma(G) = \sigma(F(G)) \cup \sigma(\mathrm{Tor}(G))$ and $\dim_G = \max(\dim_{F(G)} X, \dim_{\mathrm{Tor}(G)} X)$ (see Theorem B(b) of [7]) one gets $\dim_G X = \sup\{\dim_H X \mid H \in \sigma(G)\}$ as well. □

Remark 4.6. Notice that it is not sufficient to assume $\dim_{\mathbf{Z}_{(p)}} X = \dim_{\hat{\mathbf{Z}}_{(p)}} X$ for all primes p in Theorem 4.5. Indeed, the space X in [6] has that property as $1 = \dim_{\mathbf{Z}_{(p)}} X \geq \dim_{\hat{\mathbf{Z}}_{(p)}} X \geq 1$ for all primes p.

Theorem 4.7. *Let X be a Bockstein space. If G is nilpotent, then* $\dim_G(X) \leq 1$ *if and only if* $\sup\{\dim_H(X) | H \in \sigma(G)\} \leq 1$.

Proof. Let $\mathcal{P}_1$ be the class of all nilpotent groups and let $\mathcal{P}_2$ be the class of nilpotent groups G such that $\dim_G(X) \leq 1$ if and only if $\sup\{\dim_H(X) | H \in \sigma(G)\} \leq 1$. Since $\mathcal{P}_2$ contains all Abelian groups, in view of Lemma 2.3 it suffices to show that for any nilpotent central extension $K \to G \to I$ the conditions $K, I \in \mathcal{P}_2$ imply $G \in \mathcal{P}_2$. It is so if G is Abelian, so assume G is not Abelian. Moreover, as $\sigma(\mathrm{Ab}(G)) \subset \sigma(G)$ by Lemma 3.13 and $\dim_G X \leq 1$ implies $\dim_{\mathrm{Ab}(G)} X \leq 1$ (see Lemma 1.4), either $\dim_G(X) \leq 1$ or $\sup\{\dim_H(X) | H \in \sigma(G)\} \leq 1$ implies $\dim_{\mathrm{Ab}(G)} X \leq 1$, so we may as well assume $\dim_{\mathrm{Ab}(G)} X \leq 1$.

In view of Lemma 1.2 and the fact that $\mathrm{Ab}(G) = 1$ implies $G = 1$, the equivalence of conditions $\dim_G(X) \leq 1$ and $\sup\{\dim_H(X) | H \in \sigma(G)\} \leq 1$ may fail only if $\dim_{\mathrm{Ab}(G)} X = 1$, so assume $\dim_{\mathrm{Ab}(G)} X = 1$.

Suppose $\dim_H(X) = n > 1$ for some $H \in \sigma(K)$. If $H \neq \mathbf{Z}/p^\infty$, then $H \in \sigma_{NTD}(G) = \sigma_{NTD}(\mathrm{Ab}(G))$ (Theorem 3.14 and Lemma 3.13), a contradiction. So $H = \mathbf{Z}/p^\infty$ for some prime p and $\mathbf{Z}/p^\infty \notin \sigma(\mathrm{Ab}(G))$. Hence $\mathrm{Ab}(G)$ is p-divisible and this is equivalent to G being p-divisible [2, Lemma 5.1]. Because $\mathbf{Z}/p^\infty \notin \sigma(\mathrm{Ab}(G))$, by Lemma 3.6 $H_1(\mathrm{Ab}(G); \mathbf{Z}/p^\infty) = 0$, so also $H_1(G; \mathbf{Z}/p^\infty) = 0$. By Lemma 3.6, $H_2(G; \mathbf{Z}/p^\infty) \neq 0$ as $\mathbf{Z}/p^\infty \in \sigma(G)$. This implies that $F(H_2(G; \mathbf{Z}))$ is not p-divisible, so $\mathbf{Z}_{(p)} \in \sigma(H_2(G; \mathbf{Z}))$. If G is not a torsion group, then $\mathrm{Ab}(G)$ is not a torsion group; hence by definition $\mathbf{Q} \in \sigma(\mathrm{Ab}(G))$. Therefore $\dim_{\mathbf{Q}}(X) \leq 1$. Using that fact and the Bockstein Inequalities (BI5, BI6 [12]), we get $\dim_{\mathbf{Z}/p^\infty}(X) = \dim_{\mathbf{Z}_{(p)}}(X) - 1$. Because $\mathbf{Z}_{(p)} \in \sigma(H_2(G; \mathbf{Z}))$, the dimension $\dim_{\mathbf{Z}_{(p)}}(X) \leq \dim_{H_2(G;\mathbf{Z})}(X) \leq 2$ as X is a Bockstein space and then $\dim_{\mathbf{Z}/p^\infty}(X) = \dim_{\mathbf{Z}_{(p)}}(X) - 1 \leq 1$, a contradiction.

Thus G is a torsion group and is a product of q-groups $G = \prod_{q \in \mathbb{P}} G_q$. Hence $\mathrm{Ab}(G) = \prod_{q \in \mathbb{P}} \mathrm{Ab}(G_q)$. Because G is not $\bar{p}$-local, $G_p \neq 1$, but $\mathrm{Ab}(G)$ is uniquely p-divisible, so $\mathrm{Ab}(G_p) = 1$. Therefore G_p is a perfect nilpotent group, but such a group is trivial, a contradiction.

Thus $\dim_H X \leq 1$ for all $H \in \sigma(K)$ and $\dim_K X = \sup\{\dim_H(X) | H \in \sigma(K)\} \leq 1$ as K is Abelian and X is a Bockstein space.

Suppose $\sup\{\dim_H(X)|H \in \sigma(G)\} \leq 1$. By Theorem 3.14, $\sigma(G) = \sigma(K) \cup \sigma(I)$. Therefore $\sup\{\dim_H(X)|H \in \sigma(I)\} \leq 1$ and $\dim_I X \leq 1$ as $I \in \mathcal{P}_2$. Consequently, $\dim_G X \leq 1$ by Lemma 1.3.

Suppose $\dim_G X \leq 1$. By Lemma 1.3 one gets $\dim_I X \leq 1$ and $\sup\{\dim_H(X)|H \in \sigma(I)\} \leq 1$ as $I \in \mathcal{P}_2$. By Theorem 3.14, $\sigma(G) = \sigma(K) \cup \sigma(I)$; hence

$$\sup\{\dim_H(X)|H \in \sigma(G)\} = \sup\{\dim_H(X)|H \in \sigma(K) \cup \sigma(I)\} \leq 1. \qquad \square$$

Corollary 4.8. *Let L be a connected nilpotent CW complex. If X is a Bockstein space and $\dim_{H_n(L)} X \leq n$ for all $n \geq 1$, then $\dim_{\pi_n(L)} X \leq n$ for all $n \geq 1$.*

Proof. It is shown in [2] that $\dim_{\pi_n(L)} X \leq n$ for all $n \geq 2$, so it suffices to prove $\dim_{\pi_1(L)} X \leq 1$. If that inequality is false, then there is $H \in \sigma(\pi_1(L))$ such that $\dim_H(X) > 1$ (see Theorem 4.7). In view of Lemma 3.13, as $\dim_{H_1(L)}(X) \leq 1$, $H = \mathbf{Z}/p^\infty$ for some prime p. Also, $H_1(L)$ is not a torsion group, so $\dim_{\mathbf{Q}}(X) \leq 1$. Using Bockstein Inequalities one gets $\dim_{\mathbf{Z}_{(p)}}(X) \geq 3$. Therefore the i-th homology groups of both L and $\tilde{L}$ with coefficients in $\mathbf{Z}/p^\infty$ vanish for $i \leq 2$. From the fibration $\tilde{L} \to L \to K(\pi_1(L), 1)$ one gets that the i-th homology groups of $K(\pi_1(L), 1)$ with coefficients in $\mathbf{Z}/p^\infty$ vanish for $i \leq 2$. However, in view of Corollary 3.6, that means $\mathbf{Z}/p^\infty \notin \sigma(\pi_1(L))$, a contradiction. $\square$

Corollary 4.9. *Let L be a connected nilpotent CW complex and let X be a Bockstein space such that $\dim_{H_n(L)} X \leq n$ for all $n \geq 1$. If X is finite dimensional or $X \in ANR$, then $X \tau L$.*

Proof. By Corollary 4.8 one gets $\dim_{\pi_n(L)} X \leq n$ for all $n \geq 1$ and by Theorem G in [7] one has $X \tau L$. $\square$

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