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ON THE VALUES OF A CLASS OF DIRICHLET SERIES AT RATIONAL ARGUMENTS

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(Communicated by Ken Ono)

Dedicated to Professor Eiichi Bannai on his sixtieth birthday, with great respect and friendship

Abstract. In this paper we shall prove that the combination of the general distribution property and the functional equation for the Lipschitz-Lerch transcendent capture the whole spectrum of deeper results on the relations between the values at rational arguments of functions of a class of zeta-functions. By Theorem 1 and its corollaries, we can cover all the previous results in a rather simple and lucid way. By considering the limiting cases, we can also deduce new striking identities for Milnor's gamma functions, among which is the Gauss second formula for the digamma function.

1. Introduction and the finite value case

Srivastava [S2] (cf. also Srivastava and Choi [SC], pp. 336–344) gives two simpler proofs of interesting results of Cvijović and Klinowski [CK1] on the finite expression for the values of a Bernoulli polynomial at rational arguments in terms of the values of the Hurwitz zeta-function at integer arguments with rational values for the perturbation.

The first proof uses the Fourier series for the periodic Bernoulli polynomial and the formula for the decomposition into residue classes. The second proof uses the functional equation for the partial zeta-function

$$\zeta\left(1-s,\frac{p}{q}\right)=\frac{2\Gamma(s)}{(2\pi q)^s}\sum_{a=1}^{q}\cos\left(\frac{\pi s}{2}-\frac{2\pi ap}{q}\right)\zeta\left(s,\frac{a}{q}\right) \tag{1.1}$$

([SC, (8), p. 90], [Ap]) to be restated as (1.11) below, where

$$\zeta(s,a)=\sum_{n=0}^{\infty}\frac{1}{(n+a)^s},\ \ \sigma>1, \tag{1.2}$$

is the Hurwitz zeta-function.

We introduce the Lipschitz-Lerch transcendent

$$L(s,z,a)=\sum_{n=0}^{\infty}\frac{e^{2\pi inz}}{(n+a)^s}, \tag{1.3}$$

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absolutely convergent for $\sigma > 1$ if $z = x \in \mathbb{R}$ or $s \in \mathbb{C}$ with $\operatorname{Im} z > 0$, which generalizes not only the Hurwitz zeta-function

$$\zeta(s,a) = L(s,1,a) \tag{1.4}$$

but also the polylogarithm function (or the Lerch zeta-function)

$$l_s(x) = L(s,x,1) = \sum_{n=1}^{\infty} \frac{e^{2\pi i n x}}{n^s}, \ \ \sigma > 1. \tag{1.5}$$

The most general distribution property of the Hurwitz-Lerch transcendent (which follows from the principle of decomposition into residue classes used in Srivastava's first method),

$$L(z,s,a) = q^{-s} \sum_{j=0}^{q-1} L\left(qz, s, \frac{a+j}{q}\right) e^{2\pi i j z}, \tag{1.6}$$

([SC, (15), p. 339]) coupled with the functional equation for the Lipschitz-Lerch transcendent ([SC, (29), p. 125]),

$$L(x,1-s,a) = \frac{\Gamma(s)}{(2\pi)^s} \left\{ e^{\pi i(\frac{1}{2}s - 2ax)} L(-a,s,x) + e^{-\pi i(\frac{1}{2}s - 2a(1-x))} L(a,s,1-x) \right\}, \tag{1.7}$$

captures the whole scene and gives an immediate proof of new limiting relations as well as all these types of linear combination expressions for a class of zeta-functions and as a special case for polynomials.

Theorem 1. *For $1 < q \in \mathbb{Z}$ with $0 < x < 1$ and $\operatorname{Re} a > 0$, one has the following identity:*

$$\begin{aligned} q^{-s} \sum_{j=0}^{q-1} L\left(qx, s, \frac{a+j}{q}\right) e^{2\pi i j x} &= L(x,s,a) = \frac{\Gamma(1-s)}{(2\pi)^{1-s}} \\ &\times \left\{ e^{\pi i(\frac{1}{2}(1-s) - 2ax)} L(-a,1-s,x) + e^{-\pi i(\frac{1}{2}(1-s) - 2a(1-x))} L(a,1-s,1-x) \right\} \end{aligned} \tag{1.8}$$

except at the singularities of L, in which case, the identity is to mean the limit.

Specializing the parameter a suitably, Theorem 1 reads:

Corollary 1. *For $1 < q \in \mathbb{Z}$ and $q > p \in \mathbb{N}$, we have the identity*

$$\begin{aligned} &q^{-s} \sum_{a=1}^{q} e^{2\pi i \frac{a}{q} p} \zeta\left(s, \frac{a}{q}\right) \left(= l_s\left(\frac{p}{q}\right)\right) \\ &= i \frac{\Gamma(1-s)}{(2\pi)^{1-s}} \left\{ e^{-\frac{\pi i s}{2}} \zeta\left(1-s, \frac{p}{q}\right) - e^{\frac{\pi i s}{2}} \zeta\left(1-s, 1-\frac{p}{q}\right) \right\}, \end{aligned} \tag{1.9}$$

where, for $s = 0$ or $s = 1$, the identity is to mean the limit as $s \to 0$ or $s \to 1$.

We shall show that (1.9) and its counterpart for the polylogarithm function already entail not only all the previous results but also new non-trivial corollaries. As an example, we note that the trigonometric series ($\nu > 1$)

$$C_\nu(\pi\alpha) = \sum_{k=0}^{\infty} \frac{\cos(2k+1)\pi\alpha}{(2k+1)^\nu}, \ \ S_\nu(\pi\alpha) = \sum_{k=0}^{\infty} \frac{\sin(2k+1)\pi\alpha}{(2k+1)^\nu}, \tag{1.10}$$

introduced by Srivastava and Choi ([SC, (1), p. 341]), may be thought of as the real and imaginary parts of the odd part of

$$l_s\left(\frac{\alpha}{2}\right) - 2^{-s} l_s(\alpha).$$

Hence by the general distribution property (1.6) one has

$$\begin{aligned} C_s\left(\pi\frac{p}{q}\right) + iS_s\left(\pi\frac{p}{q}\right) &= l_s\left(\frac{p}{2q}\right) - 2^{-s} l_s\left(\frac{p}{q}\right) \\ &= (2q)^{-s}\sum_{j=1}^{q}\zeta\left(s,\ \frac{2j-1}{2q}\right)e^{2\pi i\frac{2j-1}{2q}p}. \end{aligned}$$

This proves [SC, (8), (9), p. 342], which are equivalent to Cvijović-Klinowski results [CK2] (cf. [S1]).

Also note that in view of

$$C_{2\nu-1}(\pi\alpha) = \frac{(2\pi)^{2\nu-1}(-1)^{\nu-1}}{(2\nu-1)!}\left(Cl_{2\nu-1}\left(\frac{\alpha}{2}\right) - 2^{-(2\nu-1)}Cl_{2\nu-1}(\alpha)\right)$$

and

$$S_{2\nu}(\pi\alpha) = \frac{(2\pi)^{2\nu}(-1)^{\nu}}{(2\nu)!}\left(Cl_{2\nu}\left(\frac{\alpha}{2}\right) - 2^{-2\nu}Cl_{2\nu-1}(\alpha)\right),$$

the statement by Srivastava-Choi about the slow convergence of the series is justified. Here $Cl_\nu(\alpha)$ denotes the Clausen function (log-sine integrals; cf. [Le], [Y], [K]) defined as

$$Cl_\nu(\alpha) = \begin{cases} \sum_{k=1}^{\infty}\frac{\sin k\alpha}{k^\nu}, & \text{if } 2|n, \\ \sum_{k=1}^{\infty}\frac{\cos k\alpha}{k^\nu}, & \text{if } 2\not| n. \end{cases}$$

Corollary 2 ([Ap, Theorem 128, p. 261], [SC, (8), p. 90])**.** *For all $s \neq 0, 1$, we have*

$$\zeta\left(1-s,\ \frac{p}{q}\right) = \frac{2\Gamma(s)}{(2\pi q)^s}\sum_{a=1}^{q}\cos\left(\frac{\pi s}{2} - \frac{2\pi a p}{q}\right)\zeta\left(s, \frac{a}{q}\right), \tag{1.11}$$

and for $n > 1,\ 0 \le p \le q$,

$$B_n\left(\frac{p}{q}\right) = -\frac{2n!}{(2\pi q)^n}\sum_{j=1}^{q}\cos\left(\frac{2\pi j p}{q} - \frac{n\pi}{2}\right)\zeta\left(n, \frac{j}{q}\right). \tag{1.12}$$

Proof. To prove (1.11) we sum two equalities: one is $e^{-\frac{\pi i s}{2}}$ times (1.9), and the other is $e^{\frac{\pi i s}{2}}$ times (1.9) with p replaced by $q-p$. Then we get

$$\begin{aligned} &q^{-s}\sum_{a=1}^{q}\left(e^{\left(-\frac{\pi s}{2}+\frac{2\pi a p}{q}\right)i} + e^{\left(\frac{\pi s}{2}-\frac{2\pi a p}{q}\right)i}\right)\zeta\left(s, \frac{a}{q}\right) \\ &= i\frac{\Gamma(1-s)}{(2\pi)^{1-s}}\left(e^{-\pi i s} - e^{\pi i s}\right)\zeta\left(1-s, \frac{p}{q}\right) \end{aligned}$$

or

$$2q^{-s}\sum_{a=1}^{q}\cos\left(\frac{\pi s}{2}-\frac{2\pi ap}{q}\right)\zeta\left(s,\frac{a}{q}\right)$$
$$=\frac{2\Gamma(1-s)\sin\pi s}{(2\pi)^{1-s}}\zeta\left(s,\frac{a}{q}\right)=\frac{2\pi}{(2\pi)^{1-s}\Gamma(s)}\zeta\left(s,\frac{a}{q}\right),$$

whence (1.11) follows upon using the functional equation for the gamma function.

Now (1.12) follows from (1.11) in view of the formula

$$-n\zeta(1-n,x)=B_n(x). \tag{1.13}$$ □

Remark 1. (i) We note that the above deduction of (1.11) amounts to applying the Hurwitz formula (cf. (1.7))

$$\zeta(1-s,x)=\frac{\Gamma(s)}{(2\pi)^s}\left(e^{-\frac{\pi is}{2}}l_s(x)+e^{\frac{\pi is}{2}}l_s(1-x)\right). \tag{1.14}$$

We also note that the reciprocal relation for the gamma function is equivalent to the Hurwitz formula, while (1.13) is its consequence (cf. [Vista, Chapter 5]).

(ii) As is noted in the first proof of [SC, Theorem 6.3, p. 336], it is rather simple to deduce the corresponding results on Euler polynomials from (1.12) by means of the relation

$$E_n(x)=\frac{2}{n+1}\left(B_{n+1}\left(\frac{x+1}{2}\right)-B_{n+1}\left(\frac{x}{2}\right)\right). \tag{1.15}$$

We shall therefore just state the result corresponding to (1.11),

$$2^{s-1}\left(\zeta\left(1-s,\frac{p+q}{2q}\right)-\zeta\left(1-s,\frac{p}{2q}\right)\right)$$
$$=-\frac{2\Gamma(s)}{(2\pi q)^s}\sum_{j=1}^{q}\cos\left(\frac{\pi s}{2}-2\pi\frac{2j-1}{q}a\right)\zeta\left(s,\frac{2j-1}{2q}\right), \tag{1.16}$$

which depends on the relation

$$2^{-s}\zeta\left(s,\frac{x+1}{2}\right)-q^{-s}\zeta\left(s,\frac{x}{2}\right)=\sum_{n=0}^{\infty}\frac{(-1)^n}{(n+x)^s}=L\left(\frac{1}{2},s,x\right). \tag{1.17}$$

2. Limiting cases

We now turn to the limiting case $s\to n$ of (1.9) with $1\le n\in\mathbb{Z}$. We note that recently Murty-Saradha [MS] used the Laurent expansion (2.5) at $s=1$ of (1.9) to deduce the Gauss formula for the digamma function.

To state our main result, we introduce Milnor's gamma function $\gamma_n(x)$ ([Mil]) through

$$\gamma_{1+n}(x)=\frac{\partial}{\partial n}\zeta(-n,x). \tag{2.1}$$

Theorem 2. *For $q > 1$, $n \in \mathbb{N}$ and $p \in \mathbb{Z}$, we have*

$$\begin{aligned} & q^{-n} \sum_{a=1}^{q-1} e^{2\pi i \frac{a}{q} p} \zeta\left(n, \frac{a}{q}\right) \\ & = \frac{i^{n+1}}{(2\pi)^{1-n}(n-1)!} \left\{ \pi i \frac{B_n\left(\frac{p}{q}\right)}{n} + \gamma_n\left(\frac{p}{q}\right) - (-1)^n \gamma_n\left(1 - \frac{p}{q}\right) \right\} \\ & = \begin{cases} \dfrac{(-1)^{\frac{n+2}{2}}}{(2\pi)^{1-n}(n-1)!} \left\{ \pi \dfrac{B_n\left(\frac{p}{q}\right)}{n} - i\left(\gamma_n\left(\dfrac{p}{q}\right) - \gamma_n\left(1 - \dfrac{p}{q}\right)\right) \right\}, & 2|n \\ \dfrac{(-1)^{\frac{n+1}{2}}}{(2\pi)^{1-n}(n-1)!} \left\{ i\pi \dfrac{B_n\left(\frac{p}{q}\right)}{n} + \gamma_n\left(\dfrac{p}{q}\right) + \gamma_n\left(1 - \dfrac{p}{q}\right) \right\}, & 2 \nmid n, \end{cases} \end{aligned} \tag{2.2}$$

the case $n = 1$ being a singular case to be stated in Corollary 4.

Corollary 3. *For $2 \mid n$, the real and imaginary parts of (2.2) lead to Corollary 1 respectively, while for $2 \nmid n$, the imaginary and the the real parts respectively provide:*

$$\begin{aligned} & q^{-n} \sum_{a=1}^{q} \left(\sin 2\pi \frac{a}{q} p\right) \zeta\left(n, \frac{a}{q}\right) \\ & = \frac{(-1)^{\frac{n+2}{2}}}{(2\pi)^{1-n}(n-1)!} \left(\gamma_n\left(1 - \frac{p}{q}\right) - \gamma_n\left(\frac{p}{q}\right)\right), 2|n, \end{aligned} \tag{2.3}$$

$$\begin{aligned} & q^{-n} \sum_{a=1}^{q} \left(\cos 2\pi \frac{a}{q} p\right) \zeta\left(n, \frac{a}{q}\right) \\ & = \frac{(-1)^{\frac{n+1}{2}}}{(2\pi)^{1-n}(n-1)!} \left(\gamma_n\left(\frac{p}{q}\right) - \gamma_n\left(1 - \frac{p}{q}\right)\right), 2 \nmid n. \end{aligned} \tag{2.4}$$

Remark 2. The referee has kindly pointed out Lewin's paper [Le, pp. 355-375], in which one finds [Le, (16.30)] is derived directly from (1.6), and Corollary 3 coincides with [Le, (16.30), (16.31)].

Now by the Laurent expansion of $\zeta(s, x)$ at $s = 1$,

$$\zeta(s, x) = \frac{1}{s-1} - \psi(x) + O(s-1), \tag{2.5}$$

where $\psi(x)$ designates the Euler digamma function

$$\psi(x) = \frac{\Gamma'}{\Gamma}(x), \tag{2.6}$$

and Lerch's formula ([SC, (17), p. 92])

$$\zeta'(0, x) = \log \frac{\Gamma(x)}{\sqrt{2\pi}}, \tag{2.7}$$

Theorem 2 in the case $n = 1$ reads as (see the proof in §3):

Corollary 4 (Gauss' second formula ([SC, (49), p. 19])). *We have*

$$\sum_{a=1}^{q} e^{2\pi i \frac{a}{q} k} \psi\left(\frac{a}{q}\right) = q \log\left(1 - e^{2\pi i \frac{k}{q}}\right). \tag{2.8}$$

3. Proof of Theorem 2

Proof. We equate the limits of both sides of (1.9) as $s \to n$. First, we note that the left-hand side of (1.9) in the limit as $s \to n$ $(n > 1)$ becomes

$$q^{-n} \sum_{a=1}^{q-1} e^{2\pi i \frac{a}{q} p} \zeta\left(n, \frac{a}{q}\right),$$

i.e. the left-hand side of (2.2). While if $n = 1$, by the orthogonality

$$\sum_{a=1}^{q} e^{2\pi i \frac{a}{q}} = 0,$$

the limit is

$$-q^{-1} \sum_{a=1}^{q} e^{2\pi i \frac{a}{q}} \psi\left(\frac{a}{q}\right),$$

i.e. the left-hand side of (2.8). To take the limit of the right-hand side of (1.9), we recall

$$\Gamma(1-s) = \frac{\frac{(-1)^n}{(n-1)!}}{s-n} + \frac{(-1)^{n-1}}{(n-1)!} \psi(n) + O(s-n), \quad s \to n. \tag{3.1}$$

Using (3.1) and

$$\zeta(1-s, x) = -\frac{B_n(x)}{n} + \gamma_n(x)(s-n) + \cdots, \tag{3.2}$$

we have

$$\begin{aligned} \Gamma(1-s) e^{-\frac{\pi i s}{2}} \zeta(1-s, x) &= \frac{1}{s-n} \frac{(-1)^n}{(n-1)!} (-i)^n \left(-\frac{B_n(x)}{n}\right) + (-i)^n \\ &\times \left\{ \frac{(-1)^{n-1}}{(n-1)!} \psi(n) \left(-\frac{B_n(x)}{n}\right) - \frac{\pi i}{2} \frac{(-1)^n}{(n-1)!} \left(-\frac{B_n(x)}{n}\right) + \frac{(-1)^n}{(n-1)!} \gamma_n(x) \right\} \end{aligned} \tag{3.3}$$

and correspondingly

$$\begin{aligned} \Gamma(1-s) e^{\frac{\pi i s}{2}} \zeta(1-s, 1-x) &= \frac{1}{s-n} \frac{(-1)^n}{(n-1)!} i^n \left(-\frac{B_n(x)}{n}\right) + i^n \\ &\times \left\{ \frac{(-1)^{n-1}}{(n-1)!} \psi(n) \left(-\frac{B_n(x)}{n}\right) + \frac{\pi i}{2} \frac{(-1)^n}{(n-1)!} \left(-\frac{B_n(1-x)}{n}\right) + \frac{(-1)^n}{(n-1)!} \gamma_n(x) \right\}, \end{aligned} \tag{3.4}$$

whence

$$\begin{aligned} &\lim_{s \to n} i \frac{1}{(2\pi)^{1-s}} \Gamma(1-s) \left\{ e^{-\frac{\pi i s}{2}} \zeta(1-s, x) - e^{\frac{\pi i s}{2}} \zeta(1-s, 1-x) \right\} \\ &= \frac{i^{n+1}}{(2\pi)^{1-n} (n-1)!} \left\{ \pi i \frac{B_n(x)}{n} + \gamma_n(x) - (-1)^n \gamma_n(1-x) \right\}; \end{aligned} \tag{3.5}$$

here $x = \frac{p}{q}$. Now (3.5) leads to the first equality of (2.2), and the second equality follows by distinguishing the parity of m. $\square$

Proof of Corollary 4. The right-hand side of (3.5) gives (as $s \to 1$)

$$-\pi i B_1\left(\frac{p}{q}\right) + \zeta'\left(0, \frac{p}{q}\right) + \zeta'\left(0, 1 - \frac{p}{q}\right).$$

Using (2.7) and the reciprocal relation, we get $(0 < x < 1)$

$$\begin{aligned}\zeta'(0,x) + \zeta'(0,1-x) &= \log \frac{\Gamma(x)\Gamma(1-x)}{2\pi} \\ &= \log \frac{1}{2\pi \sin \pi x}.\end{aligned}$$

Writing

$$2\sin \pi x = i\left(e^{\pi i x} - e^{-\pi i x}\right) = -ie^{-\pi i x}\left(e^{-\pi i x} - e^{2\pi i x}\right),$$

we see that

$$\zeta'(0,x) + \zeta'(0,1-x) = \pi i\left(x - \frac{1}{2}\right) - \log\left(1 - e^{2\pi i x}\right) = \pi i \overline{B_1}(x).$$

Hence, the limit of the right-hand side of (1.8) is

$$-\log\left(1 - e^{2\pi i \frac{p}{q}}\right).$$

Comparing these limit values, we conclude (2.8). □

Remark 3. Integrating [Mil, (25)], we obtain

$$\gamma_2(x) - \gamma_2(1-x) = -\int_1^x \log 2\sin \pi t dt + C.$$

Similarly, as in [Mil, p. 313], each γ_n is essentially an indefinite integral of γ_{n-1} up to a constant factor and a Bernoulli polynomial summand, and so the odd part $\gamma_n(x) - \gamma_n(1-x)$ is essentially the Clausen function mentioned in §1 (cf. [K, p. 469]).

Thus, we see that Theorem 2 captures the behavior of the Clausen functions, which escapes the seizure of Corollary 1. Indeed, $\gamma_n(x)$ are iterated integrals of $\log \Gamma(x)$ and therefore are closely connected to Barnes' multiple gamma functions.

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