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# MORPHISMS OF CLOSED RIEMANN SURFACES AND LEFSCHETZ TRACE FORMULA

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Abstract. We study the number of coincidences of two distinct morphisms $f_i : X \to Y\,(i = 1, 2)$ between closed Riemann surfaces of genera greater than zero. We give a necessary and sufficient condition for the existence of a coincidence in terms of the inner product defined on the free abelian group of homomorphisms between the Jacobian varieties $\mathrm{J}(X)$ and $\mathrm{J}(Y)$. We use the Hodge decomposition and the holomorphic Lefschetz number to study the number of coincidences in detail.

## 1. Introduction

The Lefschetz fixed-point formula has a beautiful form. It asserts that we can evaluate the number of fixed points of an endomorphism, properly counted, in terms of the action of the induced homomorphism on the cohomology. In particular, when $T$ is an automorphism of a closed Riemann surface, there is no assumption for the fixed points, and the Lefschetz number $L(T)$ is simply equal to the number of the fixed points. Y. Fuertes and G. González-Diez (see [3]) applied the Lefschetz fixed-point formula to study the number of *coincidences*, that is, the number of points $p \in X$ with $f_1(p) = f_2(p)$ for two distinct morphisms $f_i : X \to Y\,(i = 1, 2)$ between closed Riemann surfaces. They gave a sharp bound for the number of coincidences of two morphisms.

**Theorem 1.1** (Fuertes and González-Diez)**.** *Let $f_i : X \to Y$ be two distinct morphisms of degree $d_i\,(i = 1, 2)$ between closed Riemann surfaces of genera $g$ and $\gamma$, respectively, and let $L(f_1, f_2)$ denote the number of coincidences appropriately counted. We have*

*i) $L(f_1, f_2) \leq d_1 + 2\gamma\sqrt{d_1 d_2} + d_2$.*

*ii) In case $\gamma \geq 2$, this bound is attained if and only if $Y$ is hyperelliptic and $f_2 = J \circ f_1$, where $J$ denotes the hyperelliptic involution of $Y$.*

Their results generalize the well known fact concerning the number of fixed points of automorphisms; namely, $L(T) \leq 2g+2$ for an arbitrary $T \in \mathrm{Aut}(X)$. More recently, Y. Fuertes [2] showed several results concerning the number of coincidences by composing morphisms with meromorphic functions on the target $Y$.

In this paper, we will study the case where there exists no coincidence of two distinct morphisms $f_i : X \to Y\,(i = 1, 2)$ between closed Riemann surfaces of

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genera greater than zero, namely, the case where $L(f_1, f_2) = 0$. Roughly speaking, trace $(f_1^* \circ f_{2*}|_{H^1_{DR}(X)})$ defines an inner product on the space of morphisms of $X$ to $Y$, where $f_* : H^1_{DR}(X) \to H^1_{DR}(Y)$ is defined by the property $\int_Y f_* v \wedge w = \int_X v \wedge f^* w$, for any $w \in H^1_{DR}(Y)$. We will give a necessary and sufficient condition for $L(f_1, f_2) = 0$ in terms of the inner product. This is our main theorem numbered Theorem 4.1. Taking the contraposition of Theorem 4.1, we have a necessary and sufficient condition for the existence of a coincidence. This is Corollary 4.2. We will also show that if $Y$ is a torus, then $L(f_1, f_2) = 0$ holds if and only if the difference between $f_1$ and $f_2$ is only a translation on the torus $Y$. This is Corollary 4.3.

The method of the proof is to use the Hodge decomposition and the holomorphic Lefschetz number to study $L(f_1, f_2)$ in detail.

## 2. Notation, definitions and prior results

Throughout the following, all of the Riemann surfaces are closed and of genera $\geq 1$.

We follow the notation of [3] and [4]. Let $T \in \mathrm{Aut}(X)$ and let $\Gamma_T = \{(p, T(p))\} \subset X \times X$ be the graph of $T$. A fixed point of $T$ corresponds to a point of intersection of the graph $\Gamma_T$ and the diagonal submanifold $\Delta \subset X \times X$. The *Lefschetz number* of $T$ is defined to be

$$L(T) = \sharp(\Delta \cdot \Gamma_T).$$

By using the integral,

$$L(T) = \int_{\Gamma_T} \varphi_\Delta = \int_X (id. \times T)^* \varphi_\Delta,$$

where $\varphi_\Delta \in H^2_{DR}(X \times X)$ is a closed form representing the cohomology class Poincaré dual to the class of $\Delta$. For each $q$ let

$$\{\psi_{q,\mu}\}$$

be a collection of closed $q$-forms representing a basis for $H^q_{\mathrm{DR}}(X)$, and let

$$\{\psi^*_{2-q,\mu}\}$$

be closed forms representing the dual basis for $H^{2-q}_{\mathrm{DR}}(X)$, i.e., such that

$$\int_X \psi_{q,\mu} \wedge \psi^*_{2-q,\nu} = \delta_{\mu,\nu}.$$

Let $\pi_1$ and $\pi_2$ denote the two projection maps $X \times X \to X$. Then one has

$$\varphi_\Delta = \sum_q (-1)^q \sum_\mu \pi_1^* \psi_{q,\mu} \wedge \pi_2^* \psi^*_{2-q,\mu}.$$

Thus we can evaluate the Lefschetz number by

$$\begin{aligned} L(T) &= \int_{\Gamma_T} \varphi_\Delta = \int_X (id. \times T)^* \varphi_\Delta \\ &= \sum_q (-1)^q \sum_\mu \int_X \psi_{q,\mu} \wedge T^* \psi^*_{2-q,\mu} = \sum_{k=0}^{k=2} (-1)^k \mathrm{trace}\,(T^*|_{H^k_{DR}(X)}), \end{aligned} \tag{2.1}$$

where $k = 2 - q$. The obtained formula,

$$L(T) = \sum_{k=0}^{k=2} (-1)^k \mathrm{trace}\,(T^*|_{H^k_{DR}(X)}),$$

is the so-called Lefschetz trace formula (for the two-dimensional case).

Let $f_i : X \to Y$ be two distinct morphisms of degree $d_i$ $(i = 1, 2)$ between closed Riemann surfaces of genera $g$ and $\gamma$, respectively. For two distinct morphisms $f_i$ $(i = 1, 2)$, we define the Lefschetz number as follows.

**Definition 2.1.** The Lefschetz number of two distinct morphisms $f_i : X \to Y$ $(i = 1, 2)$ is defined to be

$$L(f_1, f_2) = \int_X (f_1 \times f_2)^* \varphi_\Delta,$$

where $\varphi_\Delta \in H^2_{DR}(Y \times Y)$ is the Poincaré dual of the diagonal $\Delta \subset Y \times Y$.

Thus denoting

$$\Gamma_{f_1,f_2} = \{(f_1(p), f_2(p))\} \subset Y \times Y,$$

we have

$$L(f_1, f_2) = \sharp(\Delta \cdot \Gamma_{f_1,f_2}) = \int_{\Gamma_{f_1,f_2}} \varphi_\Delta.$$

**Definition 2.2.** Let $f : X \to Y$ be a morphism between Riemann surfaces. We define a linear map

$$f_* : H^k_{DR}(X) \to H^k_{DR}(Y)$$

by the property

$$\int_Y f_* v \wedge w = \int_X v \wedge f^* w,$$

for any $w \in H^{2-k}_{DR}(Y)$.

Then the analogue to (2.1) takes the form

$$\begin{aligned} L(f_1, f_2) &= \int_X (f_1 \times f_2)^* \varphi_\Delta = \sum_q (-1)^q \sum_\mu \int_X f_1^* \psi_{q,\mu} \wedge f_2^* \psi^*_{2-q,\mu} \\ &= \sum_q (-1)^q \sum_\mu \int_Y f_{2*} \circ f_1^* \psi_{q,\mu} \wedge \psi^*_{2-q,\mu} \\ &= \sum_{q=0}^{q=2} (-1)^q \mathrm{trace}\,(f_{2*} \circ f_1^*|_{H^q_{DR}(Y)}) = \sum_{k=0}^{k=2} (-1)^k \mathrm{trace}\,(f_1^* \circ f_{2*}|_{H^k_{DR}(X)}), \end{aligned} \tag{2.2}$$

where we use the same symbols $\{\psi_{q,\mu}\}$ and $\{\psi^*_{2-q,\mu}\}$ for the basis for $H^q_{\mathrm{DR}}(Y)$ and for the dual basis for $H^{2-q}_{\mathrm{DR}}(Y)$, respectively. The last equality comes from the fact that for any two matrices $A$ and $B$, the traces of $AB$ and $BA$ agree whenever the two products make sense.

Observing (2.2), we easily have

$$L(f_1, f_2) = L(f_2, f_1).$$

Y. Fuertes and G. González-Diez [3] showed the following lemma.

**Lemma 2.3.** *i)* $f_1^* \circ f_{2*} : H^0(X) \to H^0(X)$ *is multiplication by* $d_2$.
*ii)* $f_1^* \circ f_{2*} : H^2(X) \to H^2(X)$ *is multiplication by* $d_1$.

Thus by Lemma 2.3, the Lefschetz trace formula is written as

$$\begin{aligned} L(f_1, f_2) &= \sum_{k=0}^{k=2} (-1)^k \mathrm{trace}\,(f_1^* \circ f_{2*}|_{H^k_{DR}(X)}) \\ &= d_1 - \mathrm{trace}\, f_1^* \circ f_{2*}|_{H^1_{DR}(X)} + d_2. \end{aligned} \tag{2.3}$$

**Definition 2.4.** Let $p \in X$ be a coincidence of $f_1$ and $f_2$, and let

$$f_1(z) - f_2(z) = c_k z^k + c_{k+1} z^{k+1} + \cdots, \quad c_k \neq 0$$

be the Taylor expansion of $f_1 - f_2$ with respect to small parametric discs $D$ around $p$ and $D'$ around $f_i(p)$. We define the multiplicity of $f_1$, $f_2$ at $p$ to be

$$m_p(f_1,\, f_2) = k.$$

By the definition, $m_p(f_1,\, f_2)$ is always positive. Furthermore, one can show that (see [3])

$$L(f_1, f_2) = \sum_{\{p \in X; f_1(p) = f_2(p)\}} m_p(f_1,\, f_2).$$

Thus $L(f_1, f_2)$ is always greater than or equal to the actual number of coincidences.

It is known that $\langle f_1, f_2 \rangle = \operatorname{trace} f_1^* \circ f_{2*}|_{H^1_{DR}(X)}$ gives an inner product on the free abelian group of homomorphisms between the Jacobians $\mathrm{J}(X)$ and $\mathrm{J}(Y)$. To see this, first, we will recall some notions from complex tori (for details, see e.g. [5]). Let $V$ be a complex vector space of dimension $n$ and $\Gamma$ a lattice in $V$. The quotient $T = V/\Gamma$ is called a complex torus of dimension $n$. Denote by $\widehat{T} = V^*/\widehat{\Gamma}$ the dual, where $V^*$ is the space of $\mathbb{C}$-antilinear functionals on $V$ and $\widehat{\Gamma} = \{l \in V^* : \operatorname{Im} l(\Gamma) \subseteq \mathbb{Z}\}$ is the dual lattice of $\Gamma$.

Let $T = V/\Gamma$ and $T' = V'/\Gamma'$ be two complex tori. A *homomorphism* of $T$ to $T'$ is a holomorphic map $\mathfrak{f} : T \to T'$ compatible with the group structures. The *translation* by an element $x_0 \in T$ is defined to be the holomorphic map $t_{x_0} : T \to T$, $x \mapsto x + x_0$.

**Lemma 2.5.** *Let $h : T \to T'$ be a holomorphic map.*

*i) There is a unique homomorphism $\mathfrak{f} : T \to T'$ such that $h = t_{h(0)} \circ \mathfrak{f}$, i.e., $h(x) = \mathfrak{f}(x) + h(0)$ for all $x \in T$ .*

*ii) There is a unique $\mathbb{C}$-linear map $F : V \to V'$ with $F(\Gamma) \subseteq \Gamma'$ inducing the homomorphism $\mathfrak{f}$.*

We call $F$ *the analytic representation* of $\mathfrak{f}$. The restriction $F|_\Gamma$ is $\mathbb{Z}$-linear. $F|_\Gamma$ determines $F$ and $\mathfrak{f}$ completely. Thus we have an injective homomorphism

$$\rho_r : \operatorname{Hom}(T, T') \to \operatorname{Hom}_{\mathbb{Z}}(\Gamma, \Gamma'), \quad \mathfrak{f} \mapsto F|_\Gamma,$$

*the rational representation* of $\operatorname{Hom}(T, T')$. For the analytic representation $F : V \to V'$ of a homomorphism $\mathfrak{f} : T \to T'$, the dual map ${}^tF : V'^* \to V^*$ associating to an antilinear functional $l \in V'^*$ the antilinear functional $l \circ F \in V^*$ induces a homomorphism ${}^t\mathfrak{f} : \widehat{T}' \to \widehat{T}$, since ${}^tF(\widehat{\Gamma}') \subseteq \widehat{\Gamma}$. We call ${}^t\mathfrak{f}$ *the dual map* of $\mathfrak{f}$.

Let $X$ and $Y$ be closed Riemann surfaces of genera $g$ and $\gamma$, respectively. Denote by $\mathcal{H}$ the space of holomorphic differentials on $X$. Set $\Omega = \operatorname{Hom}(\mathcal{H}, \mathbb{C})$. The Jacobian variety $\mathrm{J}(X) := \Omega/H_1(X, \mathbb{Z})$ is a complex torus of dimension $g$, and considering $\overline{\mathcal{H}}$ of $\mathbb{C}$-antilinear forms on $\Omega$, we denote by $\widehat{\mathrm{J}(X)} = \overline{\mathcal{H}}/H^1(X, \mathbb{Z})$ the dual.

There is a canonical principal polarization on $\mathrm{J}(X)$. Fix a homology basis $\lambda_1, \ldots, \lambda_{2g}$ of $H_1(X, \mathbb{Z})$ with an intersection matrix (that is, a matrix whose $(k, j)$-entry is given by the intersection number $\lambda_k \cdot \lambda_j$),

$$J_g = \begin{pmatrix} 0 & I \\ -I & 0 \end{pmatrix},$$

where each entry is $g \times g$ sized and $I$ is the identity matrix. Considered as a $\mathbb{R}$-vector space, $\lambda_1, \ldots, \lambda_{2g}$ is a basis for $\Omega$. Denote by $E_X$ the alternating form on $\Omega$ with

the matrix $J_g$ with respect to the basis $\lambda_1, \ldots, \lambda_{2g}$ for $\Omega$ and define a hermitian form $H : \Omega \times \Omega \to \mathbb{C}$ by

$$H(u, v) = E_X(iu, v) + iE_X(u, v).$$

We denote by $\langle \zeta, v \rangle$, $\zeta \in \overline{\mathcal{H}}, v \in \Omega$ the value of $\zeta$ at $v$. The $\mathbb{C}$-linear map $\phi_E : \Omega \to \overline{\mathcal{H}}$ such that $H(u, v) = \langle \phi_{E_X}(u), v \rangle$ induces a homomorphism, also denoted by $\phi_E$,

$$\phi_{E_X} : \mathrm{J}(X) \to \widehat{\mathrm{J}(X)}.$$

Then $-J_g = J_g^{-1}$ is the rational representation of $\phi_{E_X}$ with respect to the homology basis $\lambda_1, \ldots, \lambda_{2g}$ for $H_1(X, \mathbb{Z})$ above and the dual basis $\alpha_1, \ldots, \alpha_{2g}$ for $H^1(X, \mathbb{Z})$, i.e., $\alpha_j(\lambda_k) = \delta_{jk}$. Similarly we define $\phi_{E_Y} : \mathrm{J}(Y) \to \widehat{\mathrm{J}(Y)}$ for $Y$.

Choose a basis $\{\omega_i\}$ for $\mathcal{H}$. Let $\alpha_{p_0} : X \to \mathrm{J}(X)$ be the embedding defined by $\alpha_{p_0}(p) = {}^t(\int_{p_0}^p \omega_1, \ldots, \int_{p_0}^p \omega_g)$. For a morphism $f : X \to Y$, there exists a homomorphism $\mathfrak{f} : \mathrm{J}(X) \to \mathrm{J}(Y)$ such that

$$\alpha_{f(p_0)} \circ f = \mathfrak{f} \circ \alpha_{p_0}$$

holds. Furthermore, the homomorphism $\mathrm{H}_1(X, \mathbb{Z}) \to \mathrm{H}_1(Y, \mathbb{Z})$ induced by $f$ and $\rho_r(\mathfrak{f})$, the rational representation for $\mathfrak{f}$, are the same. Martens [6] showed the following.

**Theorem 2.6** (Martens)**.** *Let $X$, $Y_1$ and $Y_2$ be closed Riemann surfaces of genera $\geq 1$ and let $f_i : X \to Y_i\,(i = 1, 2)$ be morphisms. Assume that there exists a homomorphism $H$ of the first homology groups from $H_1(Y_1, \mathbb{Z})$ onto $H_1(Y_2, \mathbb{Z})$ which commutes with the induced homomorphisms $f_{i*} : H_1(X, \mathbb{Z}) \to H_1(Y_i, \mathbb{Z})\,(i = 1, 2)$, i.e. $f_{2*} = H \circ f_{1*}$. Then there exists a unique (modulo a translation in genus 1) morphism $h : Y_1 \to Y_2$ with $f_2 = h \circ f_1$.*

Thus we have

**Lemma 2.7.** *Let $f_i : X \to Y\,(i = 1, 2)$ be two morphisms between Riemann surfaces and let $\mathfrak{f}_i : \mathrm{J}(X) \to \mathrm{J}(Y)$ be induced homomorphisms. Suppose that $\mathfrak{f}_1 = \mathfrak{f}_2$ holds. Then $f_1 = f_2$ (modulo a translation in genus 1) holds.*

Under addition the set of homomorphisms $\mathrm{Hom}(\mathrm{J}(X), \mathrm{J}(Y))$ forms a free abelian group of rank $\leq 4g\gamma$. Let $\mathfrak{f}, \mathfrak{g} \in \mathrm{Hom}(\mathrm{J}(X), \mathrm{J}(Y))$.

**Definition 2.8.** The adjoint of ${}^t\mathfrak{f}$ is denoted by ${}^t\mathfrak{f}'$ and defined by

$${}^t\mathfrak{f}' = \phi_{E_Y} \circ \mathfrak{f} \circ \phi_{E_X}^{-1}.$$

By definition, ${}^t\mathfrak{f}' \in \mathrm{Hom}(\widehat{\mathrm{J}(X)}, \widehat{\mathrm{J}(Y)})$ and ${}^t\mathfrak{f} \circ {}^t\mathfrak{g}' \in \mathrm{End}(\widehat{\mathrm{J}(X)})$. Furthermore, the trace of the rational representation of ${}^t\mathfrak{f} \circ {}^t\mathfrak{g}'$ defines an inner product on $\mathrm{Hom}(\mathrm{J}(X), \mathrm{J}(Y))$ (see [7]).

**Definition 2.9.** Let $\mathfrak{f}, \mathfrak{g} \in \mathrm{Hom}(\mathrm{J}(X), \mathrm{J}(Y))$. Then define an inner product on $\mathrm{Hom}(\mathrm{J}(X), \mathrm{J}(Y))$ by

$$\langle \mathfrak{f}, \mathfrak{g} \rangle = \mathrm{trace}\, \rho_r({}^t\mathfrak{f} \circ {}^t\mathfrak{g}').$$

Put

$$\|\mathfrak{f}\| = \langle \mathfrak{f}, \mathfrak{f} \rangle^{1/2}$$

and

$$\cos(\mathfrak{f}, \mathfrak{g}) = \frac{\langle \mathfrak{f}, \mathfrak{g} \rangle}{\|\mathfrak{f}\| \cdot \|\mathfrak{g}\|}$$

as usual.

If we choose bases as above, then the matrix representation for $\rho_r({}^t\mathfrak{f} \circ {}^t\mathfrak{g}')$ is of the form ${}^tMJ_\gamma^{-1}NJ_g$, where $M$ and $N$ are the matrix representations for the homomorphisms $\mathrm{H}_1(X,\mathbb{Z}) \to \mathrm{H}_1(Y,\mathbb{Z})$ induced by $\mathfrak{f}$ and by $\mathfrak{g}$, respectively. Thus in particular if $\mathfrak{f}_1$ and $\mathfrak{f}_2$ are induced by morphisms between Riemann surfaces $f_i : X \to Y\ (i = 1, 2)$, then it is easy to see that

$$\operatorname{trace} \rho_r({}^t\mathfrak{f}_1 \circ {}^t\mathfrak{f}_2') = \operatorname{trace}(f_1^* \circ f_{2*}|_{H^1_{DR}(X)}) \tag{2.4}$$

holds. Denoting by $d_i$ the degree of $f_i\ (i = 1, 2)$, we also have

$$\|\mathfrak{f}_i\| = \sqrt{2d_i\gamma} \quad (i = 1, 2) \tag{2.5}$$

by an easy calculation.

## 3. The holomorphic Lefschetz number

In this section, we follow the notation of Chapter 3.4 of [4]. A morphism $f : X \to Y$ acts not only on the de Rham cohomology groups but on the Dolbeault cohomology groups. Let $M$ be a compact Kähler manifold. By the Hodge decomposition,

$$H^r(M,\mathbb{C}) \cong \bigoplus_{p+q=r} H^{p,q}_{\overline{\partial}}(M)$$

$$H^{p,q}_{\overline{\partial}}(M) = \overline{H^{q,p}_{\overline{\partial}}(M)}.$$

Thus, for a Riemann surface $X$,

$$H^1(X,\mathbb{C}) \cong H^{1,0}_{\overline{\partial}}(X) \oplus H^{0,1}_{\overline{\partial}}(X)$$

holds, where we may identify $H^{1,0}_{\overline{\partial}}(X)$ with the space of holomorphic 1-forms, $H^{0,1}_{\overline{\partial}}(X)$ being the complex conjugate of $H^{1,0}_{\overline{\partial}}(X)$. $H^0$ and $H^2$ are trivial in this case. Now the Lefschetz number $L(f_1, f_2)$ is written as

$$L(f_1, f_2) = \int_X (f_1 \times f_2)^*\varphi_\Delta = \sum_{p,q} (-1)^{p+q} \operatorname{trace}(f_1^* \circ f_{2*}|_{H^{p,q}_{\overline{\partial}}(X)}).$$

Let $\pi_1$ and $\pi_2$ denote the two projection maps $Y \times Y \to Y$. For each $p$ and $q$ let

$$\{\psi_{p,q,\mu}\}$$

be a collection of $\overline{\partial}$-closed $(p, q)$-forms representing a basis for $H^{p,q}_{\overline{\partial}}(Y)$, and let

$$\{\psi^*_{1-p,1-q,\mu}\}$$

be $\overline{\partial}$-closed forms representing the dual basis for $H^{1-p,1-q}_{\overline{\partial}}(Y)$ under the pairing

$$H^{p,q}_{\overline{\partial}}(Y) \otimes H^{1-p,1-q}_{\overline{\partial}}(Y) \to \mathbb{C}$$

given by

$$\psi \otimes \varphi \mapsto \int_Y \psi \wedge \varphi.$$

A basis for $H^{1,1}_{\overline{\partial}}(Y \times Y)$ is represented by the forms

$$\{\varphi_{p,q,\mu,\nu} = \pi_1^*\psi_{p,q,\mu} \wedge \pi_2^*\psi^*_{1-p,1-q,\nu}\},$$

and the dual basis for $H^{1,1}_{\overline{\partial}}(Y \times Y)$ is represented by

$$\{\varphi^*_{1-p,1-q,\mu,\nu} = \pi_1^*\psi^*_{1-p,1-q,\mu} \wedge \pi_2^*\psi_{p,q,\nu}\}.$$

The Dolbeault class of the diagonal is represented by the form

$$\varphi_\Delta = \sum_{p,q,\mu} (-1)^{p+q} \varphi_{p,q,\mu,\mu}.$$

Set

$$\varphi_\Delta^0 = \sum_{q,\mu} (-1)^q \varphi_{0,q,\mu,\mu} = \varphi_{0,0} - \varphi_{0,1,\mu,\mu}.$$

Integrating $\varphi_\Delta^0$ over $\Gamma_{f_1,f_2}$, we have

$$\int_{\Gamma_{f_1,f_2}} \varphi_\Delta^0 = \int_X (f_1 \times f_2)^* \varphi_\Delta^0 = \int_X \sum_{q,\mu} (-1)^q f_1^* \psi_{0,q,\mu} \wedge f_2^* \psi_{1,1-q,\mu}^*$$

$$= \int_Y \sum_{q,\mu} (-1)^q f_{2*} \circ f_1^* \psi_{0,q,\mu} \wedge \psi_{1,1-q,\mu}^* = \sum_{q=0}^{1} (-1)^q \operatorname{trace} f_{2*} \circ f_1^*|_{H_{\bar\partial}^{0,q}(Y)}$$

$$= \sum_{q=0}^{1} (-1)^q \operatorname{trace} f_1^* \circ f_{2*}|_{H_{\bar\partial}^{0,q}(X)}.$$

The last equality comes from the fact that for any two matrices $A$ and $B$, the trace of $AB$ and $BA$ agree whenever the two products make sense.

**Definition 3.1.** The number $\sum_{q=0}^{1} (-1)^q \operatorname{trace} f_1^* \circ f_{2*}|_{H_{\bar\partial}^{0,q}(X)}$ is called the holomorphic Lefschetz number of $(f_1, f_2)$ and is denoted $L(f_1, f_2, \mathcal{O})$.

By Lemma 2.3 i), we see that

$$L(f_1, f_2, \mathcal{O}) = d_2 - \operatorname{trace} f_1^* \circ f_{2*}|_{H_{\bar\partial}^{0,1}(X)}. \tag{3.1}$$

We also have

$$L(f_2, f_1, \mathcal{O}) = d_1 - \operatorname{trace} f_1^* \circ f_{2*}|_{H_{\bar\partial}^{1,0}(X)} \tag{3.2}$$

since

$$\int_X (f_2 \times f_1)^* \varphi_\Delta^0 = \int_X \sum_{q,\mu} (-1)^q f_2^* \psi_{0,q,\mu} \wedge f_1^* \psi_{1,1-q,\mu}^*$$

$$= \int_Y \sum_{q,\mu} (-1)^q \psi_{0,q,\mu} \wedge f_{2*} \circ f_1^* \psi_{1,1-q,\mu}^* = \sum_{q=0}^{1} (-1)^{1-q} \operatorname{trace} f_{2*} \circ f_1^*|_{H_{\bar\partial}^{1,q}(Y)}$$

$$= \sum_{q=0}^{1} (-1)^{1-q} \operatorname{trace} f_1^* \circ f_{2*}|_{H_{\bar\partial}^{1,q}(X)}.$$

Summing (3.1) and (3.2), we have

$$L(f_1, f_2) = L(f_1, f_2, \mathcal{O}) + L(f_2, f_1, \mathcal{O}).$$

We fix some additional notation. Let $A^{p,q}(X)$ denote the space of differential forms of type $(p,q)$, and let $A^{(p_1,q_1),(p_2,q_2)}(X \times X)$ denote the space of differential forms of bitype $(p_1,q_1),(p_2,q_2)$, where $(p_1,q_1)$ and $(p_2,q_2)$ come from the first and the second factor of the product $X \times X$, respectively. We have the decomposition of forms on $X \times X$ into bitype

$$A^{p,q}(X \times X) = \bigoplus_{\substack{p_1+p_2=p \\ q_1+q_2=q}} A^{(p_1,q_1),(p_2,q_2)}(X \times X). \tag{3.3}$$

## 4. The main theorem

Now we show

**Theorem 4.1.** *Let $f_i : X \to Y$ be two distinct morphisms of degree $d_i\,(i = 1, 2)$ between closed Riemann surfaces of genera $g$ and $\gamma \geq 1$, respectively. Let $L(f_1, f_2)$ denote the Lefschetz number and let $\mathfrak{f}_i$ be the homomorphisms of Jacobian varieties induced by $f_i\,(i = 1, 2)$. Then*

$$L(f_1, f_2) = 0 \Longleftrightarrow \begin{cases} d_1 = d_2, \\ \cos(\mathfrak{f}_1, \mathfrak{f}_2) = \gamma^{-1}. \end{cases}$$

*Proof.* Suppose $L(f_1, f_2) = 0$. We will show that $d_1 = d_2$ first. Let $\psi_{\Gamma_{f_1,f_2}}$ be a closed form representing the cohomology class Poincaré dual to the class of $\Gamma_{f_1,f_2}$. By the localization principle (see [1], p. 67), the support of $\psi_{\Gamma_{f_1,f_2}}$ can be shrunk into any given tubular neighborhood of $\Gamma_{f_1,f_2}$. Using the decomposition (3.3), we can write

$$\psi_{\Gamma_{f_1,f_2}} = \sum_{p_1+p_2+q_1+q_2=2} \psi^{(p_1,q_1),(p_2,q_2)}_{\Gamma_{f_1,f_2}},$$

and we see that each term of the sum is supported by a subset of the support of $\psi_{\Gamma_{f_1,f_2}}$. By the assumption $L(f_1, f_2) = 0$, there exists no correspondence. Thus taking the support of $\psi_{\Gamma_{f_1,f_2}}$ sufficiently small, we have

$$\begin{aligned} \int_{\Gamma_{f_1,f_2}} \varphi^0_\Delta &= \int_{Y\times Y} \varphi^0_\Delta \wedge \psi_{\Gamma_{f_1,f_2}} = \int_{Y\times Y} \varphi^0_\Delta \wedge \sum_q \psi^{(1,1-q),(0,q)}_{\Gamma_{f_1,f_2}} \\ &= \int_{Y\times Y} \varphi_\Delta \wedge \sum_q \psi^{(1,1-q),(0,q)}_{\Gamma_{f_1,f_2}} = \int_\Delta \sum_q \psi^{(1,1-q),(0,q)}_{\Gamma_{f_1,f_2}} \\ &= 0. \end{aligned}$$

By equations (3.1) and (3.2), we see that

$$0 = L(f_1, f_2, \mathcal{O}) = d_2 - \mathrm{trace} f_1^* \circ f_{2*}|_{H^{0,1}_{\bar\partial}(X)}$$

and

$$0 = L(f_2, f_1, \mathcal{O}) = d_1 - \mathrm{trace} f_1^* \circ f_{2*}|_{H^{1,0}_{\bar\partial}(X)}.$$

On the other hand,

$$\mathrm{trace} f_1^* \circ f_{2*}|_{H^{1,0}_{\bar\partial}(X)} = \overline{\mathrm{trace} f_1^* \circ f_{2*}|_{H^{0,1}_{\bar\partial}(X)}},$$

and so we deduce the desired relation

$$d_1 = d_2.$$

We write $d$ for $d_1$ or $d_2$. Substituting these into (2.3) with (2.4), we have

$$2d = \mathrm{trace}\, f_1^* \circ f_{2*}|_{H^1_{DR}(X)} = \langle \mathfrak{f}_1, \mathfrak{f}_2 \rangle.$$

Thus we have

$$\cos(\mathfrak{f}_1, \mathfrak{f}_2) = \frac{\langle \mathfrak{f}_1, \mathfrak{f}_2 \rangle}{\|\mathfrak{f}_1\| \cdot \|\mathfrak{f}_2\|} = \frac{2d}{2d\gamma} = \gamma^{-1}$$

by (2.5). Conversely, suppose $d_1 = d_2$ and $\cos(\mathfrak{f}_1, \mathfrak{f}_2) = \gamma^{-1}$. Then

$$\langle \mathfrak{f}_1, \mathfrak{f}_2 \rangle = \|\mathfrak{f}_1\| \cdot \|\mathfrak{f}_2\| \cos(\mathfrak{f}_1, \mathfrak{f}_2) = 2d.$$

Substituting these into (2.3) again, we have $L(f_1, f_2) = 0$. □

Theorem 4.1 is equivalent to

**Corollary 4.2.** *Let $f_i : X \to Y$ be two distinct morphisms of degree $d_i\,(i = 1, 2)$ between closed Riemann surfaces of genera $g$ and $\gamma \geq 1$, respectively. Let $\mathfrak{f}_i$ be the homomorphisms of Jacobian varieties induced by $f_i\,(i = 1, 2)$. The following two conditions are equivalent:*

*1) There exists a coincidence.*

*2) $d_1 \neq d_2$ or $\cos(\mathfrak{f}_1, \mathfrak{f}_2) \neq \gamma^{-1}$.*

If the target $Y$ is a torus, we have

**Corollary 4.3.** *Let $f_i : X \to Y$ be two distinct morphisms of degree $d_i\,(i = 1, 2)$ between closed Riemann surfaces of genera $g$ and $\gamma = 1$, respectively. The following two conditions are equivalent:*

*1) $L(f_1, f_2) = 0$.*

*2) The difference between $f_1$ and $f_2$ is only a translation on the torus $Y$.*

*Proof.* By Theorem 4.1 with $\gamma = 1$, we have

$$L(f_1, f_2) = 0 \Longleftrightarrow \begin{cases} d_1 = d_2, \\ \cos(\mathfrak{f}_1, \mathfrak{f}_2) = 1. \end{cases}$$

The right-hand side is equivalent to

$$\mathfrak{f}_1 = \mathfrak{f}_2$$

by (2.5), and this holds if and only if the difference between $f_1$ and $f_2$ is only a translation on the torus $Y$. Indeed, the "if" part is trivial and the "only if" part follows from Lemma 2.7. □

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