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ON ADDITIVE COMPLEMENTS

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Abstract. Two infinite sequences A and B of non-negative integers are called additive complements if their sum contains all sufficiently large integers. Let $A(x)$ and $B(x)$ be the counting functions of A and B. For additive complements A and B, Sárközy and Szemerédi proved that if $\limsup_{x\to\infty} \frac{A(x)B(x)}{x} \le 1$, then $A(x)B(x) - x \to +\infty$. In this paper, we prove that for additive complements A and B, if $\limsup_{x\to\infty} \frac{A(x)B(x)}{x} < \frac{5}{4}$ or $\limsup_{x\to\infty} \frac{A(x)B(x)}{x} > 2$, then $A(x)B(x) - x \to +\infty$.

1. Introduction

Two infinite sequences A and B of non-negative integers are called additive complements if their sum contains all sufficiently large integers. Let $A(x)$ and $B(x)$ be the counting functions of A and B, namely,

$$A(x) = \sum_{\substack{a\le x\\ a\in A}} 1 \quad \text{and} \quad B(x) = \sum_{\substack{b\le x\\ b\in B}} 1.$$

Motivated by a problem of Hanani and Erdős [2], [3], Danzer [1] conjectured that for additive complements A and B, if

$$\limsup_{x\to\infty} \frac{A(x)B(x)}{x} \le 1,$$

then

$$A(x)B(x) - x \to +\infty \qquad \text{as} \qquad x \to +\infty. \tag{1}$$

(See also [4, p. 10], [5, p. 75] and [6].) In [8], Sárközy and Szemerédi proved this conjecture.

In this paper, we prove the following result.

Theorem. *For additive complements A and B, if*

$$\limsup_{x\to\infty} \frac{A(x)B(x)}{x} > 2$$

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or

$$\limsup_{x\to\infty} \frac{A(x)B(x)}{x} < \frac{5}{4},$$

then (1) *must hold.*

For the construction of additive complements A and B with $A(x)B(x) \sim x$ one may refer to [1] and [7].

2. Proof of the Theorem

Let $f(n)$ be the number of solutions of $a+b=n$, for $a \in A$, $b \in B$. For additive complements A and B, there exists a constant n_0 such that

$$f(n) \geq 1 \quad \text{for} \quad n > n_0.$$

Hence $A(x)B(x) \geq [x] - n_0$.

If (1) does not hold, then $-\infty < \liminf_{x\to\infty}(A(x)B(x) - x) < +\infty$. Assume that

$$\liminf_{x\to\infty}(A(x)B(x) - x) = L.$$

By the same arguments as in Sárközy and Szemerédi [8, p. 238], there exists an integer n_1 with

$$f(n) \equiv 1 \quad \text{for} \quad n \geq n_1.$$

Then

$$A(x)B(x) \leq \sum_{n\leq 2x} f(n) \leq \sum_{n\leq n_1} f(n) + (2x - n_1) = 2x + O(1).$$

Hence

$$\limsup_{x\to\infty} \frac{A(x)B(x)}{x} \leq 2.$$

Thus, if

$$\limsup_{x\to\infty} \frac{A(x)B(x)}{x} > 2,$$

then

$$\liminf_{x\to\infty}(A(x)B(x) - x) = +\infty.$$

Now we assume that the additive complements A and B satisfy

$$\limsup_{x\to\infty} \frac{A(x)B(x)}{x} = M.$$

Then $A(x)B(x) \leq x(M + o(1))$. Since A and B are infinite, we have $A(x) = o(x)$ and $B(x) = o(x)$.

Let $x_1 < x_2 < \cdots$ be all positive integers with $A(x_k)B(x_k) - x_k = L$, $b^{(k)}$ be the largest integer in $(A \cup B) \cap [0, x_k]$ and $y_k = x_k - b^{(k)}$. By the same discussion as in [8], we know that $y_k \to +\infty$. Since

$$\begin{aligned} x_k + L = A(x_k)B(x_k) &= A(b^{(k)})B(b^{(k)}) \\ &= A(x_k - y_k)B(x_k - y_k) \leq (M + o_k(1))(x_k - y_k), \end{aligned}$$

we have

$$y_k \leq (1 - \frac{1}{M})x_k + o_k(x_k).$$

As in [8] we know that

$$A(x_k) - A(y_k) \leq L + n_0$$

and

$$A(x_k) - A(2y_k) = 0, \quad k \geq k_0.$$

Define D, D_1, D_1^+ and D_1^- in the same way as in [8], namely,

$$D = \{(b,a) : b \in B, a \in A, b \leq x_k - y_k, a \leq x_k - y_k, b - a > y_k\},$$
$$D_1 = \{(b,a) : b \in B, a \in A, 2y_k < b \leq x_k - y_k, b - a > y_k\},$$
$$D_1^+ = \{(b,a) : b \in B, a \in A, 2y_k < b \leq x_k - y_k, a \leq 2y_k\},$$
$$D_1^- = \{(b,a) : b \in B, a \in A, 2y_k < b \leq x_k - y_k, a \leq 2y_k, a \geq b - y_k\}.$$

By $A(x_k) = A(2y_k)$ for $k \geq k_0$, we have $|D_1| = |D_1^+| - |D_1^-|$. As in [8], it suffices to show that for large k, $|D| > x_k - 2y_k$.

Fix an integer t with

$$t > \frac{1}{\frac{5}{4} - M}.$$

Let

$$\alpha_i = 1 + \frac{i-1}{t} \qquad (1 \leq i \leq t+1)$$

and

$$A(\alpha_i y_k)B(\alpha_i y_k) = M_i \alpha_i y_k.$$

Obviously, $1 - o_{t,i}(1) \leq M_i \leq M + o_{t,i}(1)$. For D_1^+ we have

$$\begin{aligned} |D_1^+| &= (B(x_k - y_k) - B(2y_k))A(2y_k) = B(b^{(k)})A(2y_k) - B(2y_k)A(2y_k) \\ &= B(x_k)A(x_k) - B(2y_k)A(2y_k) = (x_k + O_k(1)) - 2M_{t+1}y_k. \end{aligned}$$

For D_1^-, as in [8] we have $|D_1^-| = o(y_k)$. Hence

$$|D_1| = x_k - 2M_{t+1}y_k + o(y_k).$$

We redefine D_2 as

$$D_2 = \{(b,a) : b \in B, a \in A, b \leq 2y_k, b - a > y_k\}.$$

Obviously, $D_1 \cap D_2 = \emptyset$ and $D = D_1 \cup D_2$. Hence $|D| = |D_1| + |D_2|$.

Now we are going to estimate $|D_2|$. It follows from the definition of D_2 that

$$\begin{aligned} |D_2| &= \sum_{y_k < b \leq 2y_k} A(b - y_k) = \sum_{i=1}^{t} \sum_{\alpha_i y_k < b \leq \alpha_{i+1} y_k} A(b - y_k) \\ &\geq \sum_{i=1}^{t} A((\alpha_i - 1)y_k)(B(\alpha_{i+1}y_k) - B(\alpha_i y_k)). \end{aligned}$$

Since

$$A((\alpha_i - 1)y_k) \geq \frac{A((\alpha_i - 1)y_k)B((\alpha_i - 1)y_k)}{A(y_k)B(y_k)} A(y_k) \geq (\frac{\alpha_i - 1}{M} - o_k(1))A(y_k)$$

and

$$\begin{aligned} B(\alpha_{i+1}y_k) - B(\alpha_i y_k) &= \frac{B(\alpha_{i+1}y_k)A(y_k) - B(\alpha_i y_k)A(y_k)}{A(y_k)} \\ &= \frac{B(\alpha_{i+1}y_k)(A(\alpha_{i+1}y_k) + O_k(1)) - B(\alpha_i y_k)(A(\alpha_i y_k) + O_k(1))}{A(y_k)} \\ &= (M_{i+1}\alpha_{i+1} - M_i\alpha_i + o_k(1))\frac{y_k}{A(y_k)}, \end{aligned}$$

we have

$$\begin{aligned}
|D_2| &\geq \sum_{i=1}^{t}\left(\frac{\alpha_i-1}{M}(M_{i+1}\alpha_{i+1}-M_i\alpha_i)\right)y_k+o_t(y_k)\\
&= \frac{1}{M}\left(-\sum_{i=2}^{t+1}M_i\alpha_i(\alpha_i-\alpha_{i-1})+M_{t+1}\alpha_{t+1}(\alpha_{t+1}-1)\right)y_k+o_t(y_k)\\
&\geq \frac{1}{M}\left(-(M+o_t(1))\sum_{i=2}^{t+1}\alpha_i(\alpha_i-\alpha_{i-1})+2M_{t+1}\right)y_k+o_t(y_k)\\
&= \frac{1}{M}\left(-\frac{3}{2}M+2M_{t+1}-\frac{M}{2t}\right)y_k+o_t(y_k).
\end{aligned}$$

Thus by $t>\frac{1}{\frac{5}{4}-M}$ we have

$$\begin{aligned}
|D| &\geq x_k+\left(-2M_{t+1}-\frac{3}{2}+\frac{2M_{t+1}}{M}-\frac{1}{2t}\right)y_k+o_t(y_k)\\
&\geq x_k+\left(-2M_{t+1}(1-\frac{1}{M})-\frac{3}{2}-\frac{1}{2t}\right)y_k+o_t(y_k)\\
&\geq x_k+\left(-2(M+o_t(1))(1-\frac{1}{M})-\frac{3}{2}-\frac{1}{2t}\right)y_k+o_t(y_k)\\
&\geq x_k+\left(-2M+\frac{1}{2}-\frac{1}{2t}\right)y_k+o_t(y_k)>x_k-2y_k,\quad k>k_1.
\end{aligned}$$

By the same discussion as in [8], we obtain a contradiction, which completes the proof of the theorem.

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