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LAMPLIGHTER GRAPHS DO NOT ADMIT HARMONIC FUNCTIONS OF FINITE ENERGY

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Abstract. We prove that a lamplighter graph of a locally finite graph over a finite graph does not admit a non-constant harmonic function of finite Dirichlet energy.

1. Introduction

The wreath product $G \wr H$ of two groups G, H is a well-known concept. Cayley graphs of $G \wr H$ can be obtained in an intuitive way by starting with a Cayley graph of G and associating with each of its vertices a lamp whose possible states are indexed by the elements of H; see below. Graphs obtained this way are called lamplighter graphs. A well-known special case are the Diestel-Leader [7] graphs $DL(n, n)$.

Kaimanovich and Vershik [11, Sections 6.1, 6.2] proved that lamplighter graphs of infinite grids $\mathbb{Z}^d, d \geq 3$ admit non-constant, bounded, harmonic functions. Their construction had an intuitive probabilistic interpretation related to random walks on these graphs, which triggered a lot of further research on lamplighter graphs. For example, spectral properties of such groups are studied in [5, 10, 13], and other properties related to random walks are studied in [8, 9, 17]. Harmonic functions on lamplighter graphs and the related Poisson boundary are further studied e.g. in [3, 12, 18]. Finally, Lyons, Pemantle and Peres [14] proved that the lamplighter graph of $\mathbb{Z}$ over $\mathbb{Z}_2$ has the surprising property that random walk with a drift towards a fixed vertex can move outwards faster than simple random walk.

It is known that the existence of a non-constant harmonic function of finite Dirichlet energy implies the existence of a non-constant bounded harmonic function [19, Theorem 3.73]. Given the aforementioned impact that bounded harmonic functions on lamplighter graphs have had, we ask whether these graphs have non-constant harmonic functions of finite Dirichlet energy. For lamplighter graphs on a grid it is known that no such harmonic functions can exist, since such graphs are amenable and thus admit no non-constant harmonic functions of finite Dirichlet energy [16]. A. Karlsson (oral communication) asked whether this is also the case for graphs of the form $T \wr \mathbb{Z}_2$, where T is any regular tree. In this paper we give an affirmative answer to this question. In fact, the actual result is much more general:

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Theorem 1.1. *Let G be a connected locally finite graph and let H be a connected finite graph with at least one edge. Then $G \wr H$ does not admit any non-constant harmonic function of finite Dirichlet energy.*

Indeed, we do not need to assume that any of the involved graphs is a Cayley graph. Lamplighter graphs on general graphs can be defined as in the usual case when all graphs are Cayley graphs; see the next section.

It is easy to prove, and well-known, that the non-existence of non-constant harmonic functions in a graph is equivalent to the uniqueness of electrical currents. Thus, in a lamplighter graph $G \wr H$ as in Theorem 1.1 electrical currents of finite energy are unique.

Classes of graphs that do admit non-constant harmonic functions of finite Dirichlet energy are known; see [1, 2, 4].

As an intermediate step to the proof of Theorem 1.1 we prove a result (Lemma 3.1 below) that strengthens a theorem of Markvorsen, McGuinness and Thomassen [15] and might be applicable in order to prove that other classes of graphs do not admit non-constant Dirichlet-finite harmonic functions.

2. Definitions

We will be using the terminology of Diestel [6]. For a finite path P we let $|P|$ denote the number of edges in P. For a graph G and a set $U \subseteq V(G)$, we let $G[U]$ denote the subgraph of G induced by the vertices in U. If G is finite, then its *diameter* $diam(G)$ is the maximum distance, in the usual graph metric, of two vertices of G.

Let G, H be connected graphs, and suppose that every vertex of G has a distinct lamp associated with it, the set of possible states of each lamp being the set of vertices $V(H)$ of H. At the beginning, all lamps have the same state $s_0 \in V(H)$, and a "*lamplighter*" is standing at some vertex of G. In each unit of time the lamplighter is allowed to choose one of two possible moves: either walk to a vertex of G adjacent to the vertex $x \in V(G)$ he is currently at or switch the current state $s \in V(H)$ of x into one of the states $s' \in V(H)$ adjacent with s. The *lamplighter graph* $G \wr H$ is, then, a graph whose vertices correspond to the possible configurations of this game and whose edges correspond to the possible moves of the lamplighter. More formally, the vertex set of $G \wr H$ is the set of pairs (C, x) where $C : V(G) \to V(H)$ is an assignment of states such that $C(v) \neq s_0$ holds for only finitely many vertices $v \in V(G)$, and x is a vertex of G (the current position of the lamplighter). Two vertices (C, x) and (C', x') of $G \wr H$ are joined by an edge if (precisely) one of the following conditions holds:

- $C = C'$ and $xx' \in E(G)$, or
- $x = x'$, all vertices except x are mapped to the same state by C and C', and $C(x)C'(x) \in E(H)$.

This definition of $G \wr H$ coincides with that of Erschler [9].

The *blow-up* of a vertex $v \in V(G)$ in $L = G \wr H$ is the set of vertices of L of the form (C, v). Similarly, the blow-up of a subgraph T of G is the subgraph of L spanned by the blow-ups of the vertices of T. Given a vertex $x \in V(L)$ we let $[x]$ denote the vertex of G, the blow-up of which contains x.

An edge of L is a *switching edge* if it corresponds to a move of the lamplighter that switches a lamp, more formally, if it is of the form $(C, v)(C', v)$. For a switching edge $e \in E(L)$, we let $[e]$ denote the corresponding edge of H. A *ray* is a 1-way infinite path; a 2-way infinite path is called a *double ray*. A *tail* of a ray R is an infinite (co-final) subpath of R.

A function $\phi : V(G) \to \mathbb{R}$ is *harmonic* if for every $x \in V(G)$ we have $\phi(x) = \frac{1}{d(x)} \sum_{xy \in E(G)} \phi(y)$, where $d(x)$ is the number of edges incident with x. Given such a function ϕ, and an edge $e = uv$, we let $w_\phi(e) := (\phi(u) - \phi(v))^2$ denote the *energy* dissipated by e. The *(Dirichlet) energy* of ϕ is defined by $W(\phi) := \sum_{e \in E(G)} w_\phi(e)$.

3. Proof of Theorem 1.1

We start with a lemma that might be applicable in order to prove that other classes of graphs do not admit non-constant Dirichlet-finite harmonic functions. This strengthens a result of [15, Theorem 7.1].

Lemma 3.1. *Let G be a connected locally finite graph such that for every two disjoint rays S, Q in G there is a constant c and a sequence $(P_i)_{i \in \mathbb{N}}$ of pairwise edge-disjoint S–Q paths such that $|P_i| \leq ci$. Then G does not admit a non-constant harmonic function of finite energy.*

Proof. Let G be a locally finite graph that admits a non-constant harmonic function ϕ of finite energy; it suffices to find two rays S, Q in G that do not satisfy the condition in the assertion.

Since ϕ is non-constant, we can find an edge x_0x_1 satisfying $\phi(x_1) > \phi(x_0)$. By the definition of a harmonic function, it is easy to see that x_0x_1 must lie in a double ray $D = \ldots x_{-1}x_0x_1 \ldots$ such that $\phi(x_i) \geq \phi(x_{i-1})$ for every $i \in \mathbb{Z}$; indeed, every vertex $x \in V(G)$ must have a neighbour y such that $\phi(y) \geq \phi(x)$.

Define the subrays $S = x_0x_1x_2 \ldots$ and $Q = x_0x_{-1}x_{-2} \ldots$ of D. Now suppose there is a sequence $(P_i)_{i \in \mathbb{N}}$ of pairwise edge-disjoint S–Q paths such that $|P_i| \leq ci$ for some constant c.

Note that by the choice of D there is a bound $u > 0$ such that $u_i := |\phi(s_i) - \phi(q_i)| \geq u$ for every i, where $s_i \in V(S)$ and $q_i \in V(Q)$ are the endvertices of P_i.

For every edge $e = xy$ let $f(e) := |\phi(y) - \phi(x)|$. Let X_i be the set of edges e in P_i such that $f(e) \geq 0.9\frac{u}{ci}$, and let Y_i be the set of all other edges in P_i. As $|P_i| \leq ci$ by assumption, the edges in Y_i contribute less than $0.9u$ to u_i; thus $\sum_{e \in X_j} f(e) > 0.1u$ must hold. But since $f(e) \geq 0.9\frac{u}{ci}$ for every $e \in X_j$, we have $\sum_{e \in X_j} w_\phi(e) > 0.1 \times 0.9\frac{u^2}{ci}$. As the sets X_j are pairwise edge-disjoint, and as the series $\sum_i 1/i$ is not convergent, this contradicts the fact that $\sum_{e \in E(G)} w_\phi(e)$ is finite. $\square$

We now apply Lemma 3.1 to prove our main result.

Proof of Theorem 1.1. We will show that $L := G \wr H$ satisfies the condition of Lemma 3.1, from which then the assertion follows. So let S, Q be any two disjoint rays of L.

Since L is connected we can find a double ray D in L that contains a tail S' of S and a tail Q' of Q. Let s_0 (respectively, q_0) be the first vertex of S' (resp. Q'). Let V_0 be the set of vertices of G, the blow-up of which meets the path s_0Dq_0. Note that V_0 induces a connected subgraph of G, because the lamplighter only moves along the edges of G. Thus we can choose a spanning tree T_0 of $G[V_0]$.

For $i = 1, 2, \ldots$ we construct an S'–Q' path P_i as follows. Let s_i be the first vertex of S' not in the blow-up of V_{i-1}, and let q_i be the first vertex of Q' not in the blow-up of V_{i-1}. Let $V_i := V_{i-1} \cup \{s_i, q_i\}$, and extend T_{i-1} into a spanning tree T_i of $G[V_i]$ by adding two edges incident with s_i and q_i, respectively; such edges do exist: their blow-up contains the edges of S', Q' leading into s_i, q_i respectively.

We now construct an s_i–q_i path P_i. Pick a switching edge $e = s_i s'_i$ incident with s_i. Then let X_i be the unique path in L from s'_i to a vertex q_i^+ with $[q_i^+] = [q_i]$ such that X_i is contained in the blow-up of T_i. Pick a switching edge $f = q_i^+ q_i^-$ incident with q_i^+. Then follow the unique path Y_i in L from q_i^- to a vertex s_i^+ with $[s_i^+] = [s_i]$ such that Y_i is contained in the blow-up of T_i. Let $e' = s_i^+ s_i^-$ be the switching edge incident with s_i^+ such that $[e'] = [e]$. Finally, let Z_i be a path from s_i^- to the unique vertex q'_i with $[q_i q'_i] = [f]$, such that the interior of Z_i is contained in the blow-up of V_{i-1} and Z_i has minimum length under all paths with these properties. Such a path exists because every lamp at a vertex in $G - V_{i-1}$ has the same state in s_i^- and q'_i; indeed, the lamps in $G - V_i$ were never switched in the above construction, the lamp at $[s_i]$ was switched twice on the way from s_i to s_i^- using the same switching edge $[e]$, which means that its state in both endpoints of Z_i coincides with that in s_i and q_i, and finally the lamp at $[q'_i]$ has the same state in both endpoints of Z_i, namely the state $[f]$ leads to. Now set $P_i := s_i s'_i X_i q_i^+ q_i^- Y_i s_i^+ s_i^- Z_i q'_i q_i$.

It is not hard to check that the paths P_i are pairwise disjoint. Indeed, let $i < j \in \mathbb{N}$. Then, by the choice of the vertices s_j, q_j and the construction of P_j, it follows that for every inner vertex x of P_j, the configuration of x differs from the configuration of any vertex in P_i in at least one of the two lamps at $[s_j]$ and $[q_j]$.

It remains to show that there is a constant c such that $|P_i| \leq ci$ for every i. To prove this, note that $|P_i| = |X_i| + |Y_i| + |Z_i| + 4$; we will show that the latter three subpaths grow at most linearly with i, which then implies that this is also true for P_i.

Firstly, note that $diam(T_i) - diam(T_{i-1}) \leq 2$ since $V(T_i) := V(T_{i-1}) \cup \{s_i, q_i\}$. By the choice of X_i we have $|X_i| \leq diam(T_i)$, from which it follows that there is a constant c_1 such that $|X_i| \leq c_1 i$. By the same argument, we have $|Y_i| \leq c_1 i$.

It remains to bound the length of Z_i. For this, note that if T is a finite tree and $v, w \in V(T)$, then there is a v-w walk W in T containing all edges of T and satisfying $|W| \leq 3|E(T)|$; indeed, starting at v, one can first walk around the "perimeter" of T traversing every edge precisely once in each direction ($2|E(T)|$ edges), and then move "straight" from v to w (at most $|E(T)|$ edges). Thus, in order to choose Z_i, we could put a lamplighter at the vertex and configuration indicated by s_i^- and let him move in $T_i \subset G$ along such a walk W from $[s_i^-]$ to $[q'_i]$, and every time he visits a new vertex x let him change the state of x to the state indicated by q'_i. This bounds the length of Z_i from above by $3|E(T_i)|diam(H)$, and since $|E(T_i)| - |E(T_{i-1})| = 2$ and H is fixed, we can find a constant c_2 such that $|Z_i| \leq c_2 i$ for every i. This completes the proof that $|P_i|$ grows at most linearly with i.

Thus we can now apply Lemma 3.1 to prove that $G \wr H$ does not admit a non-constant harmonic function of finite energy. $\square$

Problem 3.1. Does the assertion of Theorem 1.1 still hold if H is an infinite locally finite graph?

Lemma 3.1 might be applicable in order to prove that other classes of graphs also do not admit non-constant Dirichlet-finite harmonic functions. For example, it yields an easy proof of the (well-known) fact that infinite grids have this property.

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