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AN EASIER EXTRA HEAD SCHEME FOR THE POISSON PROCESS ON $\mathbf{R}^n$

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ABSTRACT. A simple construction is presented whereby a nonrandomized extra head scheme for stationary ergodic point processes on the line due to T. Liggett (2002) is lifted to $\mathbf{R}^n$ in the Poisson case. This gives the simplest construction for a nonrandomized extra head scheme in higher dimensions, yet the method has been overlooked in previous work on the subject.

1. INTRODUCTION

Let Π be a stationary ergodic point process on a Polish group G defined on some underlying probability space $(\Omega, \mathcal{A}, \mathbf{P})$. Anticipating that we will be dealing with Abelian groups, we use additive notation for G. An extra head scheme for Π is a random variable $X \in G$ defined on some extension of $(\Omega, \mathcal{A}, \mathbf{P})$ such that the translated point process $T^{-X}\Pi$, defined by $T^{-X}\Pi(A) = \Pi(A + X)$ for all bounded Borel sets $A \subset G$, has the distribution of the Palm version $\Pi^\star$, which is intuitively the process Π conditioned on there being a point at the neutral element. In particular X must then be almost surely a point of the realization Π.

A result of Thorisson in [9] showed that extra head schemes always exist under minimal assumptions. This led to the question of concretely constructing extra head schemes in special cases such as $G = \mathbf{Z}^n$ and $G = \mathbf{R}^n$. Of particular interest are extra head schemes without additional randomization, i.e. when X is a function of Π. Liggett gave necessary and sufficient conditions for a process on $\mathbf{Z}^n$ to admit a nonrandomized extra head scheme and gave a construction that works for all stationary ergodic point processes on $\mathbf{R}$.

The question for $\mathbf{R}^n$ was settled by Holroyd and Peres in [4], where the authors establish an equivalence between extra head schemes and balanced allocation rules, i.e. translation-invariant rules for allocating equal-volume areas of space to the points of a realization. For the case of a Poisson process, a much simpler construction presented in this paper gives an explicit extra head scheme for $\mathbf{R}^n$ by essentially taking a one-dimensional projection and applying a one-dimensional extra head scheme, such as that constructed in [7].

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2. The construction

We begin by presenting the construction and then verify rigorously that it indeed works. Recall that a point process on $\mathbf{R}^n$ is defined as a random integer-valued measure Π, such that $\Pi(A) < \infty$ for all bounded Borel sets A. The process is stationary if Π and $T^Z\Pi$ have the same distribution for all $Z \in \mathbf{R}^n$, where by definition $T^Z\Pi(A) = \Pi(A - Z)$. A process on $\mathbf{R}^n$ is ergodic if every translation-invariant event has probability 0 or 1.

Now for the construction. We use the fact that every ergodic stationary point process on $\mathbf{R}$ has an extra head scheme. This was proved by Liggett in [7]. Let Π be a Poisson process on $\mathbf{R}^n$ with intensity λ. Let $A \subset \mathbf{R}^{n-1}$ be a bounded Borel set with positive Lebesgue measure $\mu(A)$. Define a new point process Π_1 on $\mathbf{R}$ by the formula $\Pi_1(B) = \Pi(B \times A)$ for $B \subset \mathbf{R}$. Then Π_1 is a Poisson process with intensity $\lambda\mu(A)$. Let X_1 be an extra head scheme for Π_1. Now we construct an extra head scheme X for Π as follows: let X be the point in $\mathbf{R} \times A$ with first coordinate X_1. Now X is almost surely well defined. Let $X_2 \in A$ be the point such that $X = (X_1, X_2)$.

Lemma 2.1. *X_1 and X_2 are independent.*

Proof. Define a random element $(y_k) \in A^{\mathbf{Z}}$ by the following construction that is almost surely well defined. Index the points of Π_1 as an increasing, doubly infinite sequence (z_k), taking z_0 to be the smallest positive point. Now take the sequence (y_k) of corresponding parts in A so that $(z_k, y_k)_{k\in\mathbf{Z}}$ enumerates the points of Π in $\mathbf{R} \times A$.

It follows from elementary properties of a Poisson process that the sequences (z_k) and (y_k) are independent and that (y_k) is i.i.d. with a uniform distribution on A. Now $X_2 = y_k$ for some k, where k depends on (z_k) but not on (y_k), so X_2 has a uniform distribution on A given any condition on (z_k). On the other hand, X_1 is obviously a function of (z_k), so X_1 and X_2 are independent as claimed. □

For a Poisson process, the Palm version is simply a Poisson process with an extra point added at the origin. Thus it suffices to show that the transformation of Π consisting of translation by $-X$ and deletion of a point at 0 preserves the distribution. This transformation can be represented as $\Pi \mapsto T^{-(0,X_2)} \circ \varphi(\Pi)$, where φ corresponds to translation by $-(X_1, 0)$ and subsequent deletion of the unique point in the set $\{0\} \times A$. It is clear that $\Pi \mapsto \varphi(\Pi)$ preserves the distribution and that $\varphi(\Pi)$ is independent of X_2.

Consider the probability space $(\mathcal{M}, \mathbf{B}, \nu)$, where $\mathcal{M}$ is the set of boundedly finite counting Borel measures on $\mathbf{R}^n$, $\mathbf{B}$ is the standard sigma-algebra and $\nu(S) = \mathbf{P}(\Pi \in S)$. The following lemma implies that $\Pi \mapsto T^{-(0,X_2)} \circ \varphi(\Pi)$ is measure-preserving on $(\mathcal{M}, \mathbf{B}, \nu)$, i.e. that $T^{-(0,X_2)} \circ \varphi(\Pi)$ has the same distribution as Π.

Lemma 2.2. *Let $(G, \mathcal{G})$ be a measurable set of measure-preserving maps on a probablility space $(\Omega, \mathcal{A}, \mathbf{P})$ such that $(g, \omega) \mapsto g(\omega)$ is a measurable map $G \times \Omega \to \Omega$. Let $\varphi : \Omega \to \Omega$ be a fixed measure-preserving transformation and let g_ω be a G-valued random transformation such that g_ω and $\varphi(\omega)$ are independent random elements. Then the mapping $\Psi : \Omega \to \Omega$ defined by $\Psi(\omega) = g_\omega \circ \varphi(\omega)$ is a measure-preserving transformation.*

Proof. It is enough to show that for any $f \in L^1(\Omega)$ one has $E(f \circ \Psi) = E(f)$. Now consider $f \in L^1$ fixed. Since g_ω and $\varphi(\omega)$ are independent (see Lemma 3.11 in [6]) we find that

$$E(f \circ \Psi) = \int_\Omega f \circ g_\omega \circ \varphi(\omega) d\omega = \int_\Omega \int_\Omega f \circ g_{\omega'} \circ \varphi(\omega) d\omega d\omega' = E(f).$$

□

It has now been shown that X as defined above is an extra head scheme for the Poisson process Π. In trying to extend the construction to other processes, one faces the problem that Lemma 2.1 does not hold in general.

The tail behavior of extra head schemes has been one of the principal objects of interest within the subject. For the extra head scheme presented here, the tail behavior is exactly the same as that of the auxiliary one-dimensional extra head scheme used. It is unsurprising that using the original construction of Liggett in [7] for our auxiliary scheme, this method results in worse tail behavior than that of the extra head scheme presented in [2]. In other words, it pays to look in more than one direction.

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