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TREE METRICS AND THEIR LIPSCHITZ-FREE SPACES

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ABSTRACT. We compute the Lipschitz-free spaces of subsets of the real line and characterize subsets of metric trees by the fact that their Lipschitz-free space is isometric to a subspace of L_1.

1. INTRODUCTION

Let M be a pointed metric space (a metric space with a designated origin denoted 0). We denote by $\mathrm{Lip}_0(M)$ the space of all real-valued Lipschitz functions on M which vanish at 0 equipped with the standard Lipschitz norm

$$\|f\| = \inf \{K \in \mathbb{R} \;/\; \forall\, x, y \in M \quad |f(y) - f(x)| \leq K\, d(x, y)\}.$$

The closed unit ball of this space being compact for the topology of pointwise convergence on M, $\mathrm{Lip}_0(M)$ has a canonical predual, namely the closed linear span of the evaluation functionals δ_x (whether this predual is unique up to a linear isometry is an open question). We call Lipschitz-free space over M and denote by $\mathcal{F}(M)$ this predual. Lipschitz-free spaces are studied extensively in [12] where they are called Arens-Eells spaces. Note that for every point a of M, $f \mapsto f - f(a)$ is a weak*-weak* continuous linear isometry between $\mathrm{Lip}_0(M)$ and $\mathrm{Lip}_a(M)$: the choice of different base points therefore yields isometric Lipschitz-free spaces.

The key property of the Lipschitz-free space $\mathcal{F}(M)$ is that a Lipschitz map between two metric spaces admits a linearization between the corresponding Lipschitz-free spaces. More precisely, if M_1 and M_2 are pointed metric spaces and $L : M_1 \to M_2$ is a Lipschitz map satisfying $L(0) = 0$, there exists a unique linear map $\widehat{L} : \mathcal{F}(M_1) \to \mathcal{F}(M_2)$ such that the following diagram commutes:

$$\begin{array}{ccc} M_1 & \xrightarrow{L} & M_2 \\ {\scriptstyle\delta}\big\downarrow & & \big\downarrow{\scriptstyle\delta} \\ \mathcal{F}(M_1) & \xrightarrow{\widehat{L}} & \mathcal{F}(M_2) \end{array}$$

(we refer to [6] for details on this functorial property). In particular, $\mathcal{F}(M')$ is an isometric subspace of $\mathcal{F}(M)$ whenever M' isometrically embeds into M.

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Differentiation almost everywhere yields a weak*-weak* continuous linear isometry between $\mathrm{Lip}_0(\mathbb{R})$ and L_∞ which predualizes into a linear isometry between $\mathcal{F}(\mathbb{R})$ and L_1 (the discrete version of this argument provides $\mathcal{F}(\mathbb{N}) \equiv \ell_1$).

According to a result of Godefroy and Talagrand ([7]), any Banach space isomorphic to a subspace of L_1 is a unique isometric predual of its dual: given the open question mentioned above, it is therefore natural to highlight metric spaces which admit a subspace of L_1 as their Lipschitz-free space.

The purpose of this article is to show that $\mathcal{F}(M)$ is linearly isometric to a subspace of L_1 if and only if M isometrically embeds into an $\mathbb{R}$-tree (i.e. a metric space where any couple of points is connected by a unique arc isometric to a compact interval of the real line). This gives a complement to a result of Naor and Schechtman that $\mathcal{F}(\mathbb{R}^2)$ is not isomorphic to a subspace of L_1 ([10]).

We also compute the Lipschitz-free spaces of subsets of the real line by integrating bounded measurable functions with respect to a measure adapted to the subset considered.

2. Preliminaries

In this section we define differentiation for functions defined on a subset of a pointed $\mathbb{R}$-tree and introduce measures which enable us to retrieve the values of such absolutely continuous functions by integrating their derivative.

We first recall the definition of $\mathbb{R}$-trees and define the analogues of the Lebesgue measure for those metric spaces (this measure is called the length measure).

Definition 2.1. An $\mathbb{R}$-tree is a metric space T satisfying the following two conditions.

(1) For any points a and b in T, there exists a unique isometry ϕ of the closed interval $[\,0\,, d(a,b)\,]$ into T such that $\phi(0) = a$ and $\phi(d(a,b)) = b$.

(2) Any one-to-one continuous mapping $\varphi : [\,0\,,1\,] \to T$ has the same range as the isometry ϕ associated to the points $a = \varphi(0)$ and $b = \varphi(1)$.

If T is an $\mathbb{R}$-tree, we denote for any x and y in T, ϕ_{xy} the unique isometry associated to x and y as in definition 2.1 and write $[\,x\,,y\,]$ for the range of ϕ_{xy}; such subsets of T are called segments. We say that a subset A of T is measurable whenever $\phi_{xy}^{-1}(A)$ is Lebesgue-measurable for any x and y in T. If A is measurable and S is a segment $[\,x\,,y\,]$, we write $\lambda_S(A)$ for $\lambda\left(\phi_{xy}^{-1}(A)\right)$ where λ is the Lebesgue measure on $\mathbb{R}$. We denote by $\mathcal{R}$ the set of all subsets of T which can be written as a finite union of disjoint segments, and for $R = \bigcup_{k=1}^{n} S_k$ (with disjoint S_k) in $\mathcal{R}$, we put

$$\lambda_R(A) = \sum_{k=1}^{n} \lambda_{S_k}(A).$$

Now,

$$\lambda_T(A) = \sup_{R\in\mathcal{R}} \lambda_R(A)$$

defines a measure on the σ-algebra of T-measurable sets such that

$$\int_{[x,y]} f(u)\mathrm{d}\lambda_T(u) = \int_0^{d(x,y)} f(\phi_{xy}(t))\mathrm{d}t \tag{2.1}$$

for any $f \in L_1(T)$ and x, y in T.

Definition 2.2. Let T be a pointed $\mathbb{R}$-tree also called a rooted real tree, and let A be a closed subset of T containing 0. We denote by μ_A the positive measure defined by

$$\mu_A = \lambda_A + \sum_{a \in A} L(a)\delta_a$$

where λ_A is the restriction of the length measure on A, $L(a) = \inf_{x \in A \cap [0,a[} d(a,x)$ and δ_a is the Dirac measure on a.

This measure takes into account the gaps in A by shifting their mass to the next available point (away from the root). When $L(a) > 0$, we say that a is root-isolated in A, and we denote by $\hat{A}$ the set of root-isolated points in A.

Proposition 2.3. *If T is a pointed $\mathbb{R}$-tree and A is a closed subset of T containing 0, $L_\infty(\mu_A)$ is isometric to the dual space of $L_1(\mu_A)$.*

When T is separable, this proposition is a direct consequence of a classical theorem.

Proof. We have $L_1(\mu_A) \equiv L_1(\lambda_A) \oplus_1 \ell_1(\hat{A})$ and $L_\infty(\mu_A) \equiv L_\infty(\lambda_A) \oplus_\infty \ell_\infty(\hat{A})$. Hence all we need to prove is $L_\infty(\lambda_A) \equiv L_1(\lambda_A)^*$. We show that the canonical embedding of $L_\infty(\lambda_A)$ into $L_1(\lambda_A)^*$ is onto. Let φ be a bounded linear functional on $L_1(\lambda_A)$. Then for any x and y in T, φ defines by restriction a bounded linear functional on $L_1(A \cap [x, y])$; therefore there exists h_{xy} in $L_\infty(A \cap [x, y])$ such that

$$\varphi(f) = \int_T f h_{xy} \mathrm{d}\lambda_A$$

for any f in $L_1(A \cap [x, y])$. Any two functions h_{xy} and $h_{x'y'}$ coincide almost everywhere on $A \cap [x, y] \cap [x', y']$, and there exists h in $L_\infty(\lambda_A)$ such that

$$\varphi(f) = \int_T f h \mathrm{d}\lambda_T$$

for any $f \in L_1(\lambda_A)$ with support included in a finite union of intervals. Such functions form a dense subspace of $L_1(\lambda_A)$ and the equality above is therefore valid for any f in $L_1(\lambda_A)$. $\square$

We define differentiation on a pointed $\mathbb{R}$-tree in the following way.

Definition 2.4. Let T be a pointed $\mathbb{R}$-tree, A a subset of T containing 0 and $f : A \to \mathbb{R}$. If $a \in A$, we denote by $\hat{a}$ the unique point in $[0, a]$ satisfying $d(a, \hat{a}) = L(a)$. When

$$\lim_{\substack{x \to \hat{a} \\ x \in [0,a[\cap A}} \frac{f(a) - f(x)}{d(x,a)}$$

exists, we say that f is differentiable at a, and we set $f'(a)$ as the value of this limit.

When A is closed, a mapping $f : A \to \mathbb{R}$ is always differentiable on root-isolated points of A, $f'(a)$ being equal in this case to $(f(a) - f(\hat{a}))/L(a)$. When $A = T$, the limit corresponds to the left-derivative of $f_a = f \circ \phi_a$ at $d(0, a)$, where ϕ_a is the isometry associated to points 0 and a as in definition 2.1. If f is Lipschitz on T, the function $f_x = f \circ \phi_x$ is differentiable almost everywhere on $[0, d(0, x)]$ for any

x in T; therefore f is differentiable almost everywhere on T. Moreover, the change of variable formula (2.1) yields

$$f(x) - f(0) = \int_{[0,x]} f' \mathrm{d}\lambda_T.$$

3. Lipschitz-free spaces of metric trees

In this section we compute the Lipschitz-free spaces of a certain class of subsets of $\mathbb{R}$-trees. For this, we integrate bounded measurable functions defined on a closed subset A with respect to the measure μ_A introduced in definition 2.2. We derive the computation of Lipschitz-free spaces of subsets of the real line.

Definition 3.1. A point t of an $\mathbb{R}$-tree T is said to be a branching point of T if $T \setminus \{t\}$ is composed by at least three connected components. We denote by $\mathrm{Br}(T)$ the set of branching points of T.

Theorem 3.2. *Let T be an $\mathbb{R}$-tree and A be a subset of T such that* $\mathrm{Br}(T) \subset \bar{A}$. *Then $\mathcal{F}(A)$ is isometric to $L_1(\mu_{\bar{A}})$.*

Proof. As $\mathcal{F}(A)$ is isometric to $\mathcal{F}(\bar{A})$, we may assume that A is closed. We choose a point in A as the origin for T and we prove that $\mathrm{Lip}_0(A)$ is isometric to $L_\infty(\mu_A)$. For this, we define a linear map Φ of $L_\infty(\mu_A)$ into $\mathrm{Lip}_0(A)$ by putting

$$\Phi(g)(a) = \int_{[0,a]} g \mathrm{d}\mu_A.$$

The norm of a Lipschitz function f defined on A may be computed on intervals containing 0. Indeed if a and b are in A, there exists $x \in T$ such that $[\,0\,,a\,] \cap [\,0\,,b\,] = [\,0\,,x\,]$ and, as A contains all the branching points of T, we know that $x \in A$. One of the quantities $|f(a) - f(x)|/d(x,a)$ or $|f(b) - f(x)|/d(x,b)$ must be greater than or equal to $|f(b) - f(a)|/d(a,b)$ as x belongs to the interval $[\,a\,,b\,]$. We deduce that Φ is an isometry.

We now prove that Φ is onto. Let f be in $\mathrm{Lip}_0(A)$; we extend f to a Lipschitz mapping $\widetilde{f}$ defined on T and affine on each interval contained in the complement of A. The function $\widetilde{f}$ is differentiable almost everywhere on T (in the sense of definition 2.4); therefore f is differentiable μ_A-almost everywhere. For any $a \in A$ we have

$$f(a) = \int_{[0,a]} \widetilde{f}' \mathrm{d}\lambda_T = \int_{[0,a]} f' \mathrm{d}\lambda_A + \int_{[0,a] \setminus A} \widetilde{f}' \mathrm{d}\lambda_T.$$

The set $B_a = [\,0\,,a\,] \setminus A$ is a union of open interval,

$$B_a = \bigcup_{i \in I} \,]\,\hat{a}_i\,,a_i[,$$

so

$$\int_{B_a} \widetilde{f}' \mathrm{d}\lambda_T = \sum_{i \in I} \frac{f(a_i) - f(\hat{a}_i)}{d(\hat{a}_i, a_i)} \times L(a_i) = \sum_{i \in I} L(a_i) f'(a_i) = \int_{[0,a]} f' \mathrm{d}\nu_A$$

where $\nu_A = \sum_{u \in A} L(u)\delta_u$.

Hence

$$f(a) = \int_{[0,a]} f' \mathrm{d}\mu_A$$

for any a in A and we get $f = \Phi(f')$.

Finally, we know from Proposition 2.3 that $L_\infty(\mu_A)$ is isometrically isomorphic to the dual space of $L_1(\mu_A)$. The linear isometry Φ being weak*-weak* continuous, it is the transpose of a linear isometry between $\mathcal{F}(A)$ and $L_1(\mu_A)$. $\square$

Corollary 3.3. *If T is an $\mathbb{R}$-tree, then $\mathcal{F}(T)$ is isometric to $L_1(T)$.*

The isometry associates to each function h of $L_1(T)$ the element α of the Lipschitz-free space defined by $\alpha(f) = \int_T f' h \mathrm{d}\lambda_T$ i.e. the opposite of the derivative of h in a distribution sense. Conversely, if α is a measure with finite support, the corresponding L_1-function is defined by $g(t) = \alpha(C_t)$ where $C_t = \{x \in T \,/\, t \in [\,0\,, x\,]\}$.

For a separable $\mathbb{R}$-tree, we get $\mathcal{F}(T) \equiv L_1$. This shows that metric spaces having isometric Lipschitz-free spaces need not be homeomorphic (see [3] for examples of non-Lipschitz isomorphic Banach spaces having linearly isomorphic Lipschitz-free spaces).

Corollary 3.4. *Let T be a separable $\mathbb{R}$-tree, and A an infinite subset of T such that* $\mathrm{Br}(T) \subset \bar{A}$*. If $\bar{A}$ has length measure* 0*, then $\mathcal{F}(A)$ is isometric to ℓ_1. If $\bar{A}$ has positive length measure, then $\mathcal{F}(A)$ is isomorphic to L_1.*

Proof. As T is separable, the set of root-isolated points of $\bar{A}$ is at most countable. When $\lambda(\bar{A}) > 0$, $\mathcal{F}(A)$ is isometric to one of the spaces L_1, $L_1 \oplus_1 \ell_1^n$, $L_1 \oplus_1 \ell_1$; the fact that all these spaces are isomorphic to L_1 results from the Pełczyński decomposition method. $\square$

This result allows us to compute the Lipschitz-free space of any subset of $\mathbb{R}$. As an application, we get that infinite discrete subsets of $\mathbb{R}$ have a Lipschitz-free space isometric to ℓ_1. Also, $\mathcal{F}(K_3) \equiv \ell_1$ where K_3 is the usual Cantor set and the Lipschitz-free space of a Cantor set of positive measure is isometric to $L_1 \oplus_1 \ell_1$; this provides examples of homeomorphic metric spaces having non-isomorphic Lipschitz-free spaces.

Finite subsets of $\mathbb{R}$-trees which contain the branching points usually appear as weighted trees.

Definition 3.5. A weighted tree is a finite connected graph with no cycle, endowed with an edge weighted path metric.

Corollary 3.6. *If T is a weighted tree, then $\mathcal{F}(T)$ is isometric to ℓ_1^n where $n =$ card$(T) - 1$.*

Embedding a finite metric space into a weighted tree leads to an easy computation of the norm of its Lipschitz-free space (this is not always possible as we shall see in the following section). Let us illustrate this remark with two obvious examples. A metric space $M = \{0\,, 1\,, 2\}$ of cardinality 3 embeds isometrically into a four points weighted tree with edge lengths

$$\lambda_0 = \frac{1}{2}\,[\,d_{01} + d_{02} - d_{12}\,],\ \lambda_1 = \frac{1}{2}\,[\,d_{01} + d_{12} - d_{02}\,],\ \lambda_2 = \frac{1}{2}\,[\,d_{02} + d_{12} - d_{01}\,],$$

where d_{ij} is the distance between i and j. This embedding yields an isometry between $\mathcal{F}(M)$ and a hyperplane of ℓ_1^3, and we obtain

$$\|\alpha_1\delta_1 + \alpha_2\delta_2\| = \lambda_0\,|\alpha_1 + \alpha_2| + \lambda_1\,|\alpha_1| + \lambda_2\,|\alpha_2|.$$

If M is an $(n+1)$-point metric space equipped with the discrete distance ($d(i,j)=1$ if $i \neq j$), we obtain in the same way

$$\left\| \sum_{i=1}^{n} \alpha_i \delta_i \right\| = \frac{1}{2} \left| \sum_{i=1}^{n} \alpha_i \right| + \frac{1}{2} \sum_{i=1}^{n} |\alpha_i| \ .$$

4. Characterization of tree metrics

In this section we show that subsets of $\mathbb{R}$-trees are the only metric spaces with a Lipschitz-free space isometric to a subspace of an L_1-space. We use the fact that a metric space isometrically embeds into an $\mathbb{R}$-tree if and only if it is 0-hyperbolic, that is to say satisfies the following property.

Definition 4.1. A metric space is said to satisfy the four-point condition if

$$d(a,b)+d(c,d) \ \leq \ \max\left(d(a,c)+d(b,d)\, ,\, d(a,d)+d(b,c) \right)$$

whenever a, b, c and d are in M.

Equivalently, the maximum of the three sums $d(a,b)+d(c,d)$, $d(a,c)+d(b,d)$ and $d(a,d)+d(b,c)$ is always attained at least twice. It is fairly straightforward to check that metrics induced by $\mathbb{R}$-trees indeed satisfy this condition. For a proof of the converse, see [4], Chap. 3.

The following theorem relates the aforementioned characterization of $\mathbb{R}$-trees with Lipschitz-free spaces.

Theorem 4.2. *Let M be a metric space. The following assertions are equivalent:*
(1) *$\mathcal{F}(M)$ is isometric to a subspace of an L_1-space.*
(2) *M satisfies the four-point condition.*
(3) *M isometrically embeds into an $\mathbb{R}$-tree.*

Proof. (1) $\Rightarrow$ (2) Suppose M is a metric space with a Lipschitz-free space isometric to a subspace of an L_1-space, and consider four points in M denoted a_0, a_1, a_2 and a_3. We will henceforth write d_{ij} in place of $d(a_i,a_j)$. The unit ball of $\mathrm{Lip}_0(\{a_0,a_1,a_2,a_3\})$ is isometric to the convex set of $\mathbb{R}^3$ defined by the conditions

$$|x| \leq d_{01} \ \ ; \ \ |y| \leq d_{02} \ \ ; \ \ |z| \leq d_{03} \ \ ; \ \ |y-x| \leq d_{12} \ \ ; \ \ |z-x| \leq d_{13} \ \ ; \ \ |z-y| \leq d_{23}.$$

Since $\mathcal{F}(\{a_0,a_1,a_2,a_3\})$ is isometric to a subspace of L_1, this polytope is the projection of a cube and therefore all its faces admit a center of symmetry. The face obtained when making the third inequality to bind is characterized by

$$\begin{array}{rcccl} d_{03}-d_{13} & \leq & x & \leq & d_{01} \\ d_{03}-d_{23} & \leq & y & \leq & d_{02} \\ x-d_{12} & \leq & y & \leq & x+d_{12} \end{array}$$

in the affine plane of equation $z=d_{03}$.

For any real numbers a, b, c, d, e, f, the planar convex set defined by

$$\begin{array}{rcccl} a & \leq & x & \leq & b \\ c & \leq & y & \leq & d \\ x+e & \leq & y & \leq & x+f \end{array}$$

is either empty or centrally symmetric if and only if one of the following nine conditions holds:

$$b \le a \qquad d \le c \qquad f \le e$$

$$d \le a+e \qquad b+f \le c \qquad a+b+e+f = c+d$$

$$\begin{cases} d & \le & a+f \\ b+e & \le & c \end{cases} \qquad \begin{cases} b+f & \le & d \\ c & \le & a+e \end{cases} \qquad \begin{cases} a+f & \le & c \\ d & \le & b+e \end{cases}$$

Those different possibilities encompass segments, rectangles, parallelograms and symmetric hexagons. Applying this to $a = d_{03} - d_{13}$, $b = d_{01}$, $c = d_{03} - d_{23}$, $d = d_{02}$, $e = -d_{12}$ and $f = d_{12}$ yields that one of the following conditions holds:

$$d_{03} = d_{01} + d_{13} \tag{4.1}$$

$$d_{03} = d_{02} + d_{23} \tag{4.2}$$

$$\begin{cases} d_{01} + d_{23} & \le & d_{03} + d_{12} \\ d_{02} + d_{13} & \le & d_{03} + d_{12} \end{cases} \tag{4.3}$$

$$\begin{cases} d_{02} & = & d_{01} + d_{12} \\ d_{23} & = & d_{12} + d_{13} \end{cases} \tag{4.4}$$

$$\begin{cases} d_{01} & = & d_{02} + d_{12} \\ d_{13} & = & d_{12} + d_{23} \end{cases} \tag{4.5}$$

$$d_{01} + d_{23} = d_{02} + d_{13} \tag{4.6}$$

Both (4.1) and (4.2) imply (4.3). Also, each of (4.4) and (4.5) imply (4.6), which leaves us with

$$\begin{cases} d_{01} + d_{23} & \le & d_{03} + d_{12} \\ d_{02} + d_{13} & \le & d_{03} + d_{12} \end{cases}$$

$$d_{01} + d_{23} = d_{02} + d_{13},$$

either condition implying $d_{01} + d_{23} \le \max\,(\,d_{02} + d_{13}\,,\, d_{03} + d_{12}\,)$.
(2) $\Rightarrow$ (3) see [4].
(3) $\Rightarrow$ (1) results from Corollary 3.3. □

In the special case when M is separable, the L_1-space in condition (1) can be taken as L_1, any separable isometric subspace of an L_1-space being an isometric subspace of L_1. In case M is finite, it can be replaced by ℓ_1^N where $N = 2 \times \mathrm{Card}(M) - 3$ (it is shown in [2] that a finite metric space with the four-point property isometrically embeds into a weighted tree with at most $2 \times \mathrm{Card}(M) - 2$ vertices).

Note that a polyhedral space isometrically embeds into L_1 if and only if its 3-dimensional subspaces do so (the so-called Hlawka's inequality which characterizes this property involves only three vectors). As far as Lipschitz-free spaces

$\mathcal{F}(M)$ are concerned, embeddability of the 3-dimensional subspaces corresponding to Lipschitz-free spaces of four-point subsets of M is enough, and we obtain a canonical embedding.

5. Lipschitz-free spaces isomorphic to a subspace of L_1

We show in this section a very natural gluing result: if M is a metric space of finite radius composed of uniformly apart subsets M_γ, $\mathcal{F}(M)$ is essentially the ℓ_1-sum of the Lipschitz-free spaces $\mathcal{F}(M_\gamma)$.

Proposition 5.1. *Let Γ be a pointed set (the distinguished point being denoted 0) and $M = \bigcup_{\gamma\in\Gamma} M_\gamma$ a metric space. If there exist positive constants α and β such that $\alpha \le d(x,y) \le \beta$ whenever x and y belong to distinct M_γ's, then we have the isomorphism*

$$\mathcal{F}(M) \simeq \left(\sum_{\gamma\in\Gamma} \mathcal{F}(M_\gamma)\right)_{\ell_1} \oplus_1 \ell_1(\Gamma^*)$$

where $\Gamma^* = \Gamma \setminus \{0\}$.

The ℓ_1-space compensates for the loss of dimensions due to the choice of a base point in each M_γ: each connected component of $M \setminus M_0$ contributes for one dimension in this ℓ_1-space.

Proof. For each $\gamma \in \Gamma$, we choose a base point a_γ in M_γ in order to define $\mathcal{F}(M_\gamma)$, and use a_0 to define $\mathcal{F}(M)$. We consider the following two maps:

$$\Phi : \begin{cases} \left(\sum_{\gamma\in\Gamma} \mathrm{Lip}_0(M_\gamma)\right)_{\ell_\infty} \oplus_\infty \ell_\infty(\Gamma^*) & \to \quad \mathrm{Lip}_0(M) \\ ((f_\gamma)_{\gamma\in\Gamma}\,, (\lambda_\gamma)_{\gamma\in\Gamma^*}) & \mapsto \quad f = \begin{cases} f_0 & \text{on } M_0 \\ f_\gamma + \lambda_\gamma & \text{on } M_\gamma \end{cases} \end{cases}$$

and

$$\Psi : \begin{cases} \mathrm{Lip}_0(M) & \to \quad \left(\sum_{\gamma\in\Gamma} \mathrm{Lip}_0(M_\gamma)\right)_{\ell_\infty} \oplus_\infty \ell_\infty(\Gamma^*) \\ f & \mapsto \quad ((f|_{M_\gamma} - f(a_\gamma))_{\gamma\in\Gamma}\,, (f(a_\gamma))_{\gamma\in\Gamma^*}) \end{cases}$$

First of all let us show that Φ is well-defined and has norm less than $2(\alpha+\beta+1)/\alpha$. If $u = ((f_\gamma)_{\gamma\in\Gamma}\,, (\lambda_\gamma)_{\gamma\in\Gamma^*})$, we have for any x in M_σ and y in M_τ,

$$|\Phi(u)(y) - \Phi(u)(x)| \le |f_\sigma(x)| + |f_\tau(y)| + |\lambda_\sigma| + |\lambda_\tau|$$

(with $\lambda_0 = 0$). Now, assuming $\sigma \neq \tau$, we get

$$|f_\sigma(x)| \le \|f_\sigma\|\, d(x, a_\sigma) \le \|f_\sigma\|\,(d(x,y) + d(y, a_\sigma)) \le \|u\|\,(d(x,y) + \beta).$$

The inequality $1 \le \dfrac{1}{\alpha}\, d(x,y)$ yields

$$|f_\sigma(x)| \le \left(1 + \frac{\beta}{\alpha}\right) \|u\|\, d(x,y) \quad \text{and} \quad |\lambda_\sigma| \le \frac{1}{\alpha}\, \|u\|\, d(x,y),$$

these upper bounds being also valid for τ. Hence

$$|\Phi(u)(y) - \Phi(u)(x)| \le 2\, \frac{\alpha+\beta+1}{\alpha}\, \|u\|\, d(x,y).$$

The latter inequality holds if x and y belong to the same M_σ, and we conclude that $\Phi(u)$ is a Lipschitz map of norm less than $2\,\frac{\alpha+\beta+1}{\alpha}\,\|u\|$. The map Ψ is the inverse of Φ, and we have $\|\Psi\| \leq \max(1,\beta)$. The isomorphism Φ being weak*-weak* continuous, it is the transpose of a linear isomorphism between $\mathcal{F}(M)$ and $(\oplus_{\ell_1}\mathcal{F}(M_\gamma)) \oplus_1 \ell_1(\Gamma^*)$. □

Combined with the results of the preceding section, this gives canonical examples of Lipschitz-free spaces isomorphic to a subspace of L_1 (and therefore a unique isometric predual by [7]).

Examples of spaces $\mathcal{F}(M)$ which are unique isometric preduals are plentiful in [9]. Kalton showed the following result: if M is an arbitrary metric space and $\varepsilon > 0$, then the space $\mathcal{F}(M)$ is $(1+\varepsilon)$-isometric to a subspace of $\left(\sum_{k\in\mathbb{Z}}\ \mathcal{F}(M_k)\right)_{\ell_1}$ where $M_k = \left\{x \in M \,/\, d(x,0) \leq 2^k\right\}$. It is then deduced that when M is uniformly discrete (i.e. $\inf_{x\neq y} d(x,y) > 0$), $\mathcal{F}(M)$ is a Schur space with the Radon-Nikodým property and the approximation property. In this case, $\mathcal{F}(M)$ is a unique isometric predual since it has the Radon-Nikodým property (see [5] p. 144). Also, according to Bonic, Frampton and Tromba ([1], corrected in [12]), if K is a compact subset of a finite-dimensional normed space and $0 < \alpha < 1$, the metric space K equipped with the distance $d(x,y) = \|y-x\|^\alpha$ has a Lipschitz-free space (denoted $\mathcal{F}_\alpha(K)$) isomorphic to ℓ_1. Hence $\mathcal{F}_\alpha(A)$ is a unique isometric predual whenever A is a subset of a finite-dimensional normed space. Note that in [9] Kalton showed that if K is a compact convex subset of ℓ_2 and is infinite-dimensional, then $\mathcal{F}_\alpha(K)$ cannot be isomorphic to ℓ_1.

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