

HEINZ TYPE ESTIMATES FOR GRAPHS IN EUCLIDEAN SPACE

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To my wife, Andrea

ABSTRACT. Let M^n be an entire graph in the Euclidean $(n + 1)$ -space \mathbb{R}^{n+1} . Denote by H , R and $|A|$, respectively, the mean curvature, the scalar curvature and the length of the second fundamental form of M^n . We prove that if the mean curvature H of M^n is bounded, then $\inf_M |R| = 0$, improving results of Elbert and Hasanis-Vlachos. We also prove that if the Ricci curvature of M^n is negative, then $\inf_M |A| = 0$. The latter improves a result of Chern as well as gives a partial answer to a question raised by Smith-Xavier. Our technique is to estimate $\inf |H|$, $\inf |R|$ and $\inf |A|$ for graphs in \mathbb{R}^{n+1} of C^2 real-valued functions defined on closed balls in \mathbb{R}^n .

1. INTRODUCTION

Let $B_r \subset \mathbb{R}^n$ be an open ball of radius r , and $f : B_r \rightarrow \mathbb{R}$ a C^2 function. For $n = 2$, Heinz [11] obtained the following estimates for the mean curvature H and the Gaussian curvature K of the graph of f :

$$(1.1) \quad \inf |H| \leq \frac{1}{r},$$

$$(1.2) \quad \inf |K| \leq \frac{3e^2}{r^2},$$

where e is the basis for the natural logarithm. Chern [2] and Flanders [6], independently, obtained inequality (1.1) for any $n \geq 2$. As an immediate consequence one has that an entire graph in \mathbb{R}^{n+1} , i.e., the graph of a C^2 function from \mathbb{R}^n to \mathbb{R} , cannot have mean curvature bounded away from zero. This result of Chern and Flanders implies that an entire graph in \mathbb{R}^{n+1} with constant scalar curvature $R \geq 0$ satisfies $R = 0$ (see Section 2, equality (2.3)).

Salavessa [15] extended inequality (1.1) to graphs of smooth real-valued functions defined on oriented compact domains of Riemannian manifolds, and Barbosa-Bessa-Montenegro [1] extended it to transversally oriented codimension-one C^2 -foliations of Riemannian manifolds.

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In the theorems below, by a graph over a closed ball $\overline{B}_r \subset \mathbb{R}^n$ we mean the graph in \mathbb{R}^{n+1} of a C^2 real-valued function defined in \overline{B}_r . In our first result, we improve the estimates given by Heinz, Chern and Flanders for $\inf |H|$, by showing that the estimates can be made strict if we consider graphs over *closed* balls instead of graphs over *open* balls.

Theorem 1.1. *If $M^n \subset \mathbb{R}^{n+1}$ is a graph over a closed ball $\overline{B}_r \subset \mathbb{R}^n$, then*

$$(1.3) \quad \inf_M |H| < \frac{1}{r}.$$

The estimate (1.2) implies that an entire graph in \mathbb{R}^3 cannot have negative Gaussian curvature bounded away from zero, a result extended later to complete surfaces in \mathbb{R}^3 by Efimov [4] in a remarkable work (see the discussion after Corollary 1.4). In the next result, we obtain a version in higher dimensions of the inequality (1.2).

Theorem 1.2. *Let $M^n \subset \mathbb{R}^{n+1}$ be a graph over a closed ball $\overline{B}_r \subset \mathbb{R}^n$, and denote by R the scalar curvature of M . Then*

$$(1.4) \quad \inf_M |R| \leq \left(\sup_M |H| + \frac{1}{r} \right) \frac{2}{r}.$$

Moreover, if M has a point where the second fundamental form is semi-definite, then

$$(1.5) \quad \inf_M |R| < \frac{1}{r^2}.$$

Remark 1.3. For each $a > r$, let M_a be the graph of $f : \overline{B}_r \subset \mathbb{R}^n \rightarrow \mathbb{R}$ given by

$$f(x_1, \dots, x_n) = \left(a^2 - \sum_{i=1}^n x_i^2 \right)^{\frac{1}{2}}.$$

The mean curvature and the scalar curvature of M_a are, respectively, $1/a$ and $1/a^2$. Since a can be made arbitrarily close to r , we see that the estimates (1.3) and (1.5) are sharp.

The key to obtaining strict inequalities in (1.3) and (1.5) is a general tangency principle by Silva and the author ([7], Theorem 1.1), which establishes relatively weak sufficient conditions for two hypersurfaces of a Riemannian manifold to coincide in a neighborhood of a tangency point (see Theorems 2.1 and 2.2 in Section 2 for particular cases of this tangency principle).

It is natural to try to establish versions for the higher order mean curvatures H_k , $k \geq 2$ (see Section 2 for the definitions), of the theorem of Chern and Flanders referred to in the beginning of the introduction. In this regard, Elbert [5] proved that there is no entire graph in \mathbb{R}^{n+1} with second fundamental form of bounded length and negative 2-mean curvature H_2 bounded away from zero (for hypersurfaces of a Euclidean space, the 2-mean curvature H_2 is nothing but the scalar curvature R of the hypersurface). Hasanis-Vlachos [10] improved Elbert's result by proving that $\inf_M |R| = 0$ for all entire graphs in \mathbb{R}^{n+1} with second fundamental form of bounded length (see [5] and [10] for results regarding the other higher order mean curvatures). As an immediate consequence of the first part of Theorem 1.2, we obtain the following improvement of these results of Elbert and Hasanis-Vlachos.

Corollary 1.4. *If an entire graph $M^n \subset \mathbb{R}^{n+1}$ has bounded mean curvature, then*

$$(1.6) \quad \inf_M |R| = 0.$$

In particular, if the scalar curvature R is constant, then $R = 0$.

A classical theorem by Hilbert states that the hyperbolic plane cannot be isometrically immersed in the 3-dimensional Euclidean space. In a remarkable work, Efimov [4] extended Hilbert's theorem by proving that there is no complete immersed surface in \mathbb{R}^3 with Gaussian curvature less than a negative constant.

Reilly [14] and Yau [17] (see also [18], problem 56, p. 682) proposed the following extension of Efimov's theorem:

"There are no complete hypersurfaces in \mathbb{R}^{n+1} with negative Ricci curvature bounded away from zero."

In a well-known work, Smith-Xavier [16] showed that the above question has a positive answer for $n = 3$ and provided a partial answer for $n > 3$. This question also has a positive answer in the class of all entire graphs with negative Ricci curvature in Euclidean space, as Chern has shown [2] that $\inf_M |\text{Ric}| = 0$ for all entire graphs $M^n \subset \mathbb{R}^{n+1}$, $n \geq 3$, with negative Ricci curvature. The corollary of the following theorem improves this result of Chern.

Theorem 1.5. *Let $M^n \subset \mathbb{R}^{n+1}$, $n \geq 3$, be a graph over a closed ball $\overline{B}_r \subset \mathbb{R}^n$. If the Ricci curvature of M is negative, then*

$$(1.7) \quad \inf_M |A| < \frac{3(n-2)}{r},$$

where $|A|$ is the length of the second fundamental form of M^n in \mathbb{R}^{n+1} .

Corollary 1.6. *If the Ricci curvature of an entire graph $M^n \subset \mathbb{R}^{n+1}$, $n \geq 3$, is negative, then $\inf_M |A| = 0$.*

Okayasu [13] constructed an example of an $O(2) \times O(2)$ -invariant complete hypersurface of constant negative scalar curvature in \mathbb{R}^4 . Since the length $|A|$ of the second fundamental form in Okayasu's example is unbounded, one can then formulate the following Efimov type question: is there a complete hypersurface in \mathbb{R}^{n+1} with bounded mean curvature and negative scalar curvature bounded away from zero? Corollary 1.4 shows that if such a hypersurface does exist, then certainly it is not an entire graph. On the other hand, we do not know whether Corollary 1.4 holds without the assumption that the mean curvature is bounded.

In dimension 2, Milnor [12] conjectured (see also [18], problem 62, p. 684) the following improvement of Efimov's result: If $M^2 \subset \mathbb{R}^3$ is a complete nonflat umbilic free surface whose Gaussian curvature does not change sign, then $\inf |A| = 0$. Smyth-Xavier [16] proposed the following analogue in higher dimensions: If $M^n \subset \mathbb{R}^{n+1}$ is a complete immersed hypersurface with negative Ricci curvature, then $\inf_M |A| = 0$. Corollary 1.6 shows that this question has a positive answer for entire graphs in Euclidean spaces.

In the following theorem we obtain an estimate for $\inf_M |A|$ under another geometric condition.

Theorem 1.7. *Let $M^n \subset \mathbb{R}^{n+1}$ be a graph over a closed ball $\overline{B}_r \subset \mathbb{R}^n$. If the mean curvature of M does not change sign, then*

$$(1.8) \quad \inf_M |A| < \frac{n}{r}.$$

As immediate consequences of Theorem 1.7, we obtain the following results by Silva and the author [8]:

Corollary 1.8. *If the mean curvature of an entire graph $M^n \subset \mathbb{R}^{n+1}$ does not change sign, then $\inf_M |A| = 0$.*

Corollary 1.8 was obtained by Hasanis-Vlachos [10] under the additional assumption that the length $|A|$ of the second fundamental form A of M is bounded.

Corollary 1.9. *Let $M^n \subset \mathbb{R}^{n+1}$ be an entire graph. If $|A|$ is constant and H does not change sign, then M is a hyperplane.*

Remark 1.10. Corollary 1.8 does not hold for hypersurfaces which are not graphs. In fact, any circular cylinder satisfies $\inf |A| > 0$.

We stress that our methods in this paper are substantially different from the ones employed by Heinz [11], which were based on an ingenious use of the divergence theorem, applied to the classical formulas for the mean and Gaussian curvature of a graph in two variables. By contrast, our proofs constitute another application of our work on the tangency principle [7]. They also use a classical result of Gårding [9] on hyperbolic polynomials.

2. PRELIMINARIES

Given an oriented immersed hypersurface M^n of the $(n + 1)$ -dimensional Euclidean space \mathbb{R}^{n+1} , denote by A the shape operator associated to the second fundamental form of the immersion and by $k_1(p), \dots, k_n(p)$ the principal curvatures of M at a point p , labelled by the condition $k_1(p) \leq \dots \leq k_n(p)$. The squared length $|A|^2(p)$ of the second fundamental form at a point p is defined as the trace of $A^2(p)$. It is easy to see that

$$(2.1) \quad |A|^2(p) = \sum_{i=1}^n k_i^2(p).$$

Denote by R the scalar curvature of M and by H the mean curvature of the immersion. If e_1, \dots, e_n diagonalizes $A(p)$ with corresponding eigenvalues k_1, \dots, k_n , it follows from the Gauss equation [3] that the Ricci curvature of M at p in the direction e_i is given by

$$(2.2) \quad (n-1)\text{Ric}_p(e_i) = \sum_{j=1, j \neq i}^n k_i k_j = k_i(nH - k_i).$$

Taking the sum on i , we obtain

$$(2.3) \quad n^2 H^2 = |A|^2 + n(n-1)R.$$

For $1 \leq k \leq n$, the k -mean curvature $H_k(x)$ of M at a point x is defined by

$$(2.4) \quad H_k(x) = \frac{1}{\binom{n}{k}} \sigma_k(k_1(x), \dots, k_n(x)),$$

where $\sigma_k : \mathbb{R}^n \rightarrow \mathbb{R}$ is given by

$$(2.5) \quad \sigma_k(x_1, \dots, x_n) = \sum_{i_1 < \dots < i_k} x_{i_1} \dots x_{i_k}$$

and is called the k -elementary symmetric function. Notice that H_1 is the mean curvature H of the hypersurface and H_2 is, by the Gauss equation [3], simply the

scalar curvature R of M (more generally, for hypersurfaces of an ambient space with constant sectional curvature c , we have $R = H_2 + c$).

For $1 \leq k \leq n$, denote by Γ_k the connected component of the set $\{\sigma_k > 0\}$ that contains the vector $(1, \dots, 1)$. It follows immediately from the definitions that Γ_k contains the positive cone $\mathcal{O}^n = \{(x_1, \dots, x_n) \in \mathbb{R}^n : x_i > 0, \forall i, \}$, for all $1 \leq k \leq n$. It was proved by Gårding [9] that Γ_k is an open convex cone, $1 \leq k \leq n$, and that

$$(2.6) \quad \Gamma_1 \supset \Gamma_2 \supset \dots \supset \Gamma_n.$$

Given a hypersurface $M^n \subset \mathbb{R}^{n+1}$, a point $p \in M$ and a vector $\eta_o \perp T_pM$, $|\eta_o| = 1$, we can parametrize a neighborhood of p in M by

$$(2.7) \quad \varphi(x) = x + \mu(x)\eta_o,$$

for some smooth real-valued function $\varphi : V \rightarrow \mathbb{R}$ defined in a neighborhood of 0 in T_pM .

Let M_1^n and M_2^n be hypersurfaces of \mathbb{R}^{n+1} tangent at a point p and η_o a unitary vector normal to $T_pM_1 = T_pM_2$. Parametrize M_1 and M_2 as in (2.7), obtaining corresponding functions μ_1 and μ_2 . As in [7], we say that M_1 remains above M_2 in a neighborhood of p with respect to η_o if $\mu_1(x) \geq \mu_2(x)$ for all x in a neighborhood of zero.

In our proofs we will make use of the following theorems, which are particular cases of a general tangency principle by Silva and the author ([7], Theorem 1.1).

Theorem 2.1 (Tangency Principle for Mean Curvature). *Let M_1^n and M_2^n be hypersurfaces of \mathbb{R}^{n+1} tangent at a point p and suppose that M_1 remains above M_2 in a neighborhood of p with respect to a unit vector $\eta_o \perp T_pM_1$. If the mean curvature of M_2 at $(x, \varphi_2(x))$ is greater than or equal to the mean curvature of M_1 at $(x, \varphi_1(x))$, for all x sufficiently small, then M_1 and M_2 coincide in a neighborhood of p .*

Theorem 2.2 (Tangency Principle for Scalar Curvature). *Let M_1^n and M_2^n be hypersurfaces of \mathbb{R}^{n+1} tangent at a point p and suppose that M_1 remains above M_2 in a neighborhood of p with respect to a unit vector $\eta_o \perp T_pM_1$. If the scalar curvature of M_2 at $(x, \varphi_2(x))$ is greater than or equal to the scalar curvature of M_1 at $(x, \varphi_1(x))$, for all x sufficiently small, and if all principal curvatures $k_1(p), \dots, k_n(p)$ of M_2 at p are positive (or more generally, if $(k_1(p), \dots, k_n(p)) \in \Gamma_2$), then M_1 and M_2 coincide in a neighborhood of p .*

3. PROOFS OF THE THEOREMS

Proof of Theorem 1.1. We can suppose $c := \inf_M |H| > 0$. Otherwise, inequality (1.3) is trivial. Choose the orientation for M so that $H \geq c > 0$, and take a sphere S of radius r , disjoint from M^n and contained in the component of $(\overline{B}_r \times \mathbb{R}) \setminus M$ that contains the normals. Move S until it touches M for the first time, say at p , and denote by N the unit normal vector field in M . By our assumption that M is a graph over \overline{B}_r we have that p belongs to the interior of M . If p_o is the center of S , we have that p is a point where the function $f : M \rightarrow \mathbb{R}$, $f(x) = \frac{1}{2} \|x - p_o\|^2$, attains its minimum. If e_1, \dots, e_n is an orthonormal basis of T_pM such that $A(e_i) = k_i(p)e_i$, $i = 1, \dots, n$, we thus have

$$(3.1) \quad 0 = \text{grad}f(p) = (p - p_o)^T$$

and

$$(3.2) \quad 0 \leq \text{Hess}f(p)(e_i, e_i) = 1 + \langle \sigma(e_i, e_i), p - p_o \rangle = 1 + \langle p - p_o, N(p) \rangle k_i(p).$$

Equality (3.1) implies that

$$N(p) = \frac{p_o - p}{\|p_o - p\|} = \frac{p_o - p}{r}$$

and, by substitution of this in (3.2), we conclude that $k_i(p) \leq \frac{1}{r}$, $i = 1, \dots, n$. Thus

$$(3.3) \quad nc \leq nH(p) = k_1(p) + \dots + k_n(p) \leq \frac{n}{r},$$

from which we obtain

$$(3.4) \quad \inf_M |H| = c \leq 1/r.$$

If equality occurs in (3.4), we have $H \geq 1/r$ along M and, by Theorem 2.1, M and S coincide in a neighborhood of p . By a connectedness argument, we conclude that M is a closed hemisphere of S . In particular, the tangent planes to M along ∂M are vertical, contradicting the assumption that M is a graph over \overline{B}_r . This contradiction implies that the inequality in (3.4) is strict. \square

Proof of Theorem 1.2. We will first prove (1.4). If R changes sign, there is, by continuity, a point where the scalar curvature vanishes and (1.4) follows trivially. If $R > 0$ along M , we have from (2.3),

$$(3.5) \quad n(n - 1)|R| = n(n - 1)R = n^2H^2 - |A|^2 \leq n^2H^2,$$

which implies that

$$(3.6) \quad |R| \leq \frac{nH^2}{n - 1} \leq \frac{n}{n - 1}|H| \sup |H|.$$

Using Theorem 1.1, we obtain

$$(3.7) \quad \inf_M |R| \leq \frac{n}{n - 1} \sup |H| \inf |H| \leq \frac{n}{r(n - 1)} \sup |H|,$$

from which we easily obtain (1.4).

Now suppose that $R < 0$ everywhere and orient M by a unit normal vector field N . As in the proof of Theorem 1.1, take a sphere S of radius r , disjoint from M and contained in the component of $(\overline{B}_r \times \mathbb{R}) \setminus M$ that contains the normals, and move S until it touches M for the first time, say at p . Since $R < 0$ along M , we have principal curvatures of both signs at each point of M . Let l be the number of negative principal curvatures of M at p so that

$$(3.8) \quad k_1(p) \leq \dots \leq k_l(p) < 0 \leq k_{l+1}(p) \leq \dots \leq k_n(p).$$

By the Gauss equation, we have

$$(3.9) \quad \begin{aligned} 0 > \frac{n(n - 1)}{2}R(p) &= \sum_{1 \leq i < j \leq n} k_i k_j \\ &= \sum_{1 \leq i < j \leq l} k_i k_j + \sum_{l+1 \leq i < j \leq n} k_i k_j + \sum_{i=1, \dots, l; j=l+1, \dots, n} k_i k_j \\ &\geq \sum_{i=1, \dots, l; j=l+1, \dots, n} k_i k_j, \end{aligned}$$

and so

$$\begin{aligned}
 (3.10) \quad 0 &> \frac{n(n-1)}{2}R(p) \geq (k_1 + \cdots + k_l)(k_{l+1} + \cdots + k_n) \\
 &= \left(nH - \sum_{i=l+1}^n k_i\right) \sum_{i=l+1}^n k_i.
 \end{aligned}$$

Since $k_i(p) \leq 1/r, i = 1, \dots, n$ (see the proof of Theorem 1.1), we arrive at

$$\begin{aligned}
 (3.11) \quad \frac{n(n-1)}{2} \inf |R| &\leq \frac{n(n-1)}{2} |R(p)| \\
 &\leq \left(n \sup |H| + \sum_{i=l+1}^n k_i\right) \sum_{i=l+1}^n k_i \\
 &\leq \left(n \sup |H| + \frac{n-l}{r}\right) \frac{n-l}{r} \\
 &\leq \left(n \sup |H| + \frac{n-1}{r}\right) \frac{n-1}{r},
 \end{aligned}$$

from which we easily obtain (1.4).

We will now proceed to prove the second part of the theorem. Let q be a point where the second fundamental form is semi-definite and choose the orientation N so that all principal curvatures at q are nonnegative. We can suppose that $\inf_M |R| > 0$; otherwise there is nothing to prove. Since $k_i(q) \geq 0, i = 1, \dots, n$, we have $R > 0$ along M . From $\mathcal{O}^n \subset \Gamma_2$ (see Section 2) we infer that the principal curvature vector $\vec{k}(q) = (k_1(q), \dots, k_n(q))$ of M at q belongs to $\overline{\Gamma_2}$. Since $R(q) > 0$, we have in fact $\vec{k}(q) \in \Gamma_2$. It follows from the connectedness of both M and Γ_2 that $\vec{k}(x) \in \Gamma_2$, for all $x \in M$. In particular, $\vec{k}(p) \in \Gamma_2$, where p is as in the first part of the proof. If we had

$$(3.12) \quad \inf_M |R| \geq 1/r^2,$$

we would conclude, by Theorem 2.2, that M and S coincide in a neighborhood of p . Reasoning as in the proof of Theorem 1.1, we would conclude that M is a closed hemisphere of S , contradicting the assumption that M is a graph over $\overline{B_r}$. This contradiction implies that (3.12) does not hold and concludes the proof of the theorem. \square

Proof of Theorem 1.5. Since the Ricci curvature of M is negative, we have, by (2.2), that all principal curvatures are nonzero and that there are principal curvatures of both signs at each point of M . Let l be the number of negative principal curvatures, so that $k_1 \leq \cdots \leq k_l < 0 < k_{l+1} \leq \cdots \leq k_n$. Since $n \geq 3$, we can choose the orientation so that $n-1 \geq l \geq 2$. Let S and p be as in the proof of Theorem 1.1 and choose an orthonormal basis $\{e_1, \dots, e_n\}$ of T_pM satisfying $A(e_i) = k_i e_i, i = 1, \dots, n$. Since the Ricci curvature is negative, one has, by (2.2),

$$k_i(k_1 + \cdots + \widehat{k_i} + \cdots + k_l + k_{l+1} + \cdots + k_n) < 0, \quad i = 1, \dots, l,$$

and, since $k_i < 0$,

$$k_{l+1} + \cdots + k_n > -k_1 - \cdots - \widehat{k_i} - \cdots - k_l = |k_1| + \cdots + |\widehat{k_i}| + \cdots + |k_l|,$$

where the circumflex over k_i means that this term is omitted in the sum. Taking the sum with $i = 1, \dots, l$, we obtain

$$(3.13) \quad l(k_{l+1} + \dots + k_n) > (l-1) \sum_{m=1}^l |k_m|.$$

Since $k_i(p) \leq 1/r$, $i = 1, \dots, n$ (see the proof of Theorem 1.1), we arrive at

$$(3.14) \quad \sum_{m=1}^l |k_m| < \frac{l(n-l)}{r(l-1)}.$$

Thus

$$(3.15) \quad \sum_{m=1}^n |k_m| = \sum_{m=1}^l |k_m| + \sum_{m=l+1}^n |k_m| < \frac{l(n-l)}{r(l-1)} + \frac{n-l}{r} = \frac{(n-l)(2l-1)}{r(l-1)}.$$

Noticing that the right hand side of the above equation is strictly decreasing in l , we have

$$\sum_{m=1}^n |k_m| < \frac{3(n-2)}{r}.$$

Hence

$$|A|^2(p) = \sum_{m=1}^n |k_m|^2 < \left(\sum_{m=1}^n |k_m| \right)^2 < \left(\frac{3(n-2)}{r} \right)^2,$$

from which we obtain

$$\inf |A| \leq |A|(p) < \frac{3(n-2)}{r}. \quad \square$$

Proof of Theorem 1.7. Choose the orientation for M so that $H \geq 0$ and let S and p be as in the proof of Theorem 1.1. We have two cases to consider:

First case: All principal curvatures of M at p are nonnegative. Since $k_i(p) \leq 1/r$, $i = 1, \dots, n$ (see the proof of Theorem 1.1), we have

$$(3.16) \quad |A|^2(p) = \sum_{i=1}^n k_i^2(p) \leq \frac{n}{r^2},$$

and so

$$(3.17) \quad \inf_M |A| \leq |A|(p) \leq \frac{\sqrt{n}}{r} < \frac{n}{r}.$$

Second case: There are negative principal curvatures of M at p . Let l be the number of negative principal curvatures so that

$$(3.18) \quad k_1(p) \leq \dots \leq k_l(p) < 0 \leq k_{l+1}(p) \leq \dots \leq k_n(p).$$

Notice that $l \leq n-1$ since $H \geq 0$. From $k_i(p) \leq 1/r$, $i = 1, \dots, n$, and $H \geq 0$, we obtain

$$(3.19) \quad \frac{n-l}{r} \geq k_{l+1}(p) + \dots + k_n(p) \geq -k_1(p) - \dots - k_l(p) = |k_1|(p) + \dots + |k_l|(p).$$

Hence,

$$\begin{aligned}
 |A|^2(p) &= \sum_{i=1}^l k_i^2 + \sum_{i=l+1}^n k_i^2 \leq \left(\sum_{i=1}^l |k_i| \right)^2 + \sum_{i=l+1}^n k_i^2 \\
 (3.20) \quad &\leq \frac{(n-l)^2}{r^2} + \frac{n-l}{r^2} = \frac{(n-l)(n-l+1)}{r^2} \\
 &\leq \frac{n(n-1)}{r^2} < \frac{n^2}{r^2},
 \end{aligned}$$

from which we obtain (1.8). \square

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REFERENCES

1. J. L. M. Barbosa, G. P. Bessa and J. F. Montenegro, *On Bernstein-Heinz-Chern-Flanders inequalities*, Math. Proc. Cambridge Philos. Soc., vol. 144, 2008, pp. 457–464. MR2405902 (2009c:53039)
2. S. S. Chern, *On the curvatures of a piece of hypersurface in Euclidean space*, Abh. Math. Sem. Univ. Hamburg, vol. 29, 1965, pp. 77–91. MR0188949 (32:6376)
3. M. Dajczer, *Submanifolds and isometric immersions*, Mathematics Lectures Series, vol. 13, Houston: Publish or Perish, 1990. MR1075013 (92i:53049)
4. N. V. Efimov, *Hyperbolic problems in the theory of surfaces*, Proc. Int. Congress Math. Moscow (1966), Amer. Math. Soc. Transl. (2), vol. 70, 1968, pp. 26–38. MR0238232 (38:6508), MR0225620 (37:1213)
5. M. F. Elbert, *On complete graphs with negative r -mean curvature*, Proc. Amer. Math. Soc., vol. 128, 2000, pp. 1443–1450. MR1712913 (2000j:53081)
6. H. Flanders, *Remark on mean curvature*, J. London Math. Soc., vol. 41, 1966, pp. 364–366. MR0193600 (33:1818)
7. F. Fontenele and S. L. Silva, *A tangency principle and applications*, Illinois J. Math., vol. 45, 2001, pp. 213–228. MR1849995 (2003b:53057)
8. F. Fontenele and S. L. Silva, *Sharp estimates for the size of balls in the complement of a hypersurface*, Geom. Dedicata, vol. 115, 2005, pp. 163–179. MR2180046 (2007h:53081)
9. L. Gårding, *An inequality for hyperbolic polynomials*, J. Math. Mech., vol. 8, 1959, pp. 957–965. MR0113978 (22:4809)
10. T. Hasanis and T. Vlachos, *Curvature properties of hypersurfaces*, Arch. Math. (Basel), vol. 82, 2004, pp. 570–576. MR2080296 (2006a:53073)
11. E. Heinz, *Über Flächen mit eindeutiger Projektion auf eine Ebene, deren Krümmungen durch Ungleichungen eingeschränkt sind.*, Math. Ann., vol. 129, 1955, pp. 451–454. MR0071822 (17:189a)
12. T. Klotz and R. Osserman, *Complete surfaces in E^3 with constant mean curvature*, Comment. Math. Helv., vol. 41, 1966–67, pp. 313–318. MR0211332 (35:2213)
13. T. Okayasu, *$O(2) \times O(2)$ -invariant hypersurfaces with constant negative scalar curvature in E^4* , Proc. Amer. Math. Soc., vol. 107, 1989, pp. 1045–1050. MR990430 (90f:53094)
14. R. Reilly, *On the Hessian of a function and the curvatures of its graph*, Michigan Math. J., vol. 20, 1973, pp. 373–383. MR0334045 (48:12364)
15. I. Salavessa, *Graphs with parallel mean curvature*, Proc. Amer. Math. Soc., vol. 107, 1989, pp. 449–458. MR965247 (90a:53072)
16. B. Smith and F. Xavier, *Efimov's theorem in dimension greater than two*, Invent. Math., vol. 90, 1987, pp. 443–450. MR914845 (89h:53014)

17. S. T. Yau, *Submanifolds with constant mean curvature. II*, Amer. J. Math., vol. 97, 1975, pp. 76–100. MR0370443 (51:6670)
18. S. T. Yau, *Seminar on differential geometry*, Annals of Mathematics Studies, vol. 102. Princeton, N.J.: Princeton Univ. Press; Tokyo: University of Tokyo Press, 1982. MR645728 (83a:53002)

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