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$n!$ MATCHINGS, $n!$ POSETS

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Abstract. We show that there are $n!$ matchings on $2n$ points without so-called left (neighbor) nestings. We also define a set of naturally labeled $(\mathbf{2}+\mathbf{2})$-free posets and show that there are $n!$ such posets on n elements. Our work was inspired by Bousquet-Mélou, Claesson, Dukes and Kitaev [J. Combin. Theory Ser. A. 117 (2010) 884–909]. They gave bijections between four classes of combinatorial objects: matchings with no neighbor nestings (due to Stoimenow), unlabeled $(\mathbf{2}+\mathbf{2})$-free posets, permutations avoiding a specific pattern, and so-called ascent sequences. We believe that certain statistics on our matchings and posets could generalize the work of Bousquet-Mélou et al., and we make a conjecture to that effect. We also identify natural subsets of matchings and posets that are equinumerous to the class of unlabeled $(\mathbf{2}+\mathbf{2})$-free posets.

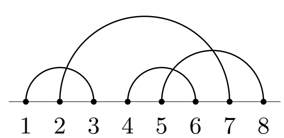
We give bijections that show the equivalence of (neighbor) restrictions on nesting arcs with (neighbor) restrictions on crossing arcs. These bijections are thought to be of independent interest. One of the bijections factors through certain upper-triangular integer matrices that have recently been studied by Dukes and Parviainen [Electron. J. Combin. 17 (2010) #R53].

1. Introduction

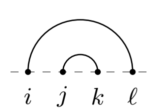
A *matching* of the integers $\{1, 2, \ldots, 2n\}$ is a partition of that set into blocks of size 2. An example of a matching is

$$M = \{(1,3), (2,7), (4,6), (5,8)\}.$$

In the diagram below there is an *arc* connecting i with j precisely when $(i,j) \in M$:



A *nesting* of M is a pair of arcs (i,ℓ) and (j,k) with $i < j < k < \ell$:



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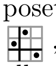
We call such a nesting a *left-nesting* if $j = i + 1$. Similarly, we call it a *right-nesting* if $\ell = k + 1$. The example matching has one nesting, formed by the two arcs $(2, 7)$ and $(4, 6)$. It is a right-nesting.

To give upper bounds on the dimension of the space of Vassiliev's knot invariants of a given degree, Stoimenow [13] was led to introduce what he calls regular linearized chord diagrams. In the terminology of this paper, Stoimenow's diagrams are matchings with no *neighbor nestings*, that is, matchings with neither left-nestings nor right-nestings. Following Stoimenow's paper, Zagier [16] derived the following beautiful generating function enumerating such matchings with respect to size:

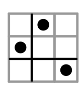
$$\sum_{n\geq 0} \prod_{i=1}^{n} \left(1 - (1 - t)^i\right).$$

Recently, Bousquet-Mélou et al. [2] gave bijections between matchings on $[2n]$ with no neighbor nestings and three other classes of combinatorial objects, thus proving that they are equinumerous. The other classes were unlabeled $(\mathbf{2}+\mathbf{2})$-free posets (or interval orders) on n nodes, permutations on $[n]$ avoiding the pattern , and ascent sequences of length n. Let $\mathfrak{f}_n$ be the cardinality of any, and thus all, of the above classes—it is the coefficient in front of t^n in Zagier's generating function. We call $\mathfrak{f}_n$ the nth *Fishburn number*; the sequence starts

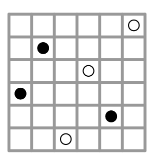
$$1, 1, 2, 5, 15, 53, 217, 1014, 5335, 31240, \ldots.$$

Fishburn [6, 7, 8] did pioneering work on interval orders; for instance, he showed the basic theorem that a poset is an interval order if and only if it is $(\mathbf{2}+\mathbf{2})$-free.

The pattern avoiding permutations and the ascent sequences were both defined by Bousquet-Mélou et al. We shall recall those definitions here. In a permutation $\pi = a_1 \ldots a_n$, an occurrence of the pattern



is a subsequence $a_i a_{i+1} a_j$ such that $a_j + 1 = a_i < a_{i+1}$. As an example, the permutation $\pi = 351426$ contains one such occurrence:



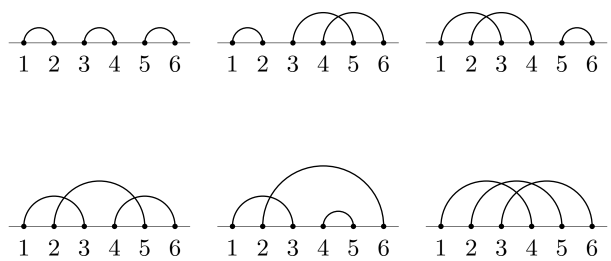
If π contains no such occurrence we say that π *avoids* the pattern. An integer sequence $(x_1, \ldots, x_n)$ is an *ascent sequence* if

$$x_1 = 0 \quad \text{and} \quad 0 \leq x_i \leq 1 + \operatorname{asc}(x_1, \ldots, x_{i-1}),$$

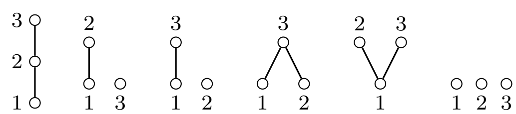
for $2 \leq i \leq n$. Here, $\operatorname{asc}(x_1, \ldots, x_k)$ denotes the number of ascents in $(x_1, \ldots, x_k)$, and an ascent is a $j \in [k-1]$ such that $x_j < x_{j+1}$. Bousquet-Mélou et al. [2] derived a closed expression for the generating function enumerating ascent sequences with respect to length and number of ascents; hence they gave a new proof of Zagier's result, or rather a refinement of it.

Recall that Stoimenow's diagrams are matchings with no neighbor nestings. The discovery that led to the present paper is that there are exactly $n!$ matchings on $[2n]$

with no left-nestings (Theorem 2.1). As an example, these are the 6 such matchings on $\{1, \dots, 6\}$:

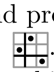


Can we also "lift" ascent sequences and unlabeled $(\mathbf{2}+\mathbf{2})$-free posets to the level of all permutations? That is, can we define "certain sequences" and "certain posets", both of cardinality $n!$, that are supersets of ascent sequences and unlabeled $(\mathbf{2}+\mathbf{2})$-free posets, respectively? For ascent sequences this is easy, and inversion tables is a natural choice. The poset case is more challenging. However, we show (Definition 3.1 and Theorem 3.4) that there are exactly $n!$ naturally labeled posets P on $[n]$ such that $i <_P k$ whenever $i < j <_P k$ for some $j \in [n]$; we call them *factorial posets*. Here is a list of the 6 factorial posets on $\{1, 2, 3\}$:



It is not hard to see (Proposition 3.3) that factorial posets are $(\mathbf{2}+\mathbf{2})$-free. Moreover, we give an additional restriction on the labeling of factorial posets under which the labeling is unique (Proposition 4.1), and thus the subset of factorial posets meeting that restriction is trivially in bijection with unlabeled $(\mathbf{2}+\mathbf{2})$-free posets.

The bijections we give to prove that inversion tables, factorial posets and matchings with no left-nesting are equinumerous do not however specialize to give back the results from [2]. This remains an interesting challenge. In Section 5 we prove that we could have studied matchings with restrictions on crossings instead of on nestings and present bijections to verify this.

Let $p =$ . As mentioned before, Bousquet-Mélou et al. [2] gave a bijection between matchings with no neighbor nestings and p-avoiding permutations. We conjecture (Conjecture 8.4) a generalization of that result. Namely, we conjecture that the distribution of right-nestings over matchings on $[2n]$ with no left-nestings coincides with the distribution of p over permutations on $[n]$.

In a recent paper, Dukes and Parviainen [5] study upper triangular matrices with non-negative integer entries such that each row and column has at least one non-zero entry and the total sum of the entries is n. They provide a recursive encoding of those matrices as ascent sequences. We have found a direct bijection (Theorem 5.2) from the same matrices to matchings with no neighbor nestings. In addition, we show (Proposition 5.6) that the subset of the matrices whose entries are 0 or 1 are in bijection with matchings with no left-nestings and no right-crossings.

2. Matchings with no left-nestings

Let $\mathcal{M}_n$ be the set of matchings on $[2n]$, and let $M \in \mathcal{M}_n$. If $i < j$ and $\alpha = (i, j)$ is an arc of M, we call i the *opener* of α and j the *closer* of α. In what follows it will be convenient to order the arcs with respect to closer. In particular, "the last arc" refers to the arc with closer $2n$. In the introduction we defined what left- and right-nestings are, and by $\mathrm{lne}(M)$ and $\mathrm{rne}(M)$ we shall denote the number of left- and right-nestings, respectively. Let

$$\mathcal{N}_n = \{\, M \in \mathcal{M}_n : \mathrm{lne}(M) = 0 \,\}$$

and $\mathcal{N} = \bigcup_{n \geq 0} \mathcal{N}_n$. Define $\mathcal{I}_n$ as the Cartesian product

$$\mathcal{I}_n = [0, 0] \times [0, 1] \times \cdots \times [0, n-1],$$

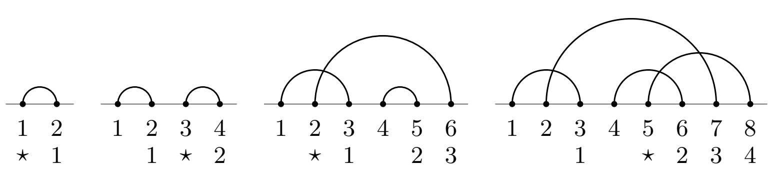
where $[i, j] = \{i, i+1, \ldots, j\}$. In other words, $\mathcal{I}_n$ is the set of inversion tables of length n. Also, let $\mathcal{I} = \bigcup_{n \geq 0} \mathcal{I}_n$.

Theorem 2.1. *Matchings of $[2n]$ with no left-nestings are in bijection with inversion tables of length n, and thus $|\mathcal{N}_n| = n!$.*

Proof. Using recursion we define a bijection $f : \mathcal{I} \to \mathcal{N}$. Let $f(\epsilon) = \emptyset$; that is, let the empty inversion table map to the empty matching. Let $w = (a_1, \ldots, a_n)$ be any inversion table in $\mathcal{I}_n$ with $n > 0$. Let $w' = (a_1, \ldots, a_{n-1})$ and let $M' = f(w')$. Now create a matching M in $\mathcal{N}_n$ by inserting a new last arc in M' whose opener is immediately to the left of the $(a_n + 1)$st closer of M' if $a_n < n - 1$ and immediately to the left of its own closer if $a_n = n - 1$. Set $f(w) = M$. Note that the opener of the last arc has to be immediately to the left of some closer; otherwise a left-nesting would be created. Also note that removing the last arc from a matching in $\mathcal{N}_n$ cannot create a left-nesting. From a simple induction argument it thus follows that the described map is a bijection.

It is also easy to give a direct, non-recursive description of the inverse of f. Indeed, $f^{-1}(M) = (a_1, \ldots, a_n)$, where a_i is the number of closers to the left of the opener of the ith arc. Here, as before, arcs are ordered by closer. □

As an example, let $w = (a_1, a_2, a_3, a_4) = (0, 1, 0, 1)$. To construct the matching corresponding to that inversion table we insert the arcs one at a time, so that—as in the proof—the opener of the new last arc is immediately to the left of the $(a_i + 1)$st closer:



Here the star marks the opener of the new arc. Reading the number to the right of the star we get $(1, 2, 1, 2)$, and subtracting one from each coordinate we recover the inversion table $(0, 1, 0, 1)$.

3. Factorial posets

A poset P of cardinality n is said to be labeled if its elements are identified with the integers $1, \ldots, n$. A poset P is *naturally labeled* if $i < j$ in P implies $i < j$ in the usual order.

Definition 3.1. We call a naturally labeled poset P on $[n]$ such that, for $i, j, k \in [n]$,

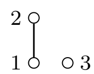
$$i < j <_P k \implies i <_P k$$

a *factorial poset*, and by $\mathcal{F}_n$ we denote the set of factorial posets on $[n]$. Similarly, we call a naturally labeled poset P on $[n]$ such that, for $i, j, k \in [n]$,

$$i > j >_P k \implies i >_P k$$

a *dually factorial poset*.

There are 6 factorial posets on $\{1, 2, 3\}$, and we listed them on page 1. It is easy to check that of those posets, exactly one is not dually factorial, namely



Denoting this poset by P we have $3 > 2 >_P 1$, but $3 \not>_P 1$.

Definition 3.2. The *predecessor set* of $j \in P$ is $\operatorname{Pred}(j) = \{i : i <_P j\}$, and we denote by $\operatorname{pred}(j) = \#\operatorname{Pred}(j)$ the number of predecessors of j. Similarly we define $\operatorname{Succ}(j) = \{i : i >_P j\}$ as the *successor set* of j and $\operatorname{succ}(j) = \#\operatorname{Succ}(j)$ as the number of successors of j.

Note that P is factorial if, and only if, for all k in P there is a j in $[0, n-1]$ such that $\operatorname{Pred}(k) = [1, j]$. It is well known—see for example Bogart [1]—that a poset is $(\mathbf{2}+\mathbf{2})$-free if, and only if, the collection $\{\operatorname{Pred}(k) : k \in P\}$ of predecessor sets can be linearly ordered by inclusion; hence the following proposition.

Proposition 3.3. *Factorial posets are* $(\mathbf{2}+\mathbf{2})$*-free.*

Theorem 3.4. *Factorial posets on* $[n]$ *are in bijection with inversion tables of length* n*, and thus* $|\mathcal{F}_n| = n!$.

Proof. Define $g : \mathcal{F}_n \to \mathcal{I}_n$ by $g(P) = (a_1, \dots, a_n)$, where $a_k = \operatorname{pred}(k)$. To see that g is a bijection we describe its inverse. Given an inversion table $w = (a_1, \dots, a_n)$ in $\mathcal{I}_n$ we construct a factorial poset $P = P(w)$ by postulating that $i <_P k$ precisely when $1 \le i \le a_k$. That this definition is consistent is easily seen by building P recursively. $\square$

We now have two bijections, f from inversion tables to matchings with no left-nestings, and g from factorial posets to inversion tables. Let $h = f \circ g$ be their composition:

$$\begin{tikzcd}
\mathcal{F}_n \arrow[rr, "h"] \arrow[dr, "g"'] & & \mathcal{N}_n \\
& \mathcal{I}_n \arrow[ur, "f"'] &
\end{tikzcd}$$

Let $P \in \mathcal{F}_n$. From the proofs of Theorems 3.4 and 2.1 it is immediate that to build $M = h(P)$ we insert the arcs one at the time so that, in the ith step, the opener of the new last arc is immediately to the left of the $(\operatorname{pred}(i)+1)$st closer.

Next we describe the inverse map, h^{-1}. Take $M \in \mathcal{N}_n$ and let $\alpha_1, \dots, \alpha_n$ be its arcs ordered by the closer. Then $i < j$ in $P = h^{-1}(M)$ if and only if the closer of α_i is to the left of the opener of α_j.

An *interval order* is a poset with the property that each element x can be assigned an interval $I(x)$ of real numbers so that $x < y$ in the poset if and only if every point in $I(x)$ is less than every point in $I(y)$. Such an assignment is called an *interval representation* of the poset. In 1970, Fishburn [7] showed that a poset is $(\mathbf{2}+\mathbf{2})$-free precisely when it has an interval representation. Let us for a moment identify the arcs of a matching with intervals of the real line. Then the function h, above, gives an interval representation of each factorial poset.

4. A unique labeling

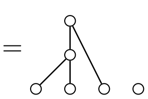
Let $M \in \mathcal{N}_n$ and let $\alpha_1, \ldots, \alpha_n$ be its arcs ordered by the closer. Let $P = h^{-1}(M)$. Assume that $1 \leq i < j \leq n$ in the usual order. Note that if α_i and α_j form a nesting, then we cannot have $\mathrm{pred}(i) = \mathrm{pred}(j)$, since then it would be a left-nesting, which can never occur by the definition of g^{-1}. Thus α_i and α_j form a nesting precisely when $\mathrm{pred}(i) > \mathrm{pred}(j)$. If, in addition, $j = i+1$ and $\mathrm{succ}(i) = \mathrm{succ}(j)$, then α_i and α_j form a right-nesting. Thus M is non-neighbor-nesting precisely when for each $i \in [n-1]$ we have $\mathrm{pred}(i) \leq \mathrm{pred}(i+1)$ or $\mathrm{succ}(i) > \mathrm{succ}(i+1)$. By applying the bijection of Bousquet-Mélou et al. [2] from non-neighbor-nesting matchings to unlabeled $(\mathbf{2}+\mathbf{2})$-free posets, we get the following result.

Proposition 4.1. *Factorial posets on $[n]$ such that for each $i \in [n-1]$ we have*

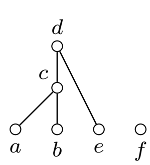
$$\mathrm{pred}(i) \leq \mathrm{pred}(i+1) \quad \textit{or} \quad \mathrm{succ}(i) > \mathrm{succ}(i+1) \tag{4.1}$$

are in bijection with unlabeled $(\mathbf{2}+\mathbf{2})$-free posets on n nodes; hence there are exactly $\mathfrak{f}_n$ such posets.

An alternative way to see the above result is that given an unlabeled $(\mathbf{2}+\mathbf{2})$-free poset P there is exactly one way to label P so that the resulting poset is factorial and satisfies (4.1). The key observation to such a labeling is that if P is factorial and (4.1) holds, then the pairs $(\mathrm{succ}(1), \mathrm{pred}(1)), \ldots, (\mathrm{succ}(n), \mathrm{pred}(n))$ are ordered weakly decreasing with respect to the first coordinate, and on equal first coordinate are weakly increasing with respect to the second coordinate. As an example we consider the unlabeled $(\mathbf{2}+\mathbf{2})$-free poset

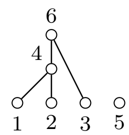
$P =$ 

We shall use the observation above to label P so that it is factorial and satisfies (4.1). We start by naming the vertices a, b, c, d, e and f. Then we calculate predecessor and successor sets:



x	$=$	a	b	c	d	e	f
$\mathrm{Pred}(x)$	$=$	$\emptyset$	$\emptyset$	ab	$abce$	$\emptyset$	$\emptyset$
$\mathrm{Succ}(x)$	$=$	cd	cd	d	$\emptyset$	d	$\emptyset$
$(\mathrm{succ}(x), \mathrm{pred}(x))$	$=$	$(2,0)$	$(2,0)$	$(1,2)$	$(0,4)$	$(1,0)$	$(0,0)$
label	$=$	1	2	4	6	3	5

Finally, we label the elements with the integers 1, 2, 3, 4 and 5 so that when we read the (succ, pred)-pair for the vertex labeled 1 and then the (succ, pred)-pair for the vertex labeled 2, and so on, those pairs are read in the prescribed order:



In this example, we could have chosen to label a by 2 and b by 1; it would not have made a difference, since the vertices named a and b are indistinguishable.

5. Crossings versus nestings

A *crossing* of a matching M is a pair of arcs (i,k) and (j,ℓ) with $i<j<k<\ell$, and we can define left- and right-crossings analogously to how they were defined for nesting arcs. With A and B as in the table below there are bijections between

$$\{M \in \mathcal{M}_n : M \text{ is non } A\} \text{ and } \{M \in \mathcal{M}_n : M \text{ is non } B\}.$$

A	B
nesting	crossing
neighbor nesting	neighbor crossing
left-nesting	left-crossing

The first case is well known: for bijections between non-nesting matchings and non-crossing matchings; see for instance [3, 4, 9]. We give bijections for the two remaining cases in this section. There exists a more complicated bijection [14] that can explain all three levels at once; see the comment at the end of this section.

The second case is the most challenging, so let us look at the third case first. The proof of Theorem 2.1 gives a bijection f from inversion tables to non-left-nesting matchings. This bijection can be modified to give a bijection f_{nc} from inversion tables to non-left-crossing matchings (Theorem 5.1), and so $f_{\mathrm{nc}} \circ f^{-1}$ is a bijection from non-left-nesting to non-left-crossing matchings.

Theorem 5.1. *Matchings of $[2n]$ with no left-crossing are in bijection with inversion tables of length n; hence there are exactly $n!$ such matchings.*

Proof. As in the proof of Theorem 2.1 we define a bijection f_{nc} recursively. The difference is that this time the opener of the new last arc is immediately to the right of the a_nth closer if $a_n > 0$ or to the extreme left if $a_n = 0$. □

For the second case, we shall give a bijection via matrices of a certain kind. Let $\mathcal{T}_n$ be the set of upper triangular matrices with non-negative integer entries, such that no row or column has only zeros and the total sum of the entries is n. These matrices have recently been studied by Dukes and Parviainen [5, §2]. They gave a recursive encoding of the matrices in $\mathcal{T}_n$ as ascent sequences, and thus they showed that $|\mathcal{T}_n| = \mathfrak{f}_n$. This fact seems to have been first observed by Vladeta Jovovic [10]. We shall give a surjection ψ from the set of matchings of $[2n]$ to $\mathcal{T}_n$. Further, we shall show that if ψ is restricted to non-neighbor-nesting matchings or to non-neighbor-crossing matchings, then ψ is a bijection.

Before we describe ψ we need a few definitions. Let M be a matching and let $O(M)$ and $C(M)$ be the set of openers and closers of M, respectively. Write

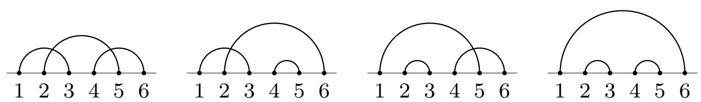
$$O(M) = O_1 \cup \cdots \cup O_k \text{ and } C(M) = C_1 \cup \cdots \cup C_\ell$$

as disjoint unions of maximal intervals. Clearly, $k = \ell$; we denote this number as $\mathrm{int}(M)$. As an example, for $M = \{(1,2),(3,5),(4,6)\}$ we have $O(M) = [1,1] \cup [3,4]$, $C(M) = [2,2] \cup [5,6]$ and $\mathrm{int}(M) = 2$.

We are now in a position to define the promised map from matchings to matrices. Assume that M is a matching and that its intervals of openers and closers are ordered in the natural order. Let $\psi(M) = T$, where $T = (t_{ij})$ is an $\mathrm{int}(M) \times \mathrm{int}(M)$ matrix and

$$t_{ij} = |M \cap O_i \times C_j|.$$

In other words, t_{ij} is the number of arcs whose opener is in O_i and closer in C_j. For instance, the preimage of $\left(\begin{smallmatrix} 1 & 1 \\ 0 & 1 \end{smallmatrix}\right)$ under ψ consists of the following 4 matchings:



Note that of these matchings exactly one has no neighbor nestings and exactly one has no neighbor crossings. We shall see that this is no coincidence.

Theorem 5.2. *When restricted to matchings of* $[2n]$ *with no neighbor nestings, the function* ψ*, defined above, is a bijection onto* $\mathcal{T}_n$*.*

Before we give the proof we need a lemma.

Lemma 5.3. *Let M be a matching. Assume that O' is an interval of openers in M and that C' is an interval of closers in M. We have:*

(1) *if* $\mathrm{lne}(M) = 0$*, then any pair of arcs with openers in O' are crossing;*
(2) *if* $\mathrm{rne}(M) = 0$*, then any pair of arcs with closers in C' are crossing;*
(3) *if* $\mathrm{lcr}(M) = 0$*, then any pair of arcs with openers in O' are nesting;*
(4) *if* $\mathrm{rcr}(M) = 0$*, then any pair of arcs with closers in C' are nesting.*

Proof. We shall prove (1). The remaining three statements are proved by similar arguments. Without loss of generality we may assume that $O' = \{o_i, o_{i+1}, \ldots, o_{i+j}\}$ is a maximal interval. If M has no left-nestings, then the arcs from o_i and o_{i+1} must cross. Similarly the arc from o_{i+2} must cross the arc from o_{i+1} and thus also the arc from o_i. So by an easy induction argument all the arcs must cross. $\square$

Proof of Theorem 5.2. Let $T \in \mathcal{T}_n$ be a $k \times k$ matrix. We shall show that there is a unique non-neighbor-nesting matching M of $[2n]$ such that $\psi(M) = T$.

Let r_i and c_i be the sum of the entries in, respectively, row i and column i of T. From the definition of ψ it is clear that

$$O(M) = O_1 \cup \cdots \cup O_k \text{ and } C(M) = C_1 \cup \cdots \cup C_k,$$

where $O_1 = [1, r_1]$, $C_1 = [r_1 + 1, r_1 + c_1]$, $O_2 = [r_1 + c_1 + 1, r_1 + c_1 + r_2]$, etc. Also, M shall have t_{ij} arcs from O_i to C_j. We must show how the arcs shall be drawn.

Since $\mathrm{lne}(M) = 0$, it follows from (1) of Lemma 5.3 that all arcs from O_i cross. Thus we know which t_{ij} openers, X_{ij}, in O_i will have arcs to C_j. Similarly, since $\mathrm{rne}(M) = 0$, it follows by (2) of Lemma 5.3 that all arcs to C_j cross. Thus we know which t_{ij} of the closers, Y_{ij}, in C_j will have arcs from O_i. So, for every $1 \le i \le j \le k$, M must have crossing arcs from X_{ij} to Y_{ij}.

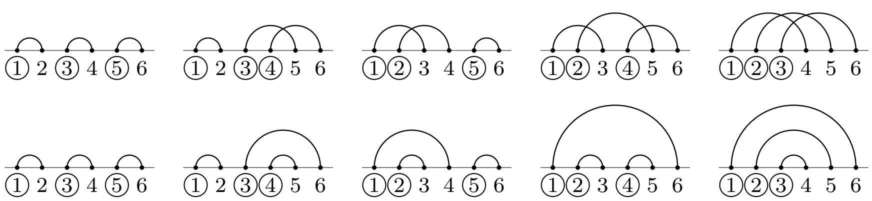
We have showed that there is exactly one M that (by construction) satisfies $\psi(M) = T$, and so the theorem follows. $\square$

Theorem 5.4. *When restricted to matchings of* $[2n]$ *with no neighbor crossings, the function* ψ*, defined above, is a bijection onto* $\mathcal{T}_n$*.*

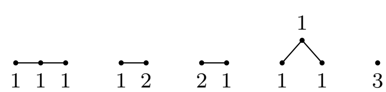
Proof. The proof is essentially the same as for Theorem 5.2. The difference is that here we use (3) and (4) of Lemma 5.3, instead of (1) and (2). $\square$

Below is an illustration of Theorems 5.2 and 5.4 for $n = 3$. We have circled the openers to make it easy to see the intervals.

$$\begin{pmatrix}1&0&0\\0&1&0\\0&0&1\end{pmatrix} \qquad \begin{pmatrix}1&0\\0&2\end{pmatrix} \qquad \begin{pmatrix}2&0\\0&1\end{pmatrix} \qquad \begin{pmatrix}1&1\\0&1\end{pmatrix} \qquad \begin{pmatrix}3\end{pmatrix}$$



We have now explained the hierarchy of nesting and crossing conditions that we set out to explain at the beginning of this section. As we pointed out, the bijections for the more general cases do not specialize to give bijections between the smaller sets. Indeed, if we specialize the map ψ to matchings with no nestings, we get the subset of matrices $(t_{ij}) \in \mathcal{T}_n$ such that for all $i, j, x, y > 0$, at least one of $t_{i,j}$ and $t_{i-x,j+y}$ must be zero. The non-zero entries in such a matrix will form a "path" with the entries as vertices, which can be seen to be equivalent to a Motzkin path. Thus, the matrices just described are in bijection with Motzkin paths with positive integer weights on the vertices of the path such that the sum of the weights is n. As an example, for $n = 3$ we have these five paths:



If we, on the other hand, specialize ψ to matchings with no crossings, we get the somewhat odd constraint that for all $i < i + x \leq j < j + y$ at least one of $t_{i,j}$ and $t_{i+x,j+y}$ must be zero.

Corollary 5.5. *The two subsets of* $\mathcal{T}_n$ *mentioned above are enumerated by the Catalan numbers.*

Before we close this section we give one more result that is almost for free given the map ψ. Let $\mathcal{T}_n^{01} \subset \mathcal{T}_n$ be the set of zero-one matrices in $\mathcal{T}_n$. For instance,

$$\mathcal{T}_3^{01} = \left\{ \begin{pmatrix}1&1\\0&1\end{pmatrix}, \begin{pmatrix}1&0&0\\0&1&0\\0&0&1\end{pmatrix} \right\}.$$

Dukes and Parviainen [5, §4] showed that the matrices in $\mathcal{T}_n^{01}$ correspond to those ascent sequences that have no two equal consecutive entries. We offer the following proposition.

Proposition 5.6. *When restricted to matchings of* $[2n]$ *with no left-nestings and no right-crossings, the function* ψ*, defined above, is a bijection onto* $\mathcal{T}_n^{01}$*.*

Proof. The proof is very similar to the proofs of Theorems 5.2 and 5.4, so we omit most of it. We do however make this key observation: Assume that $1 \le i \le j \le \mathrm{int}(M)$. From (1) and (4) of Lemma 5.3 there can be at most one arc from the O_i to O_j. □

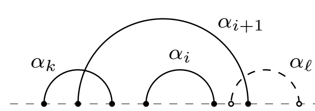
An important remark is that there exists a bijection by Sundaram [14] (see also exercise 7.24 in [12]), via certain walks in Young's lattice called oscillating tableaux, that uniformly shows all three cases above. For readers familiar with this bijection let us briefly explain why. Let M be a matching of $[2n]$ in which j and $j+1$ are two consecutive closers. Let $(\lambda^0, \dots, \lambda^{2n})$, where $\lambda^0 = \lambda^{2n} = \emptyset$, be the oscillating tableau corresponding to M. The assumption that j and $j+1$ are closers means that $\lambda^{j+1} \subset \lambda^j \subset \lambda^{j-1}$.

Let r_{j+1} be the row where λ^{j+1} has fewer elements than λ^j and let r_j be the row where λ^j has fewer elements than λ^{j-1}. Following the bumping paths of column insertion one can argue that the arcs ending in j and $j+1$ form a right-crossing if and only if $r_j \le r_{j+1}$, and thus they form a right-nesting if and only if $r_j > r_{j+1}$. Now, consider the involution M^* obtained by conjugating each partition in $(\lambda^0, \dots, \lambda^{2n})$ and then applying the inverse of Sundaram's bijection. A moment of thought gives us that the arcs ending in j and $j+1$ form a right-nesting in M if and only if they form a right-crossing in M^*. Similarly, if i and $i+1$ are two consecutive openers of M, then $\lambda^{i-1} \subset \lambda^i \subset \lambda^{i+1}$. This time let r_x be the row in which λ^x is greater than λ^{x-1}. Then the arcs with openers i and $i+1$ form a left-nesting if and only if $r_i < r_{i+1}$. Hence i and $i+1$ form a left-nesting in M if and only if they form a left-crossing in M^*.

This shows that the bijection in [14] may be used to explain all three levels discussed here at once. It also shows that by using the above restrictions we obtain two different subsets of all vacillating tableaux enumerated by $n!$ and one subset, satisfying both restrictions, that is enumerated by the Fishburn numbers.

6. Ascent and descent correcting sequences

On contemplating the picture



for a while, one realizes that condition (4.1) in Proposition 4.1 is equivalent to

$$i >_P k \text{ and } i+1 \not>_P k \implies i = \mathrm{pred}(\ell) \text{ for some } \ell \text{ in } P. \tag{6.1}$$

Let a *descent correcting sequence* be an inversion table $(a_1, \dots, a_n)$ such that

$$a_i > a_{i+1} \implies a_\ell = i \text{ for some } \ell > i.$$

That is, if there is a descent at position i, then this has to be "corrected" by the value i occurring later in the sequence. Condition (6.1) translates directly to the condition for a descent correcting sequence, and thus we have the following proposition.

Proposition 6.1. *There are exactly $\mathfrak{f}_n$ descent correcting sequences of length n, where $\mathfrak{f}_n$ is the nth Fishburn number.*

We may similarly use the map f_{nc} from matchings with no left-crossings to inversion tables. We then get that the sequences corresponding to matchings with no neighbor crossings are the inversion tables $(a_1, \dots, a_n)$ such that

$$a_i < a_{i+1} \neq i+1 \implies a_\ell = i \text{ for some } \ell > i.$$

We call them *ascent correcting sequences*. Using Theorem 5.4 we arrive at the following result.

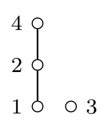
Proposition 6.2. *There are exactly $\mathfrak{f}_n$ ascent correcting sequences of length n.*

7. Posets that are both factorial and dually factorial

Note that being dually factorial entails the condition in Proposition 4.1. So, under h, matchings corresponding to dually factorial posets have no right-nestings. In fact, they do not have any nestings at all. To see this, assume that $M \in \mathcal{N}_n$ and let $\alpha_1, \dots, \alpha_n$ be its arcs ordered by closer. Also, assume that $1 \leq i < j \leq n$. Recall that the arcs α_i and α_j form a nesting precisely when $\mathrm{pred}(i) > \mathrm{pred}(j)$, which is equivalent to there being a $k <_P i$ such that $k \not<_P j$; this cannot happen in a dually factorial poset. It is easy to see that this argument works both ways, so $M = h(P)$ is non-nesting if and only if P is dually factorial. It is well known that non-nesting matchings are counted by the Catalan numbers. See for instance Stanley [12, Ex. 6.19uu]. One way to associate a given non-nesting matching with a Dyck path is to map its openers to up-steps and its closers to down-steps.

Proposition 7.1. *There are exactly $C_n = \binom{2n}{n}/(n+1)$ posets on $[n]$ that are both factorial and dually factorial.*

Let us mention an alternative way to prove this proposition. Below is the smallest example of a factorial poset that is not dually factorial but satisfies the condition of Proposition 4.1:



As stated by Proposition 3.3, factorial posets are $(\mathbf{2}+\mathbf{2})$-free; those that, in addition, are dually factorial are $(\mathbf{3}+\mathbf{1})$-free.

Proposition 7.2. *If P is a factorial poset satisfying* (4.1) *from Proposition* 4.1*, then P is dually factorial if and only if P is $(\mathbf{3}+\mathbf{1})$-free.*

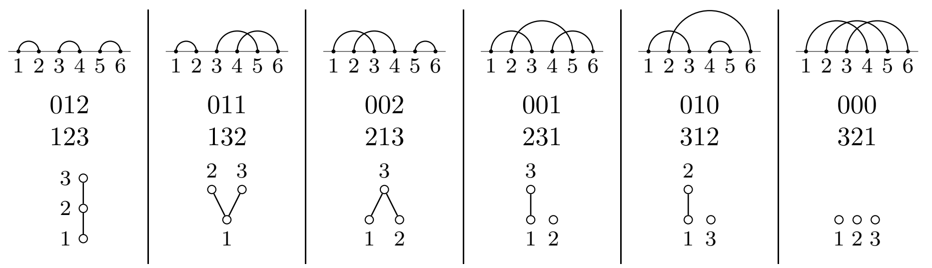
Proof. For factorial posets P on less than 4 elements the result is trivial. We shall assume that P has at least 4 elements and prove the contra-positive statement: P is not dually factorial if and only if P contains an induced subposet isomorphic to $\mathbf{3}+\mathbf{1}$. Assume that the elements $x <_P y <_P z$ and w form an induced subposet of P that is isomorphic to $\mathbf{3}+\mathbf{1}$. Since P is factorial we cannot have $w < y <_P z$ and $w \not<_P z$; thus $w > y$. Then $w > y >_P x$ and $w \not>_P x$, so P is not dually factorial.

Conversely, assume that P is not dually factorial. Then there exists x, y and w with $w > y >_P x$ but $w \not>_P x$. Assume further that y is maximal and then w is minimal with this property. We claim that $w = y+1$. If not, then $y+1 >_P x$ by the minimality of w. Using transitivity and $w \not>_P x$ this implies $w \not>_P y+1$, which contradicts the maximality of y. Since $w = y+1$ and $\mathrm{pred}(y) > \mathrm{pred}(w)$, it follows from property (4.1) that $\mathrm{succ}(y) > \mathrm{succ}(w)$. Thus there exists a z such that $z >_P y$ and $z \not>_P w$, and so the induced subposet on $\{x, y, z, w\}$ is isomorphic to $\mathbf{3} + \mathbf{1}$. $\square$

Since posets that are both factorial and dually factorial have a unique labeling, we can regard them as unlabeled. Further, unlabeled posets that are both $(\mathbf{2}+\mathbf{2})$- and $(\mathbf{3}+\mathbf{1})$-free (also called semiorders) are known to be enumerated by the Catalan numbers; see [12, Ex. 6.19ddd] and [15].

8. Statistics and equidistributions

One question we shall consider in this section is what statistics are respected by the bijections f, g and h. For reference, we list the size 3 matchings, inversion tables, permutations and posets that correspond to each other under those bijections:



There are several well known ways of translating between permutations and inversion tables. Here we have chosen the following way: Given $\pi \in \mathcal{S}_n$, we build the corresponding inversion table w from right to left. The rightmost letter of w is $\pi^{-1}(n) - 1$. The remaining letters of w are obtained by repeating this procedure on the length $n-1$ permutation that results from π by deleting n.

We shall now define the relevant statistics, and we start with statistics on posets. The ordinal sum [11, §3.2] of two posets P and Q is the poset $P \oplus Q$ on the union $P \cup Q$ such that $x \leq y$ in $P \oplus Q$ if $x \leq_P y$ or $x \leq_Q y$, or $x \in P$ and $y \in Q$. Let us say that P has k *components*, and write $\mathrm{comp}(P) = k$, if P is the ordinal sum of k, but not of $k+1$, non-empty posets. The number of minimal elements of a poset P is denoted by $\min(P)$. The number of levels of P—in other words, the number of distinct predecessor sets in P—is denoted by $\mathrm{lev}(P)$. A pair of elements x and y in P are said to be *incomparable* if $x \not\leq_P y$ and $y \not\leq_P x$. The number of incomparable pairs in P we denote by $\mathrm{ip}(P)$.

Let π be a permutation. An *ascent* in π is a letter followed by a larger letter; a *descent* in π is a letter followed by a smaller letter. The number of ascents and descents is denoted by $\mathrm{asc}(\pi)$ and $\mathrm{des}(\pi)$, respectively. An inversion is a pair $i < j$ such that $\pi(i) > \pi(j)$. The number of inversions is denoted by $\mathrm{inv}(\pi)$. A *left-to-right minimum* of π is a letter with no smaller letter to the left of it; the number of left-to-right minima is denoted by $\mathrm{lmin}(\pi)$. The statistics *right-to-left minima* (rmin), *left-to-right maxima* (lmax), and *right-to-left maxima* (rmax) are defined similarly. For permutations π and σ, let $\pi \oplus \sigma = \pi\sigma'$, where σ' is obtained from σ by adding $|\pi|$ to each of its letters, and juxtaposition denotes concatenation. We say that π has k components, and write $\mathrm{comp}(\pi) = k$, if π is the sum of k, but not of $k+1$, non-empty permutations. Let $\mathrm{dent}(\pi)$ denote the number of distinct entries of the inversion table associated with π.

For M a matching on $[2m]$ and N a matching on $[2n]$, let $M \oplus N = M \cup N'$, where N' is the matching on $[2m+1, 2m+2n]$ obtained from N by adding $2m$

to all of its openers and closers. Let us say that M has k *components*, and write $\mathrm{comp}(M) = k$, if M is the sum of k, but not of $k+1$, non-empty matchings. Let $\min(M) = j - 1$, where j is the smallest closer of M. For a matching with no left nestings, j is the closer of the arc with opener 1. Let $\mathrm{last}(M)$ be the number of closers that are smaller than the opener of the last arc. Recall from Section 5 that $\mathrm{int}(M)$ denotes the number of intervals in the list of openers of M. Let us assume that k is the closer of some arc of M, and let $\alpha = (i, j)$ be another arc of M. If $i < k < j$ we say that k is *embraced* by α, and by $\mathrm{emb}(M)$ we denote the number of pairs (k, α) in M such that the closer k is embraced by α.

Proposition 8.1. *Let f and g be as in the proofs of Theorems* 2.1 *and* 3.4. *Let P be a factorial poset on $[n]$. Let $w = g(P)$ and $M = f(w)$ be the corresponding inversion table and matching, respectively. Let π be the permutation corresponding to w. Then*

$$\begin{aligned}
&(\ \mathrm{comp}(P),\quad \min(P),\quad \mathrm{pred}(n),\quad \mathrm{lev}(P),\quad \mathrm{ip}(P)\ )\\
=\ &(\ \mathrm{comp}(\pi),\quad \mathrm{lmin}(\pi),\quad \pi^{-1}(n)-1,\quad \mathrm{dent}(\pi),\quad \mathrm{inv}(\pi)\ )\\
=\ &(\ \mathrm{comp}(M),\quad \min(M),\quad \mathrm{last}(M),\quad \mathrm{int}(M),\quad \mathrm{emb}(M)\ )
\end{aligned}$$

Proof. For inversion tables u and v, let $u \oplus v = uv'$, where v' is obtained from v by adding $1 + \max(u)$ to each of its letters and juxtaposition denotes concatenation. It is easy to see that if σ and τ are the permutations corresponding to u and v, respectively, then the permutation corresponding to $u \oplus v$ is $\sigma \oplus \tau$. Also, $f(u \oplus v) = f(u) \oplus f(v)$ and, for factorial posets Q and R, $g(Q \oplus R) = g(Q) \oplus g(R)$. It follows that $\mathrm{comp}(P) = \mathrm{comp}(\pi) = \mathrm{comp}(M)$.

The numbers $\min(P)$, $\mathrm{lmin}(\pi)$ and $\min(M)$ are all equal to the number of zeros in w. It is plain that $\mathrm{pred}(n) = \pi^{-1}(n) - 1 = \mathrm{last}(M)$ and $\mathrm{lev}(P) = \mathrm{dent}(\pi) = \mathrm{int}(M)$. It remains to show that $\mathrm{ip}(P) = \mathrm{inv}(\pi) = \mathrm{emb}(M)$. Note that

$$\begin{aligned}
\mathrm{ip}(P) &= \tbinom{n}{2} - \#\{(i,j) : i <_P j\},\\
\mathrm{inv}(\pi) &= \tbinom{n}{2} - \#\{(i,j) : i < j \text{ and } \pi(i) < \pi(j)\},
\end{aligned}$$

and, if $\alpha_1, \dots, \alpha_n$ are M's arcs ordered by closer, then

$$\mathrm{emb}(M) = \tbinom{n}{2} - \#\{(i,j) : i < j \text{ and closer of } \alpha_i < \text{opener of } \alpha_j\}.$$

It follows that $\mathrm{ip}(P)$, $\mathrm{inv}(\pi)$ and $\mathrm{emb}(M)$ are all equal to $\binom{n}{2}$ minus the sum of entries in the inversion table w. ☐

Let us note a few direct consequences of the above proposition.

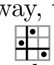
Corollary 8.2. *The statistic* ip *is Mahonian on $\mathcal{F}_n$. That is, it has the same distribution as* inv *on $\mathcal{S}_n$. Also, the statistic* emb *is Mahonian on $\mathcal{N}_n$.*

Corollary 8.3. *The statistic* lev *is Eulerian on the set $\mathcal{F}_n$. That is, it has the same distribution as* des *on $\mathcal{S}_n$. Also, the statistic* int *is Eulerian on $\mathcal{N}_n$.*

Proof. It suffices to show that the statistic dent is Eulerian. The following proof is due to Emeric Deutsch (personal communication, May 2009). Let $d(n,k)$ be the number of inversion tables of length n with k distinct entries. Clearly, $d(n,0) = 0$ for $n > 0$ and $d(n,k) = 0$ for $k > n$. We shall show that, for $0 < k \leq n$,

$$d(n,k) = kd(n-1,k) + (n-k+1)d(n-1,k-1).$$

(This recursion characterizes the Eulerian numbers.) Inversion tables of length n with k distinct entries fall into two disjoint classes: those whose last entry is equal to at least one of the preceding $n-1$ entries (there are $kd(n-1,k)$ such inversion tables) and those whose last entry is different from the preceding $n-1$ entries (there are $(n-(k-1))d(n-1,k-1)$ such inversion tables). □

Recall that $\operatorname{lne}(M)$ and $\operatorname{rne}(M)$ denote the number of left- and right-nestings, respectively. Let $\operatorname{lcr}(M)$ and $\operatorname{rcr}(M)$ denote the number of left- and right-crossings, respectively. The bijections $f : \mathcal{I}_n \to \mathcal{N}_n$ and $g : \mathcal{F}_n \to \mathcal{I}_n$ that we have presented do not specialize to the bijections presented by Bousquet-Mélou et al. [2]. If one were to find bijections that do specialize in the desired way, then one could also hope to prove the following conjecture. Here we view $p =$  as a function counting the occurrences of the pattern p. Also, for posets P, we define

$$\operatorname{rne}(P) = \#\{\, x \in P : \operatorname{pred}(x) > \operatorname{pred}(x+1) \text{ and } \operatorname{succ}(x) = \operatorname{succ}(x+1) \,\}.$$

In other words, $\operatorname{rne}(P)$ is the number of violations of property (4.1) of Proposition 4.1.

Conjecture 8.4. *These three triples of statistics are equidistributed:*

$$\begin{array}{llllll} (& \operatorname{rne}, & \operatorname{comp}, & \min &) & \text{on } \mathcal{F}_n, \\ (& p, & \operatorname{comp}, & \operatorname{lmin} &) & \text{on } \mathcal{S}_n, \\ (& \operatorname{rne}, & \operatorname{comp}, & \min &) & \text{on } \mathcal{N}_n. \end{array}$$

We also conjecture these additional equidistributions:

Conjecture 8.5. *These three triples of statistics are equidistributed:*

$$\begin{array}{llllll} (& \operatorname{rne}, & \min, & \operatorname{lev}-1, &) & \text{on } \mathcal{F}_n, \\ (& p, & \operatorname{lmax}, & \operatorname{des}, &) & \text{on } \mathcal{S}_n, \\ (& \operatorname{rne}, & \min, & \operatorname{int}-1, &) & \text{on } \mathcal{N}_n. \end{array}$$

Conjectures 8.4 and 8.5 have been checked by computer for $n \le 7$.

9. Two additional conjectures and a generalization

Conjecture 9.1. *Assume that $i < j < k < \ell$. Let us say that the arcs (i,ℓ) and (j,k) are m-left-nesting if $j - i \le m$. Note that a 1-left-nesting is the same as a left-nesting. This conjecture claims that among all the matchings on $[2n]$ there are exactly $\mathfrak{f}_n$ that have no 2-left-nestings.*

Conjecture 9.2. *The distribution of* lne *over the set of all matchings on* $[2n]$ *is given by the "Second-order Eulerian triangle", entry A008517 in OEIS* [10].

Conjectures 9.1 and 9.2 have been checked by computer for $n \le 7$.

Note added in proof. Paul Levande has found proofs for Conjectures 9.1 and 9.2. See his preprint arXiv:1006.3013.

Problem 9.3. Consider the following generalization of factorial posets. Let P and Q be labeled posets on $[n]$ such that $i <_P j \implies i <_Q j$. If, in addition,

$$i <_Q j <_P k \implies i <_P k,$$

then we say that P is *Q-factorial*. Note that $\mathbf{n}$-factorial coincides with factorial, where $\mathbf{n}$ is the n-chain. Note also that Q itself is always a Q-factorial poset and is the only one if Q is an antichain. Is this generalization useful? How many Q-factorial posets are there?

Acknowledgment

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