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NEUMANN PROBLEM ON A HALF-SPACE

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Abstract. In this paper, a solution of the Neumann problem on a half-space for a slowly growing continuous boundary function is constructed by the generalized Neumann integral with this boundary function. The relation between this particular solution and certain general solutions is discussed. A solution of the Neumann problem for any continuous boundary function is also given explicitly by the Neumann integral with the generalized Neumann kernel depending on this boundary function.

1. Introduction

Let n be a positive integer satisfying $n \geq 2$. Let $\mathbf{R}^{n+1}$ be the $(n+1)$-dimensional Euclidean space. A point in $\mathbf{R}^{n+1}$ is represented by

$$M = (X, y) = (x_1, \ldots, x_n, y)$$

with

$$|M| = (x_1^2 + \ldots + x_n^2 + y^2)^{\frac{1}{2}}.$$

By ∂E we denote the boundary of a subset E of $\mathbf{R}^{n+1}$. The sphere of radius r centered at the origin of $\mathbf{R}^{n+1}$ is represented by $S_{n+1}(r)$. By $\mathbf{T}_{n+1}$ we denote the open half-space

$$\{M = (X, y) \in \mathbf{R}^{n+1} : y > 0\}.$$

Then $\partial \mathbf{T}_{n+1}$ is identified with $\mathbf{R}^n$ and the n-dimensional Lebesgue measure at $N \in \partial \mathbf{T}_{n+1}$ is denoted by dN. When g is a function defined on

$$\sigma_{n+1}(r) = \mathbf{T}_{n+1} \cap S_{n+1}(r) \quad (r > 0),$$

we define the mean of g as follows:

$$\mathcal{M}(g; r) = 2(s_{n+1} r^n)^{-1} \int_{\sigma_{n+1}(r)} g(M) d\sigma_M \qquad (r > 0),$$

where s_{n+1} is the surface area of $S_{n+1}(1)$ (the $(n+1)$-dimensional unit sphere $\mathbf{S}^n$) and $d\sigma_M$ is the surface element on $S_{n+1}(r)$ at $M \in \sigma_{n+1}(r)$.

Let f be a continuous function defined on $\partial \mathbf{T}_{n+1}$. A solution of the Neumann problem on $\mathbf{T}_{n+1}$ for f is a harmonic function h in $\mathbf{T}_{n+1}$ such that

$$\lim_{M \in \mathbf{T}_{n+1}, M \to N} \frac{\partial}{\partial y} h(M) = f(N)$$

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for every point $N \in \partial\mathbf{T}_{n+1}$. Armitage proved

Theorem A (Armitage [1], Theorem 1 and Remarks)**.** *Let f be a continuous function on $\partial\mathbf{T}_{n+1} = \mathbf{R}^n$ such that*

$$\int_{\mathbf{R}^n} (1+|N|)^{1-n}|f(N)|dN < \infty. \tag{1.1}$$

Then a solution of the Neumann problem on $\mathbf{T}_{n+1}$ for f is given by the Neumann integral I_f for f,

$$I_f(M) = -\alpha_{n+1}\int_{\mathbf{R}^n} |M-N|^{1-n} f(N)dN \qquad (M \in \mathbf{T}_{n+1}),$$

which satisfies

$$\mathcal{M}(|I_f|; r) = O(1) \qquad (r \to \infty),$$

where $\alpha_{n+1} = 2\{(n-1)s_{n+1}\}^{-1}$.

The following result deals with a type of uniqueness of solutions for the Neumann problem on $\mathbf{T}_{n+1}$.

Theorem B (Armitage [1], Theorem 3)**.** *Let k be a positive integer and f be a continuous function on $\partial\mathbf{T}_{n+1}$ satisfying* (1.1)*. If h is a solution of the Neumann problem on $\mathbf{T}_{n+1}$ for f satisfying*

$$\mathcal{M}(h^+; r) = o(r^k) \qquad (r \to \infty),$$

then h is given by

$$h(M) = I_f(M) + \begin{cases} C & (k=1), \\ \Pi(X) + \displaystyle\sum_{j=1}^{[\frac{k}{2}]} \tfrac{(-1)^j}{(2j)!} y^{2j}\Delta^j \Pi(X) & (k \geq 2) \end{cases}$$

for any $M = (X, y) \in \mathbf{T}_{n+1}$, where h^+ is the positive part of h,

$$\Delta^j = \left(\frac{\partial^2}{\partial x_1^2} + \frac{\partial^2}{\partial x_2^2} + \cdots + \frac{\partial^2}{\partial x_n^2}\right)^j \qquad (j = 1, 2, \ldots),$$

C is a constant and Π is a polynomial of $X = (x_1, x_2, \ldots, x_n) \in \mathbf{R}^n$ of degree less than k in $\partial\mathbf{T}_{n+1}$.

Gardiner [7, Theorem 1] gave a solution of the Neumann problem for any continuous function on $\partial\mathbf{T}_{n+1}$. His solution is constructed using approximation of functions, and hence it is not explicit. In this paper, we will explicitly give a solution of the Neumann problem for any continuous function on $\partial\mathbf{T}_{n+1}$ in the same way as Finkelstein and Scheinberg [6] and Yoshida [10] did in the case of the Dirichlet Problem. To do this, Theorem A will be extended by defining generalized Neumann integrals for continuous functions under less restricted conditions than (1.1) (Theorem 1). Siegel and Talvila [9] defined a more complicated generalized Neumann integral for their purpose. But our generalized Neumann integral is much simpler than theirs. By using Theorem 1, we shall give a solution of the Neumann problem for any continuous function on $\partial\mathbf{T}_{n+1}$. Our solution is much simpler than the solution given by Gardiner (Theorem 2). We shall also extend Theorem B (Theorem 3).

2. Statements of results

Let M and N be two points in $\mathbf{T}_{n+1}$ and $\partial\mathbf{T}_{n+1}$, respectively. By $\langle M, N\rangle$ we denote the usual inner product in $\mathbf{R}^{n+1}$. We note that

$$|M-N|^{1-n}=\sum_{k=0}^{\infty}c_{k,n+1}|N|^{1-k-n}|M|^{k}L_{k,n+1}(\rho)\qquad (|M|<|N|),$$

where

$$\rho=\frac{\langle M,N\rangle}{|M||N|},\qquad c_{k,n+1}=\begin{pmatrix} k+n-2 \\ k \end{pmatrix} \tag{2.1}$$

and $L_{k,n+1}$ is the $(n+1)$-dimensional Legendre polynomial of degree k. We remark that $L_{k,n+1}(1)=1$, $L_{k,n+1}(-1)=(-1)^k$, $L_{0,n+1}=1$ and $L_{k,n+1}(t)=t$ (see Armitage [3, p. 55]).

Let l be a non-negative integer. We set

$$V_{l,n+1}(M,N)=\begin{cases} -\alpha_{n+1}\displaystyle\sum_{k=0}^{l-1}c_{k,n+1}|N|^{1-k-n}|M|^{k}L_{k,n+1}(\rho) & (|N|\ge 1,\quad l\ge 1),\\ 0 & (|N|<1,\quad l\ge 1),\\ 0 & (l=0)\end{cases}$$

for any $M\in\mathbf{T}_{n+1}$ and any $N\in\partial\mathbf{T}_{n+1}$. The generalized Neumann kernel $K_{l,n+1}(M,N)$ $(M\in\mathbf{T}_{n+1}, N\in\partial\mathbf{T}_{n+1})$ is defined by

$$K_{l,n+1}(M,N)=K_{0,n+1}(M,N)-V_{l,n+1}(M,N)\qquad (l\ge 0),$$

where

$$K_{0,n+1}(M,N)=-\alpha_{n+1}|M-N|^{1-n}.$$

Since $|M|^kL_{k,n+1}(\rho)$ $(k\ge 0)$ is harmonic in $\mathbf{T}_{n+1}$ (Armitage [3, Theorem D]), $K_{l,n+1}(\cdot,N)$ is also harmonic in $\mathbf{T}_{n+1}$ for any fixed $N\in\partial\mathbf{T}_{n+1}$.

By $F_{l,n+1}$ we denote the set of continuous functions f on $\partial\mathbf{T}_{n+1}=\mathbf{R}^n$ such that

$$\int_{\mathbf{R}^n}\frac{|f(N)|}{1+|N|^{n+l-1}}dN<\infty. \tag{2.2}$$

The following Theorem 1 generalizes Theorem A, which is our result in the case $l=0$.

Theorem 1. *Let l be a non-negative integer and $f\in F_{l,n+1}$. Then the generalized Neumann integral $H_{l,n+1}f$ of f, defined in $\mathbf{T}_{n+1}$ by*

$$H_{l,n+1}f(M)=\int_{\mathbf{R}^n}K_{l,n+1}(M,N)f(N)dN,$$

is a solution of the Neumann problem for f and

$$\mathcal{M}(|H_{l,n+1}f|;r)=O(r^l)\qquad (r\to\infty). \tag{2.3}$$

Remark 1. We remark that Theorem 1 yields multiple representations in the case that f satisfies (2.2) for more than one l. For example, if f is bounded with bounded support, then (2.2) is satisfied for every non-negative integer l and hence many generalized Neumann integrals $H_{l,n+1}f$ $(l=0,1,2,\dots)$ of f are obtained.

We shall define another Neumann kernel. The construction of our Neumann kernel is similar in spirit to Finkelstein and Scheinberg's construction for the Poisson kernel [6]. Let $\varphi(t)$ be a positive continuous function of $t \geq 1$ satisfying

$$\varphi(1) = c_n/2,$$

where $c_n = 3(n-1)2^n \alpha_{n+1}$. Denote the set

$$\{t \geq 1 \ : \ t^{n-1}\varphi(t) = 2^{-i}c_n\}$$

by $U_n(\varphi, i)$ $(i = 1, 2, 3, \cdots)$. Then $1 \in U_n(\varphi, 1)$. When there is an integer L such that $U_n(\varphi, L) \neq \emptyset$ and $U_n(\varphi, L+1) = \emptyset$, we denote the set $\{i \ : \ 1 \leq i \leq L\}$ of integers by $E_n(\varphi)$. Otherwise, we denote the set of all positive integers by $E_n(\varphi)$. Let $t_n(i) = t_n(\varphi, i)$ be the minimum of elements in $U_n(\varphi, i)$ for each $i \in E_n(\varphi)$. In the former case, we put $t_n(L+1) = \infty$. We remark that $t_n(1) = 1$. We define $V_{\varphi,n+1}(M, N)$ $(M \in \mathbf{T}_{n+1}, N \in \partial\mathbf{T}_{n+1})$ by

$$V_{\varphi,n+1}(M,N) = \begin{cases} 0 & |N| < t_n(1), \\ V_{i,n+1}(M,N) & t_n(i) \leq |N| < t_n(i+1) \quad (i \in E_n(\varphi)). \end{cases}$$

We put

$$K_{\varphi,n+1}(M,N) = K_{0,n+1}(M,N) - V_{\varphi,n+1}(M,N) \quad (M \in \mathbf{T}_{n+1}, N \in \partial\mathbf{T}_{n+1}).$$

It is evident that $K_{\varphi,n+1}(\cdot, N)$ is also harmonic on $\mathbf{T}_{n+1}$ for any fixed $N \in \partial\mathbf{T}_{n+1}$.

To solve the Neumann problem on $\mathbf{T}_{n+1}$ for any continuous function f on $\partial\mathbf{T}_{n+1} = \mathbf{R}^n$, we have

Theorem 2. *Let f be any continuous function on $\partial\mathbf{T}_{n+1} = \mathbf{R}^n$. Then there is a positive continuous function $\varphi(t)$ of $t \geq 1$, given explicitly in terms of the growth of f, such that*

$$H_{\varphi,n+1}f(M) = \int_{\mathbf{R}^n} K_{\varphi,n+1}(M,N)f(N)dN$$

is a solution of the Neumann problem on $\mathbf{T}_{n+1}$ for f.

The following Theorem 3 extends Theorem B, which is our result in the case $l = 0$.

Theorem 3. *Let k be a positive integer and l be a non-negative integer. Let $f \in F_{l,n+1}$ and h be a solution of the Neumann problem on $\mathbf{T}_{n+1}$ for f such that*

$$\mathcal{M}(h^+; r) = o(r^{k+l}) \quad (r \to \infty). \tag{2.4}$$

Then

$$h(M) = \begin{cases} H_{l,n+1}f(M) + C & (k = 1), \\ H_{l,n+1}f(M) + \Pi(X) + \sum_{j=1}^{[\frac{k+l}{2}]} \frac{(-1)^j}{(2j)!} y^{2j} \Delta^j \Pi(X) & (k \geq 2) \end{cases}$$

for any $M = (X, y) \in \mathbf{T}_{n+1}$, where C is a constant and Π is a polynomial of $X = (x_1, x_2, \ldots, x_n) \in \mathbf{R}^n$ of degree less than $k + l$.

3. Proofs of Theorems 1, 2 and 3

In this section we use the following notation:

$$B_m(Q,r)=\{P\in \mathbf{R}^m : |P-Q|<r\} \quad (Q\in \mathbf{R}^m, r>0)$$

and

$$B_m(r)=\{P\in \mathbf{R}^m : |P|<r\} \quad (r>0).$$

First of all, we note two facts concerning $L_{k,n+1}(\rho)$. If we observe that

$$\frac{d}{d\rho}L_{k,n+1}(\rho)=\frac{k(n+k-1)}{n}L_{k-1,n+3}(\rho) \quad (k\geq 1)$$

from Müller [8, Lemma 13], then we have

$$\begin{aligned}\frac{\partial}{\partial y}(c_{k,n+1}|M|^k L_{k,n+1}(\rho)) &= (n-1)c_{k-1,n+2}y|M|^{k-2}L_{k,n+1}(\rho)\\ &\quad -(n-1)c_{k-1,n+3}y|M|^{k-2}\rho L_{k-1,n+3}(\rho) \quad (k\geq 1).\end{aligned} \tag{3.1}$$

We also know that

$$|L_{k,m}(\rho)|\leq 1 \tag{3.2}$$

for any ρ in (2.1), any non-negative integer k and any positive integer $m\geq 2$ (see Armitage [3, Theorems C and D]).

Lemma 1. *Let l be a non-negative integer. For any $M\in \mathbf{T}_{n+1}$ and any $N\in \partial\mathbf{T}_{n+1}$ satisfying $2|M|<|N|$ and $|N|\geq 1$, we have*

$$|K_{l,n+1}(M,N)|\leq C_1(l,n)|M|^l|N|^{1-n-l} \tag{3.3}$$

and

$$\left|\frac{\partial}{\partial y}K_{l,n+1}(M,N)\right|\leq \begin{cases} C_2(l,n)|M|^{l-1}|N|^{1-n-l} & (l\geq 1),\\ C_2(0,n)|N|^{1-n} & (l=0),\end{cases} \tag{3.4}$$

where $C_1(l,n)=2^{n+l-1}\alpha_{n+1}$, $C_2(l,n)=3(n-1)2^{n+l-1}\alpha_{n+1}$ and $C_2(0,n)=3(n-1)2^n\alpha_{n+1}$.

Proof. Take any $M\in \mathbf{T}_{n+1}$ and any $N\in \partial\mathbf{T}_{n+1}$ satisfying $2|M|<|N|$ and $|N|\geq 1$. Then

$$\begin{aligned}|K_{l,n+1}(M,N)| &= \alpha_{n+1}\left|\sum_{k=l}^{\infty}c_{k,n+1}|N|^{1-n-k}|M|^k L_{k,n+1}(\rho)\right|\\ &\leq \alpha_{n+1}\sum_{k=l}^{\infty}c_{k,n+1}|N|^{1-n}2^{-k}\left(\frac{2|M|}{|N|}\right)^k|L_{k,n+1}(\rho)|\\ &\leq \alpha_{n+1}\left(\frac{2|M|}{|N|}\right)^l|N|^{1-n}\sum_{k=l}^{\infty}c_{k,n+1}2^{-k}\end{aligned}$$

from (3.2). If we put $C_1(l,n)=2^{n+l-1}\alpha_{n+1}$, then we have (3.3).

If $l \geq 2$, we similarly have

$$\begin{aligned}
&\left|\frac{\partial}{\partial y}K_{l,n+1}(M,N)\right| \\
&\leq \alpha_{n+1}\sum_{k=l}^{\infty}(n-1)c_{k-1,n+2}y|M|^{k-2}|N|^{1-n-k}|L_{k,n+1}(\rho)| \\
&\quad + \alpha_{n+1}\sum_{k=l}^{\infty}(n-1)c_{k-1,n+3}y|M|^{k-2}|N|^{1-n-k}|\rho||L_{k-1,n+3}(\rho)| \\
&\leq (n-1)\alpha_{n+1}|N|^{-n}\sum_{k=l}^{\infty}2^{1-k}\left(\frac{2|M|}{|N|}\right)^{k-1}(c_{k-1,n+2}+c_{k-1,n+3}) \\
&\leq (n-1)\alpha_{n+1}|N|^{-n}\left(\frac{2|M|}{|N|}\right)^{l-1}\sum_{k=l}^{\infty}2^{1-k}(c_{k-1,n+2}+c_{k-1,n+3})
\end{aligned}$$

from (3.1). By putting $C_2(l,n) = 3(n-1)2^{n+l-1}\alpha_{n+1}$, we also obtain (3.4) in the case $l \geq 2$. Since for $l = 1$ or 0,

$$\frac{\partial}{\partial y}K_{l,n+1}(M,N) = -\alpha_{n+1}\sum_{k=2}^{\infty}c_{k,n+1}|N|^{1-n-k}\frac{\partial}{\partial y}|M|^k L_{k,n+1}(\rho),$$

we have

$$\begin{aligned}
\left|\frac{\partial}{\partial y}K_{l,n+1}(M,N)\right| &\leq (n-1)\alpha_{n+1}|N|^{-n}\sum_{k=2}^{\infty}2^{1-k}(c_{k-1,n+2}+c_{k-1,n+3}) \\
&\leq 3(n-1)2^n\alpha_{n+1}|N|^{-n} \\
&\leq 3(n-1)2^n\alpha_{n+1}|N|^{1-n}.
\end{aligned}$$

This gives (3.4) in the case $l = 1$ or 0. □

Lemma 2. *Let l be a non-negative integer, δ be any positive number satisfying $0 < \delta < 1$, and N^* be any fixed point of $\partial\mathbf{T}_{n+1}$. Then*

$$\left|\frac{\partial}{\partial y}V_{l,n+1}(M,N)\right| \leq C(l,\delta,N^*)y$$

for any $M \in B_{n+1}(N^*,\delta) \cap \mathbf{T}_{n+1}$ and any $N \in \partial\mathbf{T}_{n+1}$, where $C(l,\delta,N^*)$ is a constant depending only on l, δ and N^*.

Proof. From the definition of $V_{l,n+1}(M,N)$ and (3.1), we can evidently assume that $l \geq 3$ and $|N| \geq 1$. Then we have from (3.2) that

$$\begin{aligned}
\left|\frac{\partial}{\partial y}V_{l,n+1}(M,N)\right| &\leq \alpha_{n+1}\sum_{k=2}^{l-1}(n-1)c_{k-1,n+2}y|M|^{k-2}|N|^{1-k-n}|L_{k,n+1}(\rho)| \\
&\quad + \alpha_{n+1}\sum_{k=2}^{l-1}(n-1)c_{k-1,n+3}y|M|^{k-2}|N|^{1-k-n}|\rho||L_{k,n+3}(\rho)| \\
&\leq \frac{2y}{s_{n+1}}\sum_{k=2}^{l-1}(c_{k-1,n+2}+c_{k-1,n+3})(|N^*|+\delta)^{k-2} \\
&= C(l,\delta,N^*)y,
\end{aligned}$$

where

$$C(l,\delta,N^*)=\frac{2}{s_{n+1}}\sum_{k=2}^{l-1}(c_{k-1,n+2}+c_{k-1,n+3})(|N^*|+\delta)^{k-2}. \qquad \square$$

Lemma 3. *Let l be any non-negative integer. Let f be a locally integrable function on $\partial\mathbf{T}_{n+1}$ satisfying (2.2). Then $H_{l,n+1}f$ is a harmonic function on $\mathbf{T}_{n+1}$.*

Proof. For any fixed $M\in\mathbf{T}_{n+1}$, take a number R satisfying $R\geq\max\{1,2|M|\}$. Then from Lemma 1 we have

$$\int_{\mathbf{R}^n\setminus B_n(R)}|K_{l,n+1}(M,N)||f(N)|dN\leq C_1(l,n)|M|^l\int_{\mathbf{R}^n\setminus B_n(R)}\frac{|f(N)|}{|N|^{n+l-1}}dN<\infty.$$

Thus $H_{l,n+1}f(M)$ is finite for any $M\in\mathbf{T}_{n+1}$. Since the mean value equality for $H_{l,n+1}f$ follows from Fubini's theorem, $H_{l,n+1}f(M)$ is harmonic in $\mathbf{T}_{n+1}$. $\square$

Lemma 4. *Let l be any non-negative integer. Let f be a locally integrable and upper semicontinuous function on $\partial\mathbf{T}_{n+1}$ satisfying (2.2). Then*

$$\limsup_{M\in\mathbf{T}_{n+1},M\to N^*}\frac{\partial}{\partial y}H_{l,n+1}f(M)\leq f(N^*)$$

for any fixed $N^*\in\partial\mathbf{T}_{n+1}$.

Proof. Let N^* be any fixed point on $\partial\mathbf{T}_{n+1}=\mathbf{R}^n$ and ε be any positive number. Take a positive number δ, $\delta<1$, such that

$$f(N)<f(N^*)+\varepsilon \tag{3.5}$$

for any $N\in B_n(N^*,\delta)$. From (2.2) and (3.4), we can choose a number R^*, $R^*>2(|N^*|+1)$, such that

$$\int_{\mathbf{R}^n\setminus B_n(R^*)}\left|\frac{\partial}{\partial y}K_{l,n+1}(M,N)\right||f(N)|dN<\varepsilon, \tag{3.6}$$

for any $M\in\mathbf{T}_{n+1}\cap B_{n+1}(N^*,\delta)$. Put

$$J(M)=\int_{B_n(R^*)}f(N)\frac{\partial}{\partial y}K_{0,n+1}(M,N)dN$$

and

$$J_l(M)=-\int_{B_n(R^*)}f(N)\frac{\partial}{\partial y}V_{l,n+1}(M,N)dN \quad (l\geq 0).$$

Since

$$\frac{\partial}{\partial y}K_{0,n+1}(M,N)=\frac{2y}{s_{n+1}}|M-N|^{-n-1} \quad (M=(X,y)\in\mathbf{T}_{n+1},N\in\partial\mathbf{T}_{n+1}),$$

we observe that

$$\begin{aligned}&\left|\int_{B_n(R^*)\setminus B_n(N^*,\delta)}f(N)\frac{\partial}{\partial y}K_{0,n+1}(M,N)dN\right| \\ \leq&\frac{2y}{s_{n+1}}\int_{B_n(R^*)\setminus B_n(N^*,\delta)}|M-N|^{-n-1}|f(N)|dN \\ \leq&\frac{2y}{s_{n+1}}\left(\frac{\delta}{2}\right)^{-n-1}\int_{B_n(R^*)\setminus B_n(N^*,\delta)}|f(N)|dN\end{aligned} \tag{3.7}$$

for any $M \in \mathbf{T}_{n+1} \cap B_{n+1}(N^*, \delta/2)$. Since

$$\begin{aligned} 1 - \int_{B_n(N^*,\delta)} \frac{\partial}{\partial y} K_{0,n+1}(M,N)dN &= \int_{\mathbf{R}^n \setminus B_n(N^*,\delta)} \frac{\partial}{\partial y} K_{0,n+1}(M,N)dN \\ &= \frac{2y}{s_{n+1}} \int_{\mathbf{R}^n \setminus B_n(N^*,\delta)} |M-N|^{-n-1}dN \end{aligned}$$

for any $M \in \mathbf{T}_{n+1}$ (see Armitage and Gardiner [4, p. 24]), we have

$$\lim_{M \to N^*, M \in \mathbf{T}_{n+1}} \int_{B_n(N^*,\delta)} \frac{\partial}{\partial y} K_{0,n+1}(M,N)dN = 1. \tag{3.8}$$

Finally (3.5), (3.7) and (3.8) yield

$$\limsup_{M \to N^*, M \in \mathbf{T}_{n+1}} J(M) \le f(N^*) + \varepsilon.$$

From Lemma 2 we obtain

$$\begin{aligned} |J_l(M)| &\le \int_{B_n(R^*)} |f(N)| \left| \frac{\partial}{\partial y} V_{l,n+1}(M,N) \right| dN \\ &\le \int_{B_n(R^*)} C(l,\delta,N^*) y |f(N)| dN \\ &\le C_3 y \end{aligned} \tag{3.9}$$

for any $M \in \mathbf{T}_{n+1} \cap B_{n+1}(N^*, \delta)$, where

$$C_3 = C(l,\delta,N^*) \int_{B_n(R^*)} |f(N)|dN.$$

These and (3.6) yield

$$\begin{aligned} & \limsup_{M \to N^*, M \in \mathbf{T}_{n+1}} \frac{\partial}{\partial y} H_{l,n+1} f(M) \\ = & \limsup_{M \to N^*, M \in \mathbf{T}_{n+1}} \int_{\mathbf{R}^n} f(N) \frac{\partial}{\partial y} K_{l,n+1}(M,N)dN \\ = & \limsup_{M \to N^*, M \in \mathbf{T}_{n+1}} \left(J(M) + J_l(M) + \int_{\mathbf{R}^n \setminus B_n(R^*)} f(N) \frac{\partial}{\partial y} K_{l,n+1}(M,N)dN \right) \\ \le & f(N^*) + 2\varepsilon. \end{aligned}$$

Now the conclusion immediately follows. □

Proof of Theorem 1. It immediately follows from Lemmas 3 and 4 that $H_{l,n+1}f$ is a harmonic function on $\mathbf{T}_{n+1}$ and

$$\lim_{M \to N^*, M \in \mathbf{T}_{n+1}} \frac{\partial}{\partial y} H_{l,n+1} f(M) = f(N^*)$$

for any $N^* \in \partial \mathbf{T}_{n+1}$.

We now turn to the proof of (2.3). For any positive number $r>1$ we have

$$\begin{aligned}\frac{s_{n+1}r^n}{2}\mathcal{M}(|H_{l,n+1}f|;r) &= \int_{\sigma_{n+1}(r)}\left|\int_{\mathbf{R}^n}K_{l,n+1}(M,N)f(N)dN\right|d\sigma_M\\ &\le \int_{\sigma_{n+1}(r)}\int_{\mathbf{R}^n}|K_{l,n+1}(M,N)f(N)|dNd\sigma_M\\ &= \int_{\mathbf{R}^n}\int_{\sigma_{n+1}(r)}|K_{l,n+1}(M,N)f(N)|d\sigma_M dN\\ &= T_{1,l}(r)+T_{2,l}(r),\end{aligned}$$

where

$$T_{1,l}(r)=\int_{\mathbf{R}^n\setminus B_n(2r)}\int_{\sigma_{n+1}(r)}|K_{l,n+1}(M,N)f(N)|\,d\sigma_M dN$$

and

$$T_{2,l}(r)=\int_{B_n(2r)}\int_{\sigma_{n+1}(r)}|K_{l,n+1}(M,N)f(N)|\,d\sigma_M dN.$$

We note that if $l\ge 1$ and $1\le |N|<2|M|$, then

$$\begin{aligned}|V_{l,n+1}(M,N)| &\le \alpha_{n+1}\sum_{k=0}^{l-1}c_{k,n+1}|N|^{1-k-n}|M|^k|L_{k,n+1}(\rho)|\\ &\le \alpha_{n+1}|N|^{1-n}\sum_{k=0}^{l-1}2^{-k}c_{k,n+1}\left(\frac{2|M|}{|N|}\right)^k\\ &\le C_4|N|^{2-l-n}|M|^{l-1},\end{aligned}$$

where

$$C_4=2^{l-1}\alpha_{n+1}l\max_{0\le k\le l-1}2^{-k}c_{k,n+1}.$$

Hence we have

$$\begin{aligned}&\int_{B_n(2r)}|f(N)|\int_{\sigma_{n+1}(r)}|V_{l,n+1}(M,N)|d\sigma_M dN\\ &\le 2^{-1}C_4 s_{n+1}r^{n+l-1}\int_{B_n(2r)\setminus B_n(1)}\frac{|f(N)|}{|N|^{n+l-2}}dN=C_5r^{n+l},\end{aligned}$$

where

$$C_5=C_4s_{n+1}\int_{\mathbf{R}^n\setminus B_n(1)}\frac{|f(N)|}{|N|^{n+l-1}}dN\quad(<\infty).$$

Since

$$\frac{1}{s_{n+1}r^n}\int_{S_{n+1}(r)}|M-N|^{1-n}d\sigma_M\le r^{1-n}$$

(see Armitage and Gardiner [4, p. 99]), we obtain

$$\begin{aligned}\int_{B_n(2r)}|f(N)|\int_{\sigma_{n+1}(r)}|K_{0,n+1}(M,N)|d\sigma_M dN &\le 2^{-1}\alpha_{n+1}s_{n+1}r\int_{B_n(2r)}|f(N)|dN\\ &\le 2^{-1}(n-1)^{-1}r\int_{B_n(2r)}\frac{2(2r)^{n+l-1}}{1+|N|^{n+l-1}}|f(N)|dN\le C_6r^{n+l},\end{aligned}$$

where

$$C_6=2^{n+l-1}(n-1)^{-1}\int_{\mathbf{R}^n}\frac{|f(N)|}{1+|N|^{n+l-1}}dN.$$

These immediately yield

$$T_{2,l}(r) \le \int_{B_n(2r)} |f(N)| \int_{\sigma_{n+1}(r)} \left(|K_{0,n+1}(M,N)| + |V_{l,n+1}(M,N)|\right) d\sigma_M dN$$
$$\le (C_5 + C_6) r^{n+l}.$$

From Lemma 1 we easily see that

$$T_{1,l}(r) \le 2^{-1} C_1(l,n) s_{n+1} r^{n+l} \int_{\mathbf{R}^n \setminus B_n(2r)} \frac{|f(N)|}{|N|^{n+l-1}} dN \le C_7 r^{n+l},$$

where

$$C_7 = 2^{-1} C_1(l,n) s_{n+1} \int_{\mathbf{R}^n \setminus B_n(1)} \frac{|f(N)|}{|N|^{n+l-1}} dN.$$

These give (2.3). □

To prove Theorem 2, we need

Lemma 5. *Let $\varphi(t)$ be a positive continuous function of $t \ge 1$ satisfying $\varphi(1) = c_n/2$. Then for any $M \in \mathbf{T}_{n+1}$ and any $N \in \partial \mathbf{T}_{n+1}$ satisfying $|N| > \max\{1, 4|M|\}$,*

$$|K_{\varphi,n+1}(M,N)| < \varphi(|N|) \tag{3.10}$$

and

$$\left| \frac{\partial}{\partial y} K_{\varphi,n+1}(M,N) \right| < 4\varphi(|N|). \tag{3.11}$$

Proof. Take any $M \in \mathbf{T}_{n+1}$ and any $N \in \partial \mathbf{T}_{n+1}$ satisfying $|N| > \max\{1, 4|M|\}$. Choose an integer $i_0 \in E_n(\varphi)$ such that $t_n(i_0) \le |N| < t_n(i_0 + 1)$. Then

$$K_{\varphi,n+1}(M,N) = K_{i_0,n+1}(M,N).$$

From Lemma 1 we easily see that

$$|K_{i_0,n+1}(M,N)| \le C_1(i_0,n) |M|^{i_0} |N|^{1-n-i_0} \le C_1(i_0,n) 2^{-2i_0} |N|^{1-n}.$$

Hence

$$|K_{\varphi,n+1}(M,N)| \le C_1(i_0,n) 2^{-2i_0} |N|^{1-n} \le \varphi(|N|).$$

In the same way we can also see (3.11) by applying Lemma 1 to $\frac{\partial}{\partial y} K_{i_0,n+1}(M,N)$. □

Proof of Theorem 2. Let (t, Θ) be the spherical coordinates in $\mathbf{R}^n$. We identify $(1,\Theta) \in \mathbf{S}^{n-1}$ with Θ. Put

$$C_8 = \frac{c_n}{2} \max\left\{1, \int_{\mathbf{S}^{n-1}} |f(1,\Theta)| d\Theta \right\}$$

and

$$\psi(t) = \begin{cases} C_8 t^{-n-1} \left(\int_{\mathbf{S}^{n-1}} |f(t,\Theta)| d\Theta \right)^{-1} & \left(\int_{\mathbf{S}^{n-1}} |f(t,\Theta)| d\Theta > 0 \right), \\ \infty & \left(\int_{\mathbf{S}^{n-1}} |f(t,\Theta)| d\Theta = 0 \right) \end{cases}$$

for $t \ge 1$, where $d\Theta$ is the surface element of $\mathbf{S}^{n-1}$ at $(1,\Theta) \in \mathbf{S}^{n-1}$. If we define $\varphi(t)$ $(t \ge 1)$ by

$$\varphi(t) = \min\left\{ \frac{c_n}{2}, \psi(t) \right\},$$

then $\varphi(t)$ is a positive continuous function satisfying $\varphi(1) = c_n/2$. For any fixed $M \in \mathbf{T}_{n+1}$ we can choose a number $R_1 > \max\{1, 4|M|\}$ such that

$$\int_{\mathbf{R}^n \setminus B_n(R_1)} |K_{\varphi,n+1}(M,N)f(N)|dN \tag{3.12}$$

$$\leq \int_{R_1}^{\infty} \left(\int_{\mathbf{S}^{n-1}} |f(t,\Theta)|d\Theta \right) \varphi(t) t^{n-1} dt \leq C_8 \int_{R_1}^{\infty} t^{-2} dt < \infty$$

from Lemma 5. It is evident that

$$\int_{B_n(R_1)} |K_{\varphi,n+1}(M,N)f(N)|dN < \infty.$$

These give that

$$\int_{\mathbf{R}^n} |K_{\varphi,n+1}(M,N)f(N)|dN < \infty.$$

To see that $H_{\varphi,n+1}f(M)$ is harmonic in $\mathbf{T}_{n+1}$, we observe from Fubini's theorem that $H_{\varphi,n+1}f(M)$ has the locally mean-value property.

Finally we shall show that

$$\lim_{M \in \mathbf{T}_{n+1},\ M \to N^*} \frac{\partial}{\partial y} H_{\varphi,n+1} f(M) = f(N^*) \tag{3.13}$$

for any fixed $N^* \in \partial \mathbf{T}_{n+1}$. In a similar way to (3.12) we also have

$$\int_{\mathbf{R}^n \setminus B_n(R_1)} \left| \frac{\partial}{\partial y} K_{\varphi,n+1}(M,N) f(N) \right| dN < \infty$$

for any fixed $M \in \mathbf{T}_{n+1}$ and any number $R_1 > \max\{1, 4|M|\}$. Let ε be any positive number. Choose a sufficiently large number R^* ($R^* > 4(|N^*| + 1)$) such that

$$\int_{\mathbf{R}^n \setminus B_n(R^*)} \left| \frac{\partial}{\partial y} K_{\varphi,n+1}(M,N) f(N) \right| dN < \varepsilon.$$

Since f is continuous on $\partial \mathbf{T}_{n+1}$, take a positive number δ ($\delta < 1$) such that

$$f(N) < f(N^*) + \varepsilon$$

for any $N \in B_n(N^*, \delta)$. In the completely same way as the proof of Lemma 4, we also obtain

$$\limsup_{M \in \mathbf{T}_{n+1},\ M \to N^*} \int_{B_n(R^*)} f(N) \frac{\partial}{\partial y} K_{0,n+1}(M,N) dN \leq f(N^*) + \varepsilon.$$

If we take an integer $i_0 \in E_n(\varphi)$ satisfying $t_n(i_0) \leq R^* < t_n(i_0 + 1)$, then we see from Lemma 2 that

$$\int_{B_n(R^*)} |f(N)| \left| \frac{\partial}{\partial y} V_{\varphi,n+1}(M,N) \right| dN \leq \int_{B_n(R^*)} \sum_{i=1}^{i_0} \left| \frac{\partial}{\partial y} V_{i,n+1}(M,N) f(N) \right| dN$$

$$\leq y \int_{B_n(R^*)} \sum_{i=1}^{i_0} C(i, \delta, N^*) |f(N)| dN = C_9 y$$

for any $M \in B_{n+1}(N^*, \delta) \cap \mathbf{T}_{n+1}$, where C_9 is a constant. These yield

$$\limsup_{M \in \mathbf{T}_{n+1},\ M \to N^*} \int_{\mathbf{R}^n} f(N) \frac{\partial}{\partial y} K_{\varphi,n+1}(M,N) dN \leq f(N^*) + 2\varepsilon.$$

By replacing f with $-f$, we also have

$$\liminf_{M\in\mathbf{T}_{n+1},\ M\to N^*}\int_{\mathbf{R}^n} f(N)\frac{\partial}{\partial y}K_{\varphi,n+1}(M,N)dN \geq f(N^*)-2\varepsilon.$$

From these, (3.13) follows immediately. □

To prove Theorem 3 completely we shall first give an easy proof of the following lemma, which is proved in a different way from Armitage [1].

Lemma 6 (Armitage [1, Lemma 2])**.** *If $H(M)$ is a harmonic polynomial of $M=(X,y)\in\mathbf{R}^{n+1}$ of degree m and $\partial H/\partial y$ vanishes on $\partial\mathbf{T}_{n+1}$, then there is a polynomial Π of $X\in\mathbf{R}^n$ of degree m such that*

$$H(X,y)=\begin{cases}\Pi(X)+\sum_{j=1}^{[\frac{1}{2}m]}\frac{(-1)^j}{(2j)!}y^{2j}\Delta^j\Pi(X) & (m\geq 2),\\ \Pi(X) & (m=0,1).\end{cases}$$

Proof. Put

$$H(X,y)=\Pi_0(X)+\Pi_1(X)y+\cdots+\Pi_m(X)y^m \quad ((X,y)\in\mathbf{R}^{n+1}), \tag{3.14}$$

where $\Pi_j(X)$ is a polynomial of $X\in\mathbf{R}^n$ of degree at most $m-j$. We remark that a sequence of the equations

$$\Pi_j(X)=-j^{-1}(j-1)^{-1}\Delta\Pi_{j-2}(X) \quad (j=2,3,\cdots,m) \tag{3.15}$$

and

$$\Pi_1(X)=0 \tag{3.16}$$

follows from

$$\Delta H=0 \quad \text{on } \mathbf{R}^{n+1} \qquad \text{and} \qquad \partial H/\partial y=0 \quad \text{on } \mathbf{R}^n,$$

respectively. If we set $\Pi(X)=\Pi_0(X)$ on $\mathbf{R}^n$, then

$$H(X,y)=\Pi(X)+\sum_{j=1}^{[\frac{1}{2}m]}\frac{(-1)^j}{(2j)!}y^{2j}\Delta^j\Pi(X)$$

from (3.14), (3.15) and (3.16). □

Proof of Theorem 3. Suppose that f and h are two functions given in Theorem 3. Then we know from Theorem 1 that $h-H_{l,n+1}f$ has a harmonic continuation H to $\mathbf{R}^{n+1}$ such that H is an even function of y (see Armitage [2, §8.2]). Now we have

$$\begin{aligned}\mathcal{M}(H^+;r)&=\mathcal{M}((h-H_{l,n+1}f)^+;r)\\ &\leq \mathcal{M}(h^+;r)+\mathcal{M}(|H_{l,n+1}f|;r)\\ &=o(r^{k+l})+o(r^{l+1})=o(r^{k+l}) \quad (r\to\infty),\end{aligned}$$

by (2.4) and (2.3) of Theorem 1. This implies that H is a polynomial of degree less than $k+l$ (see Brelot [5, Appendix]). The conclusions of the theorem follow immediately from Lemma 6. □

References

1. D.H. Armitage, *The Neumann problem for a function harmonic in* $\mathbf{R}^n \times (0, \infty)$*,* Arch. Rational Mech. Anal., **63** (1976), 89-105. MR0427656 (55:687)
2. ———, *On harmonic polynomials,* Proc. London Math. Soc., (3) **34** (1977), 173-192. MR0427661 (55:692)
3. ———, *Representations of harmonic functions in half-spaces,* Proc. London Math. Soc., (3) **38** (1979), 53-71. MR520972 (80d:31003)
4. D.H. Armitage and S.J. Gardiner, *Classical Potential Theory,* Springer-Verlag London Ltd., 2001. MR1801253 (2001m:31001)
5. M. Brelot, *Éléments de la théorie classique du potentiel,* Centre de Documentation Universitaire, Paris, 1965. MR0178154 (31:2412)
6. M. Finkelstein and S. Scheinberg, *Kernels for solving problems of Dirichlet type in a half-plane,* Advances in Math., **18** (1975), 108-113. MR0382677 (52:3559)
7. S.J. Gardiner, *The Dirichlet and Neumann problems for harmonic functions in half-spaces,* J. London Math. Soc., (2) **24** (1981), 502-512. MR635881 (82m:31006)
8. C. Müller, *Spherical harmonics,* Springer-Verlag, Berlin, 1966. MR0199449 (33:7593)
9. D. Siegel and E. Talvila, *Sharp growth estimates for modified Poisson integrals in a half space,* Potential Anal., **15** (2001), 333-360. MR1856153 (2002h:31008)
10. H. Yoshida, *A type of uniqueness for the Dirichlet problem on a half-space with continuous data,* Pacific J. Math., **172** (1996), 591-609. MR1386632 (97g:31007)

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