A COMPARISON INEQUALITY FOR RATIONAL FUNCTIONS

XIN LI

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ABSTRACT. We establish a new inequality for rational functions and show that it implies many inequalities for polynomials and their polar derivatives.

1. INTRODUCTION

Let $p(z)$ be a polynomial of degree at most n of a complex variable z. According to the well-known Bernstein's inequality ([3]),

$$\max_{|z|=1} |p'(z)| \le n \max_{|z|=1} |p(z)|.$$

The inequality is sharp in the sense that the equality holds if $p(z) = z^n$. Let P_n denote the set of all polynomials of degree at most n and let $\|f\| = \max_{|z|=1} |f(z)|$, the sup-norm of f on the unit circle. Then Bernstein's inequality can be restated as the following extremal problem:

$$\max_{p \in P_n} \frac{\|p'\|}{\|p\|} = n.$$

Since the polynomial $p(z) = z^n$ yielding the maximum value has all its zeros at $z = 0$, by restricting the zeros of polynomials, the maximum value may be smaller. Indeed, if P_n^1 denotes the set of polynomials with no zero inside the unit circle $|z| < 1$, Erdös conjectured and Lax verified the following Erdös-Lax inequality ([8]):

$$\|p'\| \le \frac{n}{2} \|p\|.$$

Aziz was among the first to extend these results by replacing the derivatives with the polar derivatives of polynomials. For a complex number α and for $p \in P_n$, let

$$D_\alpha p(z) = np(z) + (\alpha - z)p'(z).$$

Note that $D_\alpha p(z)$ is a polynomial of degree at most $n - 1$. This is the so-called polar derivative of $p(z)$ with respect to point α ([11]). It generalizes the ordinary derivative in the following sense:

$$\lim_{\alpha \to \infty} \frac{D_\alpha p(z)}{\alpha} = p'(z).$$

It is proved by Aziz in 1988 ([1]) that, for $p \in P_n^1$ and $|\alpha| \geq 1$,

$$|D_\alpha p(z)| \leq \frac{n}{2}(|\alpha z^{n-1}| + 1)\|p\| \text{ for } |z| \geq 1.$$

The result is sharp and the equality holds at special values of z if $p(z) = az^n + b$ with $|a| = |b|$. Dividing both sides by $|\alpha|$ and letting $\alpha \to \infty$ yields the Erdös-Lax inequality.

Bernstein's inequality was extended by Aziz and Shah in 1998 ([2]):

$$|D_\alpha p(z)| \leq n|\alpha z^{n-1}|\|p\| \text{ for } |z| \geq 1.$$

In fact, the special and equivalent case of this result was known much earlier. For example, in [14], Smirnov and Lebedev established the above inequality for $z = \alpha$ (p. 393) and derived the inequality for $|z| = |\alpha| = 1$ (p. 396).

In 1930, Bernstein revisited his inequality and established the following comparison result ([3]): Assume that p and q are polynomials with $\partial p \leq \partial q$, where ∂g denotes the exact degree of a polynomial g. If $q(z)$ has all its zeros in $|z| \leq 1$ and

$$|p(z)| \leq |q(z)| \text{ for } |z| = 1,$$

then

(1.1) $$|p'(z)| \leq |q'(z)| \text{ for } |z| = 1.$$

In 1985, Malik and Vong ([10]) improved this result of Bernstein further by showing that

(1.2) $$\left| \frac{zp'(z)}{n} + \beta\frac{p(z)}{2} \right| \leq \left| \frac{zq'(z)}{n} + \beta\frac{q(z)}{2} \right|$$

for all $|z| = 1$ and $|\beta| \leq 1$. Recently, this inequality has been extended to polar derivative by Mohapatra and Shah ([12]):

(1.3) $$\left| zD_\alpha p(z) + n\beta\left(\frac{|\alpha| - 1}{2}\right)p(z) \right| \leq \left| zD_\alpha q(z) + n\beta\left(\frac{|\alpha| - 1}{2}\right)q(z) \right|$$

for all $|z| = 1$, $|\alpha| \geq 1$, and $|\beta| \leq 1$. The author is curious to know how to interpret the expressions in these last two inequalities. Indeed, this paper is mainly motivated by the desire to understand these expressions. We believe that a proper perspective has been found using rational functions. We will show how the above mentioned results, as well as some new inequalities, can be obtained in a natural way from a general inequality for rational functions with prescribed poles. Our method of proof may be useful for proving other inequalities for polynomials and rational functions.

2. RATIONAL FUNCTIONS

Let a_1, a_2, ... , a_n be n given points in $|z| > 1$. We will consider the following space of rational functions with prescribed poles:

$$R_n := R_n(a_1, a_2, ..., a_n) = \left\{ \frac{p(z)}{w(z)} \; : \; p \in P_n \right\},$$

where $w(z) := (z - a_1)(z - a_2) \cdots (z - a_n)$.

Denote

$$B(z) = \frac{z^n \overline{w(1/\overline{z})}}{w(z)} = \prod_{k=1}^{n} \frac{1 - \overline{a_k}z}{z - a_k}.$$

Note that $|B(z)| = 1$ when $|z| = 1$.

The inequalities of Bernstein and Erdös-Lax have been extended to the rational functions ([4], [9]): If $|z| = 1$, then, for any $r \in R_n$,

$$(2.1) \qquad |r'(z)| \le |B'(z)| \|r\|.$$

Furthermore, the inequality is sharp and the equality holds if $r(z) = \alpha B(z)$ with $|\alpha| = 1$. If we assume $r \in R_n$ has all its zeros in $|z| \ge 1$, the inequality can be strengthened to

$$(2.2) \qquad |r'(z)| \le \frac{1}{2} |B'(z)| \|r\| \text{ for } |z| = 1.$$

The inequality is sharp and the equality holds if $r(z) = \alpha B(z) + \beta$ with $|\alpha| = |\beta|$.

In this paper, our main result is an extension of Bernstein's inequality (1.1) to the rational functions.

3. Main results

Theorem 3.1. *Let $r, s \in R_n$ and assume s has all its n zeros in $|z| \le 1$ and*

$$|r(z)| \le |s(z)| \text{ for } |z| = 1.$$

Then

$$|r'(z)| \le |s'(z)| \text{ for } |z| = 1.$$

Taking $s(z) = B(z)$ in the theorem, we immediately obtain Bernstein's inequality (2.1) for rational functions.

The statement of the theorem is almost the same as the polynomial case, and the only difference is that polynomials are replaced by rational functions. Despite the similarity in the statements, the proofs used in the polynomial case (see [1, 12]) could not be directly applied in the rational case. We will explain more on the differences in the proofs in the next section.

Theorem 3.1 is a consequence of the following more general inequality.

Theorem 3.2. *Let $r, s \in R_n$ and assume s has all its n zeros in $|z| \le 1$ and*

$$|r(z)| \le |s(z)| \text{ for } |z| = 1.$$

Then, for any ρ with $|\rho| \le 1/2$,

$$|r'(z) + \rho B'(z) r(z)| \le |s'(z) + \rho B'(z) s(z)| \text{ for } |z| = 1.$$

We remark that the sharpness of the result is obvious: The equality holds if $r(z) \equiv s(z)$.

This inequality should be compared with the polynomial inequality (1.2) of Malik and Vong.

Now we present and discuss some consequences of these results. First, we point out that inequalities involving polynomials and their polar derivatives are a special case of the inequalities for rational functions. For example, taking $a_i = a$ for all $i = 1, 2, ..., n$ in Theorem 3.1 gives us the following inequality on the polar derivatives.

Corollary 3.3. *For $p, q \in P_n$ where q has all its n zeros in $|z| \le 1$, if*

$$|p(z)| \le |q(z)| \text{ for } |z| = 1,$$

then, for $|a| \ge 1$,

$$(3.1) \qquad |D_a p(z)| \le |D_a q(z)| \text{ for } |z| = 1.$$

Indeed, for $|a| > 1$, applying Theorem 3.1 to rational functions $r(z) = p(z)/(z-a)^n$ and $s(z) = q(z)/(z-a)^n$ with poles all at one point $z = a$ gives us

$$(3.2) \qquad \left|\left(\frac{p(z)}{(z-a)^n}\right)'\right| \leq \left|\left(\frac{q(z)}{(z-a)^n}\right)'\right|$$

for all $|z| = 1$. But, it is easy to see that

$$\left(\frac{p(z)}{(z-a)^n}\right)' = -\frac{D_a p(z)}{(z-a)^{n+1}}.$$

Therefore, the inequality (3.1) follows from the inequality in (3.2) when $|a| > 1$. Taking the limit as $|a| \to 1$ will give us the inequality when $|a| \geq 1$.

Similarly, applying Theorem 3.1 to rational functions $r(z) = p(z)/(z-a)^n$ and $s(z) = p^*(z)/(z-a)^n$ with $|a| \geq 1$ yields the following results of Aziz ([1, p. 190]). Recall that for a polynomial $p(z)$ of degree n, $p^*(z) = z^n \overline{p(1/\overline{z})}$.

Corollary 3.4. *For $p \in P_n$ with its zeros in $|z| \geq 1$ (i.e., $p \in P_n^1$), for any a with $|a| \geq 1$,*

$$(3.3) \qquad |D_a p(z)| \leq |D_a p^*(z)| \text{ for } |z| = 1.$$

In fact, the inequality in (1.3) of Mohapatra and Shah is a special case of Theorem 3.2 applied to the single pole (when $a_i = a$ with $|a| \geq 1$, $i = 1, 2, ..., n$) case as well. In this case, Theorem 3.2 implies that, for $|z| = 1$,

$$\left|-\frac{D_a p(z)}{(z-a)^{n+1}} + \rho \frac{n(|a|^2-1)}{|z-a|^2} \frac{p(z)}{(z-a)^n}\right| \leq \left|-\frac{D_a q(z)}{(z-a)^{n+1}} + \rho \frac{n(|a|^2-1)}{|z-a|^2} \frac{q(z)}{(z-a)^n}\right|.$$

Now, defining β such that

$$\beta\left(\frac{|a|-1}{2}\right) = \rho \frac{|a|^2-1}{\overline{z}-\overline{a}},$$

we recover the inequality of Mohapatra and Shah, and this gives us a better understanding of the expressions in (1.3).

Next, we show that Theorem 3.2 implies other inequalities of rational functions.

By taking suitable ρ in the inequality of Theorem 3.2, the expression inside $|\cdots|$ can be written as

$$r'(z) + c r^{*'}(z),$$

where r^* is defined as: If $r = p/w$, then

$$r^*(z) = p^*(z)/w(z).$$

Corollary 3.5. *Let $r, s \in R_n$ and assume that s has n zeros, all lying in $|z| \leq 1$, and*

$$|r(z)| \leq |s(z)| \text{ for } |z| = 1.$$

Then, for any c with $0 \leq c \leq 1/3$,

$$|r'(z)| + c|r^{*'}(z)| \leq |s'(z)| + c|s^{*'}(z)| \text{ for } |z| = 1.$$

Corollary 3.6. *Let $r \in R_n$.*
(i) If r has n zeros all lying in $|z| \leq 1$, then for $|\rho| \leq 1/2$,

$$|(r^*(z))' + \rho B'(z) r^*(z)| \leq |r'(z) + \rho B'(z) r(z)| \text{ for } |z| = 1.$$

(ii) If r has all its zeros in $|z| \geq 1$, then

$$|r'(z) + \rho B'(z)r(z)| \leq |(r^*(z))' + \rho B'(z)r^*(z)| \text{ for } |z| = 1.$$

The two parts of the above result are equivalent. When $\rho = 0$, the second part recovers a known result (see, e.g., the proof of of Theorem 3 in [9, p. 530]), and we now state both parts explicitly in this case:

Corollary 3.7. *Let $r \in R_n$.*
(i) If r has n zeros all lying in $|z| \leq 1$, then

$$|(r^*(z))'| \leq |r'(z)| \text{ for } |z| = 1.$$

(ii) If r has all its zeros in $|z| \geq 1$, then

$$|r'(z)| \leq |(r^*(z))'| \text{ for } |z| = 1.$$

4. PROOFS

Our proof is different from its polynomial counterpart. The main reason is that the polynomial results are often proved through the application of some forms of Laguerre's Theorem or Grace's Theorem that are not readily available to rational functions. Indeed, according to the result of Bonsall and Marden [5], the counting of the critical points of $r \in R_n$ depends on the distinct number of poles of r, in addition to its zeros. Therefore, there is no direct extension of Laguerre's Theorem and Grace's Theorem to the rational functions.

We need a rational inequality proved by Li, Mohapatra, and Rodriguez in [9]. It is a rational version of an inequality of Turan for polynomials.

Lemma 4.1 ([9, Thm. 4]). *If $r \in R_n$ has exactly n zeros and they are all lying in $|z| \leq 1$, then*

$$|r'(z)| \geq \frac{1}{2}|B'(z)| \, |r(z)| \text{ for } |z| = 1.$$

We also need the following simple statement.

Lemma 4.2. *Let A and B be any two complex numbers. Then*
(i) If $|A| \geq |B|$ and $B \neq 0$, then $A \neq \delta B$ for all complex numbers δ satisfying $|\delta| < 1$.
(ii) Conversely, if $A \neq \delta B$ for all complex numbers δ satisfying $|\delta| < 1$, then $|A| \geq |B|$.

Proof. (i) Assume $|A| \geq |B|$ and $B \neq 0$. If $A = \delta B$ for some δ with $|\delta| < 1$, then $|A| = |\delta| \, |B| < |B|$, a contradiction.
(ii) Assume $A \neq \delta B$ for any δ with $|\delta| < 1$. If $|A| < |B|$, then $B \neq 0$. Let $\hat{\delta} = A/B$. Then

$$A = \hat{\delta}B \text{ and } |\hat{\delta}| = \frac{|A|}{|B|} < 1,$$

contradicting the assumption.   □

Proof of Theorem 3.2. First assume that no zeros of $s(z)$ are on the unit circle $|z| = 1$ and therefore that all zeros of $s(z)$ are in $|z| < 1$.

Let α be an arbitrary number satisfying $|\alpha| < 1$. Consider function $\alpha r(z) + s(z)$. This is a rational function with no poles in $|z| \leq 1$. Since $|r(z)| \leq |s(z)|$ for $|z| = 1$,

by Rouche's Theorem, $\alpha r(z) + s(z)$ has the same number of zeros in $|z| < 1$ as $s(z)$. Thus, $\alpha r(z) + s(z)$ also has n zeros in $|z| < 1$. By Lemma 4.1,

$$|\alpha r'(z) + s'(z)| \geq \frac{1}{2}|B'(z)| \, |\alpha r(z) + s(z)| \text{ for } |z| = 1.$$

Now, note that $B'(z) \neq 0$ (e.g., see formula (14) in [9]). So, the right hand side is nonzero. Thus, by using (i) of Lemma 4.2, we have, for all β satisfying $|\beta| < 1$,

$$\alpha r'(z) + s'(z) \neq \beta \frac{1}{2} B'(z)[\alpha r(z) + s(z)] \text{ for } |z| = 1$$

or, equivalently, for $|z| = 1$,

$$\alpha[r'(z) - \frac{\beta}{2}B'(z)r(z)] \neq -[s'(z) - \frac{\beta}{2}B'(z)s(z)] \text{ for } |\alpha| < 1, \ |\beta| < 1.$$

Using (ii) of Lemma 4.2, we have

$$\left| s'(z) - \frac{\beta}{2}B'(z)s(z) \right| \geq \left| r'(z) - \frac{\beta}{2}B'(z)r(z)] \right| \text{ for } |z| = 1, \ |\beta| < 1.$$

Taking $\rho := \beta/2$ gives us the desired inequality when $|\rho| < 1/2$.

Finally, using the continuity in zeros and ρ, we can obtain the inequality when some zeros of $s(z)$ lie on the unit circle and for $|\rho| \leq 1/2$. $\qquad\square$

Proof of Corollary 3.5. First, by a direct calculation (see, for example, [9, p. 529]) one can obtain

$$|(r^*(z))'| = |B'(z)r(z) - r'(z)B(z)| \text{ for } |z| = 1.$$

Let λ satisfy $|\lambda| \geq 3$. Take $\rho = -1/(B(z) + \lambda)$. Then $|\rho| \leq 1/2$. So, Theorem 3.2 gives us

$$|\lambda r'(z) + (B(z)r'(z) - B'(z)r(z))| \leq |\lambda s'(z) + (B(z)s'(z) - B'(z)s(z))|.$$

Now, choose the argument of λ such that

$$|\lambda r'(z) + (B(z)r'(z) - B'(z)r(z))| = |\lambda r'(z)| + |B(z)r'(z) - B'(z)r(z)|.$$

Then we have

$$|\lambda| \, |r'(z)| + |(r^*(z))'| \leq |\lambda s'(z) + B(z)s'(z) - B'(z)s(z)| \leq |\lambda| \, |s'(z)| + |(s^*(z))'|.$$

Taking $c = 1/|\lambda|$ gives us the desired inequality. $\qquad\square$

Proof of Corollary 3.6. For (i), since

$$|r^*(z)| = |r(z)| \text{ for } |z| = 1$$

and $r(z)$ has all its n zeros in $|z| \leq 1$, we apply Theorem 3.2 with $r(z)$ and $s(z)$ being replaced by $r^*(z)$ and $r(z)$, respectively.

For (ii), we apply Theorem 3.2 with $r(z)$ and $s(z)$ being replaced by $r(z)$ and $r^*(z)$, respectively. $\qquad\square$

REFERENCES

1. A. Aziz, *Inequalities for the polar derivative of a polynomial*, J. Approx. Theory, **55** (1988), 183–193. MR965215 (89m:30010)
2. A. Aziz and W.M. Shah, *Inequalities for the polar derivative of a polynomial*, Indian J. Pure Appl. Math., **29** (1998), 163–173. MR1623254 (2000b:30002)
3. P. Borwein and T. Erdélyi, *Polynomial Inequalities*, Springer-Verlag, New York, 1995. MR1367960 (97e:41001)
4. P. Borwein and T. Erdélyi, *Sharp extensions of Bernstein inequalities to rational spaces*, Mathematika, **43** (1996), 413–423. MR1433285 (97k:26014)
5. F.F. Bonsall and M. Marden, *Critical points of rational functions with self-inversive polynomial factors*, Proc. Amer. Soc., **5** (1954), 111–114. MR0060016 (15:613a)
6. N.K. Govil, *On the derivative of a polynomial*, Proc. Amer. Math. Soc., **41** (1973), 543–546. MR0325932 (48:4278)
7. N.K. Govil, G. Nyuydinkong, and B. Tameru, *Some L^p inequalities for the polar derivative of a polynomial*, J. Math. Anal. Appl., **254** (2001), 618–626. MR1805528 (2001m:41007)
8. P.D. Lax, *Proof of a conjecture of P. Erdös on the derivative of a polynomial*, Bull. Amer. Math. Soc., **50** (1944), 509–513. MR0010731 (6:61f)
9. X. Li, R.N. Mohapatra, and R.S. Rodgriguez, *Bernstein-type inequalities for rational functions with prescribed poles*, J. London Math. Soc., **51** (1995), 523–531. MR1332889 (96b:30005)
10. M.A. Malik and M.C. Vong, *Inequalities concerning the derivative of polynomials*, Rendiconts Del Circolo Matematico Di Palermo Serie II, **34** (1985), 422–426. MR848119 (88c:41026)
11. M. Marden, *Geometry of Polynomials*, Math. Surveys, No. 3, Amer. Math. Soc., Providence, Rhode Island, 1966. MR0225972 (37:1562)
12. R.N. Mohapatra and W.M. Shah, *Inequalities for the polar derivative of a polynomial*, preprint, 2008.
13. Q.I. Rahman and G. Schmeisser, *Analytic theory of polynomials*, Oxford University Press, Oxford, 2002. MR1954841 (2004b:30015)
14. V.I. Smirnov and N.A. Lebedev, *Function of a Complex Variable*, English edition, Iliffe Books, Landon, 1968.

Department of Mathematics, University of Central Florida, Orlando, Florida 32816
E-mail address: xli@math.ucf.edu