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# ALMOST REGULAR INVOLUTORY AUTOMORPHISMS OF UNIQUELY 2-DIVISIBLE GROUPS

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Abstract. We prove that a uniquely 2-divisible group that admits an almost regular involutory automorphism is solvable.

## 1. Introduction

Recall that an automorphism $\nu$ of a group $H$ is called *involutory* if $\nu \neq id$ and $\nu^2 = id$. The automorphism $\nu$ is called *almost regular* if $C_H(\nu)$ is finite. Recall that a group $U$ is *uniquely* 2*-divisible* if for each $u \in U$ there exists a unique $v \in U$ such that $v^2 = u$. Note that in particular a uniquely 2-divisible group contains no involutions (i.e. elements of order 2).

The purpose of this paper is to use the techniques introduced in the impressive paper [Sh] of Shunkov, where he proves that a periodic group that admits an almost regular involutory automorphism is virtually solvable (i.e. it has a solvable subgroup of finite index). We prove

**Theorem 1.1.** *Let $U$ be a uniquely 2-divisible group. If $U$ admits an involutory almost regular automorphism, then $U$ is solvable.*

Our main motivation for dealing with automorphisms of uniquely 2-divisible groups comes from questions about the root groups of special Moufang sets, and those tend to be uniquely 2-divisible; see, e.g., [S]. Indeed, using Theorem 1.1 it immediately follows that

**Corollary 1.2.** *Let $\mathbb{M}(U, \tau)$ be a special Moufang set. If the Hua subgroup contains an involution $\nu$ such that $C_U(\nu)$ is finite, then $U$ is abelian.*

*Proof.* If $U$ contains involutions, then $U$ is abelian by [DST, Theorem 5.5, p. 782]. If $U$ does not contain involutions, then by [DS, Proposition 4.6, p. 5840], $U$ is uniquely 2-divisible, and then by Theorem 1.1 and by the main theorem of [SW], $U$ is abelian. □

The proof of Theorem 1.1 is obtained as follows. First note that if $U$ is finite, then $U$ has odd order, so by the Feit-Thompson theorem $U$ is solvable. Hence we may assume that $U$ is infinite.

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We let $A$ be a maximal abelian subgroup of $U$ (with respect to inclusion) inverted by $\nu$ (i.e. each element of $A$ is inverted by $\nu$). In Lemma 3.1(2) we show that we can take $A$ to be infinite. We then show that for elements $u_1, \dots, u_n \in U$, the involutions $u_1\nu u_1^{-1}, \dots, u_n\nu u_n^{-1}$ in the semi-direct product $U \rtimes \langle\nu\rangle$ invert a subgroup $D \leq A$ with $|A : D| < \infty$ (Proposition 3.3). The next step is to show that $C_U(D)/D$ is finite and solvable (Lemma 3.5). Since $K := \langle \nu u_1\nu u_1^{-1}, \dots, \nu u_n\nu u_n^{-1}\rangle \leq C_U(D)$, the subgroup $K$ is solvable and $K/Z(K)$ is finite.

Next let $S := \{x \in U \mid x^\nu = x^{-1}\}$. It is easy to see that an element $y \in U$ is in $S$ iff $y = \nu u\nu u^{-1}$, for some $u \in U$, so by the above each finitely generated subgroup $H$ of $R := \langle S\rangle$ is solvable and satisfies the fact that $H/Z(H)$ is finite. It follows that $R'$ is periodic (Proposition 3.6). Using the above mentioned result of Shunkov, we see that $R'$ is solvable, so $R$ is solvable.

As is well known (see [K]), $U = RC_U(\nu)$ and $R \trianglelefteq U$. Since $C_U(\nu)$ is finite and uniquely 2-divisible it has odd order. By the Feit-Thomson theorem, $C_U(\nu)$ is solvable, and this at last shows that $U$ is solvable.

We remark that it is possible that with the aid of the Theorem on page 286 of [HM], one can get even more delicate information on $U$. However, we do not need that, so we do not pursue this avenue further.

## 2. Notation and preliminary results

*Notation* 2.1.
(1) Throughout this paper $U$ is an infinite uniquely 2-divisible group and $\nu \in \mathrm{Aut}(U)$ is an involutory automorphism which is almost regular.
(2) We denote by $G$ the semi-direct product of $U$ by $\nu$, and we identify $U$ and $\nu$ with their images in $G$. We let $\mathrm{Inv}(G)$ denote the set of involutions of $G$.
(3) We let $S := \{x \in U \mid x^\nu = x^{-1}\}$.
(4) The letter $A$ always denotes a fixed infinite maximal (with respect to inclusion) abelian subgroup of $U$ which is inverted by $\nu$ (i.e. all of whose elements are inverted by $\nu$). The existence of $A$ is guaranteed by Lemma 3.1(2) and by Zorn's lemma.
(5) For each $u \in U$ we denote by $A_u$ the subgroup of $A$ inverted by $u\nu u^{-1}$.

*Remark* 2.2.
(1) Note that for any non-empty subset $T \subseteq U$, the centralizer $C_U(T)$ is a uniquely 2-divisible subgroup of $U$.
(2) Notice that $A$ is uniquely 2-divisible. Also, for any $u \in U$, the subgroup $A_u$ is uniquely 2-divisible.
(3) It is easy to check that $S = \{\nu\nu^x \mid x \in U\}$.

**Lemma 2.3** ([N], Lemma 4.1, p. 239)**.** *Let the group $H$ be the union of finitely many, let us say $n$, cosets of subgroups $C_1, C_2, \dots, C_n$ :*

$$H = \bigcup_{i=1}^n C_i g_i.$$

*Then the index of (at least) one of these subgroups in $H$ does not exceed $n$.*

**Corollary 2.4.** *Let the group $H$ be the union of finitely many, let us say $n$, subsets $S_1, S_2, \dots, S_n$ :*

$$H = \bigcup_{i=1}^n S_i.$$

*For each $i$ set $C_i := \langle ab^{-1} \mid a, b \in S_i\rangle$. Then the index of (at least) one of the subgroups $C_1, \dots, C_n$ in $H$ does not exceed $n$.*

*Proof.* For each $i = 1, \ldots, n$, pick an arbitrary $g_i \in S_i$. Notice that $S_i \subseteq C_i g_i$ for all $i$, so $H = \bigcup_{i=1}^{n} C_i g_i$ and Corollary 2.4 follows from Lemma 2.3. □

**Lemma 2.5.** (1) *All involutions in $G$ are conjugate;*
(2) $S = \{\nu\tau \mid \tau \in \mathrm{Inv}(G)\}$.

*Proof.* Let $\tau \in \mathrm{Inv}(G)$. Then $\tau = x\nu$ for some $x \in U$. Since $\tau$ is an involution, $x \in S$. Let $y \in U$ be the unique element with $y^2 = x$. Then $y \in S$ and $\tau = x\nu = y^2\nu = y\nu y^{-1}$. This shows (1). Part (2) is Remark 2.2(3). □

**Lemma 2.6.** *Let $D$ be an abelian uniquely 2-divisible subgroup of $U$. Then the following hold:*

1. *$C_U(D)/D$ is a uniquely 2-divisible group.*
2. *If $D$ is inverted by $\nu$, then $\nu D$ is an almost regular involutory automorphism of $C_U(D)/D$.*
3. *Assume that $D$ is inverted by $\nu$ and let $E/D$ be a subgroup of $C_U(D)/D$ which is inverted by $\nu D$. Then $E$ is inverted by $\nu$, so, in particular, $E$ is abelian.*

*Proof.* (1) Set $C := C_U(D)$. Assume that $a, b \in C$ and $a^2D = b^2D$. Let $x, y \in D$ with $a^2x = b^2y$ and let $u, v \in D$ with $u^2 = x$ and $v^2 = y$. Then $a^2u^2 = b^2v^2$, and since $a, b$ commute with $u, v$ we see that $(au)^2 = (bv)^2$. Hence $au = bv$, so $aD = bD$.

Furthermore let $aD \in C/D$. Let $b \in U$ with $b^2 = a$. Then $b \in C$ and $bD$ is the square root of $aD$ in $C/D$.

(2) Clearly $\nu D$ is an involutory automorphism of $C/D$ (acting via conjugation). Assume that $aD \in C/D$ centralizes $\nu D$. Then $\nu^a = \nu d$ for some $d \in D$. Let $x \in D$ with $x^2 = d$. Then $\nu$ inverts $x$, and we see that $\nu^a = \nu^x$ and $ax^{-1} \in C_U(\nu)$. It follows that $C_{C/D}(\nu D) = C_C(\nu)D/D$, and since $\nu$ is almost regular, so is $\nu D$.

(3) Let $xD \in C/D$ be an element inverted by $\nu D$. Then $x^\nu = x^{-1}d$ for some $d \in D$, and conjugating by $\nu$ we see that $x = x^{-\nu}d^{-1}$, which implies that $x^\nu = x^{-1}d^{-1}$. Thus $d = d^{-1}$, so $d = 1$.

Now let $e \in E$. Then, by hypothesis, $eD$ is inverted by $\nu D$, so $e^\nu = e^{-1}$. □

## 3. The proof of Theorem 1.1

**Lemma 3.1.** *Let $D$ be an abelian subgroup of $U$ (we allow $D = 1$) such that $D$ is inverted by $\nu$ and such that $C_U(D)$ is infinite. Assume that*

$$(S \cap C_U(D)) \setminus D \neq \emptyset.$$

*Then the following hold:*

1. *There exists an element $w \in C_U(D) \setminus D$ which is inverted by $\nu$ and such that $C_U(\langle D, w\rangle)$ is infinite.*
2. *There exists an infinite abelian subgroup of $U$ which is inverted by $\nu$.*

*Proof.* (1) Set $V := C_U(D)$. Then $V$ is an infinite uniquely 2-divisible group, and $\nu$ acts on $V$, so without loss we may assume that $U = V$ and that $D \leq Z(U)$.

Pick $b \in S \setminus D$ (note that $b$ exists by hypothesis), and write $b = \nu\tau$ with $\tau \in \mathrm{Inv}(G)$. Let

$$u \in U \text{ with } u^{-2} = \nu\tau,$$

and note that since $u$ is inverted by both $\nu$ and $\tau$, we have

$$\nu = \tau^u.$$

We claim that there exists $h \in C_U(\tau)$ such that $hu$ is inverted by infinitely many involutions of $G$. Suppose for a moment that the claim holds. Note that $hu \notin D$, for all $h \in C_U(\tau)$. Indeed, if $h = 1$, then $hu = u$, and since $b \notin D$ also $u \notin D$. Otherwise if $hu \in D$ and $h \neq 1$, then

$$u^{-1}h^{-1} = (hu)^\tau = h^\tau u^\tau = hu^{-1},$$

and it follows that $u$ inverts $h$, which is not possible in a uniquely 2-divisible group.

Since all involutions in $G$ are conjugate, conjugating $hu$ by an appropriate element we may assume that $\nu$ inverts $hu$, and since $hu$ is inverted by infinitely many involutions, we see that $C_U(hu)$ is infinite. Taking $w = hu$, we are done.

It remains to show the existence of $h$. For each $a \in S$, let

$$s_a := \nu\tau^a \text{ and } \ell_a^{-2} = s_a.$$

It is easy to check that since $\ell_a$ is inverted by $\nu$ and $\tau^a$, we have $\tau^{a\ell_a} = \nu$. Hence

$$\tau^{a\ell_a} = \tau^u, \text{ and hence } h_a := a\ell_a u^{-1} \in C_U(\tau).$$

It follows that $\ell_a = a^{-1}h_a u$. Since both $\ell_a$ and $a$ are inverted by $\nu$ we get after conjugating by $\nu$ that $\ell_a^{-1} = a(h_a u)^\nu = (h_a u)^{-1}a$. Notice now that $a\nu \in \mathrm{Inv}(G)$, and it follows that

$$(h_a u)^{a\nu} = (h_a u)^{-1}.$$

By hypothesis the set $\{h_a \mid a \in S\}$ is finite since it is contained in $C_U(\tau)$. Further, the set $S$ is infinite. This implies the existence of $h \in C_U(\tau)$ such that the number of involutions $a\nu$ that invert $hu$ is infinite. This proves (1).

(2) If $D$ is finite and $C_U(D)$ is infinite, then, since $\nu$ is almost regular, $(S \cap C_U(D)) \smallsetminus D \neq \emptyset$ (because $S \cap C_U(D)$ is infinite; see Remark 2.2(3)). Hence (2) follows from (1) by starting with $D = 1$ and iterating the process as long as the subgroup $\langle D, w\rangle$ is finite. $\square$

**Lemma 3.2.** *Let $x \in U$ and let $s \in U$ be the unique element such that $s^{-2} = \nu x^{-1}\nu x$. Then $xs \in C_U(\nu)$.*

*Proof.* Notice that $s$ is inverted by $\nu$ and $\nu^x$. Hence

$$1 = s^2\nu\nu^x = \nu s^{-2}\nu^x = \nu s^{-1}\nu^x s,$$

so the lemma holds. $\square$

**Proposition 3.3.** *Let $A$ be as in Notation* 2.1(4) *and let $u \in U$. Let $A_u$ be as in Notation* 2.1(5). *Then $|A : A_u| < \infty$.*

*Proof.* Fix $a \in A$ and consider the element

$$\nu\nu^{au}.$$

This element is in $U$. Let $s \in U$ with $s^{-2} = \nu\nu^{au}$. By Lemma 3.2 we get that

$$v_a := aus \in C_U(\nu). \tag{3.1}$$

Now set

$$\mathcal{M}_a := \{b \in A \mid v_b = v_a\}.$$

Notice that since $|C_U(\nu)| < \infty$,

$$\text{the set } \{\mathcal{M}_c \mid c \in A\} \text{ is finite and } A = \textstyle\bigcup_{c\in A}\mathcal{M}_c. \tag{3.2}$$

By equation (3.1) we get $s^{-1} = v_a^{-1}au$, and conjugating by $\nu$ and noticing that $\nu$ inverts $a$ and $s$ and centralizes $v_a$, we see that $s^{-1} = u^{-\nu}av_a$. So we get the equality

$$v_a^{-1}au = u^{-\nu}av_a,$$

from which it follows that

$$u^{-1}\nu b v_a u^{-1} = \nu v_a^{-1}b, \quad \forall b \in \mathcal{M}_a. \tag{3.3}$$

Let $c \in \mathcal{M}_a$. Then as in equation (3.3) we get that $u^{-1}\nu c v_a u^{-1} = \nu v_a^{-1}c$, and this together with equation (3.3) yields

$$uv_a^{-1}c^{-1}bv_a u^{-1} = c^{-1}b, \quad \forall b, c \in \mathcal{M}_a.$$

Since $\nu$ inverts $c^{-1}b \in A$, it follows that $uv_a^{-1}\nu v_a u^{-1} = u\nu u^{-1}$ inverts $c^{-1}b$. We thus can conclude that

$$u\nu u^{-1} \text{ inverts } \langle bc^{-1} \mid b, c \in \mathcal{M}_a\rangle, \quad \forall a \in A. \tag{3.4}$$

By equation (3.2) and by Corollary 2.4, one of the subgroups $\langle bc^{-1} \mid b, c \in \mathcal{M}_a\rangle$ has finite index in $A$, so $|A : A_u| < \infty$ as asserted. □

**Lemma 3.4.** *Let $B$ be a finitely generated abelian subgroup of $U$ which is inverted by $\nu$. Then $A$ contains a subgroup $A_1$ of finite index such that $\langle A_1, B\rangle$ is abelian.*

*Proof.* Recall the definition of $A$ from Notation 2.1(4) and for $b \in B$ the definition of $A_b$ from Notation 2.1(5). Let $\mathcal{B}$ be a finite set of generators for $B$ and set $A_1 := \bigcap_{b\in\mathcal{B}} A_b$. By Proposition 3.3 and since $\mathcal{B}$ is finite, $|A : A_1| < \infty$. Further, for each $b \in \mathcal{B}$, $\nu$ and $b\nu b^{-1}$ invert $A_1$, so $b^2 = b\nu b^{-1}\nu \in C_U(A_1)$ (recall that $\nu$ inverts $b$). Since $U$ is uniquely 2-divisible, $b \in C_U(A_1)$. Hence $\mathcal{B} \le C_U(A_1)$ and the lemma holds. □

**Lemma 3.5.** *Let $D$ be a uniquely 2-divisible subgroup of $A$ of finite index. Then $C_U(D)/D$ is finite and solvable.*

*Proof.* Set $C := C_U(D)$ and $\overline{C} := C/D$. Assume that $\overline{C}$ is infinite. By Lemma 2.6(1), $\overline{C}$ is uniquely 2-divisible, and by hypothesis $\overline{A} := A/D$ is a finite subgroup of $\overline{C}$.

Let $\overline{\mathcal{A}}$ be an infinite maximal abelian subgroup of $\overline{C}$ inverted by $\nu D$. The existence of $\overline{\mathcal{A}}$ is guaranteed by Lemma 2.6(2) and by Lemma 3.1(2) (with $\overline{C}$ in place of $U$). By Lemma 3.4 (with $\overline{C}$ in place of $U$ and $\overline{A}$ in place of $B$), there exists a finite index $\overline{\mathcal{A}_1} \le \overline{\mathcal{A}}$ such that $\overline{\mathcal{A}_2} := \langle \overline{\mathcal{A}_1}, \overline{A}\rangle$ is abelian. Note that $\overline{\mathcal{A}_2}$ is inverted by $\nu D$, so by Lemma 2.6(3) the inverse image $\mathcal{A}_2$ of $\overline{\mathcal{A}_2}$ in $C_U(D)$ is an abelian subgroup inverted by $\nu$. Clearly $\mathcal{A}_2$ properly contains $A$. This contradicts the maximality of $A$ and shows that $\overline{C}$ is finite.

Suppose $tD \in \overline{C}$ is an involution. Then $t^2 \in D$, so also $t \in D$, and we see that $\overline{C}$ has odd order. By the Feit-Thompson theorem, $\overline{C}$ is solvable, and the proof of the lemma is complete. □

**Proposition 3.6.** *Let $R := \langle S\rangle$. Then:*

(1) *$R'$ is a periodic group;*
(2) *$R$ is solvable.*

*Proof.* (1) We first show that for elements $u_1, \dots, u_n \in U$ the subgroup $K := \langle \nu u_1 \nu u_1^{-1}, \dots, \nu u_n \nu u_n^{-1} \rangle$ is solvable and $K/Z(K)$ is finite. By Remark 2.2(3), this will show that

($*$) if $H$ is a finitely generated subgroup of $R$,
then $H$ is solvable, and $H/Z(H)$ is finite.

Let $D := \bigcap_{i=1}^n A_{u_i}$. By the definition of $A_{u_i}$ and by Proposition 3.3, $|A : D| < \infty$ and $D$ is inverted by $\nu, u_1\nu u_1^{-1}, \dots, u_n \nu u_n^{-1}$. Also, by Remark 2.2(2), $D$ is uniquely 2-divisible. By Lemma 3.5, $C_U(D)/D$ is finite and solvable, so since $K \leq C_U(D)$, we see that $K/Z(K)$ is finite and solvable. Hence ($*$) holds.

Next let $g \in R'$. Then there exists a finitely generated subgroup $H$ of $R$ such that $g \in H'$. By ($*$) and by [A, (33.9), p. 168], $H'$ is finite, so the order of $g$ is finite. This completes the proof of part (1).

(2) By (1), $R'$ is a periodic group, and since $R$ is $\nu$-invariant, $\nu$ is an almost regular automorphism of $R'$. By the main result of Shunkov in [Sh], $R'$ is virtually solvable. But by ($*$), $R'$ is also locally solvable, so this shows that $R'$ is solvable and hence so is $R$. $\square$

*Proof of Theorem* 1.1. By Proposition 3.6, $\langle S \rangle$ is solvable. By [K, (3.4), p. 281] (see also [S, Lemma 2.1(1) and Lemma 2.2(1)]), $U = \langle S \rangle C_U(\nu)$ and $\langle S \rangle \trianglelefteq U$. Since $C_U(\nu)$ is a finite uniquely 2-divisible group, it has odd order. By the Feit-Thompson theorem it is solvable. Hence $U$ is solvable. $\square$

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