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LANGLANDS RECIPROCITY FOR THE EVEN-DIMENSIONAL NONCOMMUTATIVE TORI

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Abstract. We conjecture an explicit formula for the higher-dimensional Dirichlet character; the formula is based on the K-theory of the so-called noncommutative tori. It is proved that our conjecture is true for the two-dimensional and one-dimensional (degenerate) noncommutative tori. In the second case, one gets a noncommutative analog of the Artin reciprocity law.

Introduction

The aim of this paper is to bring some evidence in favor of the following analog of the Langlands reciprocity [5]:

Conjecture 1 (Langlands conjecture for noncommutative tori)**.** *Let K be a finite extension of the rational numbers $\mathbb{Q}$ with the Galois group $Gal\ (K|\mathbb{Q})$. For an irreducible representation $\sigma_{n+1} : Gal\ (K|\mathbb{Q}) \to GL_{n+1}(\mathbb{C})$, there exists a $2n$-dimensional noncommutative torus with real multiplication, $\mathcal{A}^{2n}_{RM}$, such that $L(\sigma_{n+1}, s) \equiv L(\mathcal{A}^{2n}_{RM}, s)$, where $L(\sigma_{n+1}, s)$ is the Artin L-function and $L(\mathcal{A}^{2n}_{RM}, s)$ an L-function attached to the $\mathcal{A}^{2n}_{RM}$. Moreover, $\mathcal{A}^{2n}_{RM}$ is the image of an n-dimensional abelian variety $V_n(K)$ under the (generalized) Teichmüller functor F_n.*

For the notation and terminology we refer the reader to sections 1 and 3; the noncommutative torus $\mathcal{A}^{2n}_{RM}$ can be regarded as a substitute of the "automorphic cuspidal representation $\pi_{\sigma_{n+1}}$ of the group $GL(n+1)$" in terms of the Langlands theory. Roughly speaking, Conjecture 1 says, that the Galois extensions of the field of rational numbers come from the even-dimensional noncommutative tori with real multiplication. Note that the noncommutative tori are intrinsic to the problem, since they classify the irreducible (infinite-dimensional) representations of the Lie group $GL(n+1)$ [11]. Such representations are the heart of the Langlands program [5]. Our conjecture is supported by the following evidence.

Theorem 1. *Conjecture 1 is true for $n = 1$ (resp., $n = 0$) and the K abelian extension of an imaginary quadratic field k (resp., the rational field $\mathbb{Q}$).*

The structure of the paper is as follows. Minimal necessary notation is introduced in section 1, and a brief summary of the Teichmüller functor(s) is given in section 3. Theorem 1 is proved in section 2.

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1. Preliminaries

1.1. Noncommutative tori.

A. The k-dimensional noncommutative tori ([4], [12]). A *noncommutative k-torus* is the universal C^*-algebra generated by k unitary operators $u_1, \dots, u_k$. The operators do not commute with each other, but their commutators $u_i u_j u_i^{-1} u_j^{-1}$ are fixed scalar multiples $\exp(2\pi i \theta_{ij})$, $\theta_{ij} \in \mathbb{R}$, of the identity operator. The k-dimensional noncommutative torus, $\mathcal{A}^k_\Theta$, is defined by a skew symmetric real matrix $\Theta = (\theta_{ij})$, $1 \le i, j \le k$. Further, we think of the $\mathcal{A}^k_\Theta$ as a noncommutative topological space whose algebraic K-theory yields $K_0(\mathcal{A}^k_\Theta) \cong \mathbb{Z}^{2^{k-1}}$ and $K_1(\mathcal{A}^k_\Theta) \cong \mathbb{Z}^{2^{k-1}}$. The canonical trace τ on the C^*-algebra $\mathcal{A}^k_\Theta$ defines a homomorphism from $K_0(\mathcal{A}^k_\Theta)$ to the real line $\mathbb{R}$. Under the homomorphism, the image of $K_0(\mathcal{A}^k_\Theta)$ is a $\mathbb{Z}$-module whose generators $\tau = (\tau_i)$ are polynomials in θ_{ij}. (More precisely, $\tau = \exp(\Theta)$, where the exterior algebra of θ_{ij} is nilpotent.) Recall that the C^*-algebras $\mathcal{A}$ and $\mathcal{A}'$ are said to be stably isomorphic (Morita equivalent) if $\mathcal{A} \otimes \mathcal{K} \cong \mathcal{A}' \otimes \mathcal{K}$ for the C^*-algebra $\mathcal{K}$ of compact operators. Such an isomorphism indicates that the C^*-algebras are homeomorphic as noncommutative topological spaces. By a result of Rieffel and Schwarz [13], the noncommutative tori $\mathcal{A}^k_\Theta$ and $\mathcal{A}^k_{\Theta'}$ are stably isomorphic if the matrices Θ and Θ' belong to the same orbit of a subgroup $SO(k, k \mid \mathbb{Z})$ of the group $GL_{2k}(\mathbb{Z})$, which acts on Θ by the formula $\Theta' = (A\Theta + B) \ / \ (C\Theta + D)$, where $(A, B, C, D) \in GL_{2k}(\mathbb{Z})$ and the matrices $A, B, C, D \in GL_k(\mathbb{Z})$ satisfy the conditions

$$A^t D + C^t B = I, \quad A^t C + C^t A = 0 = B^t D + D^t B. \tag{1}$$

(Here I is the unit matrix, and t at the upper right of a matrix means a transpose of the matrix.) The group $SO(k, k \mid \mathbb{Z})$ can be equivalently defined as a subgroup of the group $SO(k, k \mid \mathbb{R})$ consisting of linear transformations of the space $\mathbb{R}^{2k}$ which preserve the quadratic form $x_1 x_{k+1} + x_2 x_{k+2} + \dots + x_k x_{2k}$.

B. The even-dimensional normal tori. Further, we restrict ourselves to the case $k = 2n$ (the even-dimensional noncommutative tori). It is known that by the orthogonal linear transformations every (generic) real even-dimensional skew symmetric matrix can be brought to the normal form

$$\Theta_0 = \begin{pmatrix} 0 & \theta_1 & & & \\ -\theta_1 & 0 & & & \\ & & \ddots & & \\ & & & 0 & \theta_n \\ & & & -\theta_n & 0 \end{pmatrix}, \tag{2}$$

where $\theta_i > 0$ are linearly independent over $\mathbb{Q}$. We shall consider the noncommutative tori $\mathcal{A}^{2n}_{\Theta_0}$, given by the matrix (2); we refer to the family as a *normal family*. Recall that any noncommutative torus has a canonical trace τ which defines a homomorphism from $K_0(\mathcal{A}^k_\Theta) \cong \mathbb{Z}^{2^{k-1}}$ to $\mathbb{R}$. It follows from [4] that the image of $K_0(\mathcal{A}^{2n}_{\Theta_0})$ under the homomorphism has a basis, given by the formula $\tau(K_0(\mathcal{A}^{2n}_{\Theta_0})) = \mathbb{Z} + \theta_1 \mathbb{Z} + \dots + \theta_n \mathbb{Z} + \sum_{i=n+1}^{2^{2n-1}} p_i(\theta)\mathbb{Z}$, where $p_i(\theta) \in \mathbb{Z}[1, \theta_1, \dots, \theta_n]$.

C. The real multiplication ([8]). The noncommutative torus $\mathcal{A}^k_\Theta$ is said to have a *real multiplication* if the endomorphism ring $End\ (\tau(K_0(\mathcal{A}^k_\Theta)))$ exceeds the trivial ring $\mathbb{Z}$. Since any endomorphism of the $\mathbb{Z}$-module $\tau(K_0(\mathcal{A}^k_\Theta))$ is the

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multiplication by a real number, it is easy to deduce that all the entries of $\Theta = (\theta_{ij})$ are algebraic integers. (Indeed, the endomorphism is described by an integer matrix which defines a polynomial equation involving θ_{ij}.) Thus, the noncommutative tori with real multiplication is a countable subset of all k-dimensional tori; any element of the set we shall denote by $\mathcal{A}^k_{RM}$. Notice that for the even-dimensional normal tori with real multiplication, the polynomials $p_i(\theta)$ produce the algebraic integers in the extension of $\mathbb{Q}$ by θ_i. Any such integer is a linear combination (over $\mathbb{Z}$) of the θ_i. Thus, the trace formula reduces to $\tau(K_0(\mathcal{A}^{2n}_{RM})) = \mathbb{Z} + \theta_1\mathbb{Z} + \cdots + \theta_n\mathbb{Z}$.

1.2. ***L*-function of noncommutative tori.** We consider even-dimensional normal tori with real multiplication. Denote by A a positive integer matrix whose (normalized) Perron-Frobenius eigenvector coincides with the vector $\theta = (1, \theta_1, \dots, \theta_n)$ such that A is not a power of a positive integer matrix. In other words, $A\theta = \lambda_A\theta$, where $A \in GL_{n+1}(\mathbb{Z})$ and λ_A is the corresponding eigenvalue. (Explicitly, A can be obtained from vector θ as the matrix of minimal period of the Jacobi-Perron continued fraction of θ [2].) Let p be a prime number; take the matrix A^p and consider its characteristic polynomial $char\ (A^p) = x^{n+1} + a_1x^n + \cdots + a_nx + 1$. We introduce the following notation:

$$L^{n+1}_p := \begin{pmatrix} a_1 & a_2 & \dots & a_n & p \\ -1 & 0 & \dots & 0 & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & \dots & -1 & 0 \end{pmatrix}. \tag{3}$$

A *local* zeta function of the $\mathcal{A}^{2n}_{RM}$ is defined as the reciprocal of $det\ (I_{n+1} - L^{n+1}_p z)$; in other words,

$$\zeta_p(\mathcal{A}^{2n}_{RM}, z) := \frac{1}{1 - a_1z + a_2z^2 - \cdots - a_nz^n + pz^{n+1}}, \quad z \in \mathbb{C}. \tag{4}$$

An *L-function* of $\mathcal{A}^{2n}_{RM}$ is a product of the local zetas over all but a finite number, of primes $L(\mathcal{A}^{2n}_{RM}, s) = \prod_{p\ \nmid\ tr^2(A)-(n+1)^2} \zeta_p(\mathcal{A}^{2n}_{RM}, p^{-s}),\ s \in \mathbb{C}$.

Remark 1. It will be shown that for $n = 0$ and $n = 1$ formula (4) fits Conjecture 1. For $n \geq 2$ it is an open problem based on an observation that the crossed product $\mathcal{A}^{2n}_{RM} \rtimes_{L^{n+1}_p} \mathbb{Z}$ is a proper noncommutative analog of the (higher-dimensional) Tate module, where the matrix L^{n+1}_p corresponds to the Frobenius automorphism of the module [14], p. 172.

2. Proof of Theorem 1

2.1. **Case $n = 1$.** Each one-dimensional abelian variety is a nonsingular elliptic curve. Choose this curve to have complex multiplication by (an order in) the imaginary quadratic field k and denote such a curve by E_{CM}. Then, by theory of complex multiplication, the (maximal) abelian extension of k coincides with the minimal field of definition of the curve E_{CM}, i.e. $E_{CM} \cong E(K)$ [14]. The Teichmüller functor $F := F_1$ maps $E(K)$ into a two-dimensional noncommutative torus with real multiplication (section 3); we shall denote the torus by $\mathcal{A}^2_{RM}$. To calculate the corresponding L-function $L(\mathcal{A}^2_{RM}, s)$, let A be a 2×2 positive integer matrix whose normalized Perron-Frobenius eigenvenctor is $(1, \theta_1)$. For a prime

p, the characteristic polynomial of the matrix A^p takes the form $char\ (A^p) = x^2 + tr\ (A^p)x + 1$, and the matrix L_p^2 takes the form

$$L_p^2 = \begin{pmatrix} tr\ (A^p) & p \\ -1 & 0 \end{pmatrix}. \tag{5}$$

The corresponding local zeta function $\zeta_p(\mathcal{A}_{RM}^2, z) = (1 - tr\ (A^p)z + pz^2)^{-1}$. We have to prove that $\zeta_p(\mathcal{A}_{RM}^2, z) = \zeta_p(E_{CM}, z)$, where $\zeta_p(E_{CM}, z)$ is the local zeta function for the elliptic curve E_{CM}. The proof will be arranged into a series of lemmas (Lemmas 1-5).

Recall that $\zeta_p(E_{CM}, z) = (1 - tr\ (\psi_{E(K)}(\mathfrak{P}))z + pz^2)^{-1}$, where $\psi_{E(K)}$ is the Grössencharacter on K, $\mathfrak{P}$ the prime ideal of K over p and tr is the trace of algebraic number [14], Ch. 2, §9. Roughly, our proof consists in the construction of the representation ρ of $\psi_{E(K)}$ into the group of invertible elements (units) of $End\ (\tau(K_0(\mathcal{A}_{RM}^2)))$ such that $tr\ (\psi_{E(K)}(\mathfrak{P})) = tr\ (\rho(\psi_{E(K)}(\mathfrak{P}))) = tr\ (A^p)$. This will be achieved with the help of an explicit formula for the Teichmüller functor F ([10], p. 524):

$$F : \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in End\ (E_{CM}) \longmapsto \begin{pmatrix} a & b \\ -c & -d \end{pmatrix} \in End\ (\mathbb{A}_{RM}). \tag{6}$$

Lemma 1. *Let $A = (a, b, c, d)$ be an integer matrix with $ad - bc \neq 0$ and $b = 1$. Then A is similar to the matrix $(a + d, 1, c - ad, 0)$.*

Proof. Indeed, consider a matrix $(1, 0, d, 1) \in SL_2(\mathbb{Z})$. It is verified directly that the matrix realizes the required similarity. □

Lemma 2. *The matrix $A = (a + d, 1, c - ad, 0)$ is similar to its transpose $A^t = (a + d, c - ad, 1, 0)$.*

Proof. We shall use the following criterion: the (integer) matrices A and B are similar if and only if the characteristic matrices $xI - A$ and $xI - B$ have the same Smith normal form. The calculation for the matrix $xI - A$ gives

$$\begin{pmatrix} x-a-d & -1 \\ ad-c & x \end{pmatrix} \sim \begin{pmatrix} x-a-d & -1 \\ x^2-(a+d)x+ad-c & 0 \end{pmatrix}$$
$$\sim \begin{pmatrix} 1 & 0 \\ 0 & x^2-(a+d)x+ad-c \end{pmatrix},$$

where the $\sim$ are the elementary operations between the rows (columns) of the matrix. Similarly, a calculation for the matrix $xI - A^t$ gives

$$\begin{pmatrix} x-a-d & ad-c \\ -1 & x \end{pmatrix} \sim \begin{pmatrix} x-a-d & x^2-(a+d)x+ad-c \\ -1 & 0 \end{pmatrix}$$
$$\sim \begin{pmatrix} 1 & 0 \\ 0 & x^2-(a+d)x+ad-c \end{pmatrix}.$$

Thus, $(xI - A) \sim (xI - A^t)$, and Lemma 2 follows. □

Corollary 1. *The matrices $(a, 1, c, d)$ and $(a + d, c - ad, 1, 0)$ are similar.*

Let E_{CM} be an elliptic curve with complex multiplication by an order R in the ring of integers of the imaginary quadratic field k. Then $\mathbb{A}_{RM} = F(E_{CM})$ is a noncommutative torus with real multiplication by the order $\mathfrak{R}$ in the ring of integers of a real quadratic field $\mathfrak{k}$ (section 3). Let $tr\ (\alpha) = \alpha + \bar{\alpha}$ be the trace function of a (quadratic) algebraic number field.

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Lemma 3. *Each $\alpha\in R$ goes under F into an $\omega\in\mathfrak{R}$ such that $tr\ (\alpha)=tr\ (\omega)$.*

Proof. Recall that each $\alpha\in R$ can be written in a matrix form for a given base $\{\omega_1,\omega_2\}$ of the lattice Λ. Namely, $\alpha\omega_1=a\omega_1+b\omega_2$ and $\alpha\omega_2=c\omega_1+d\omega_2$, where (a,b,c,d) is an integer matrix with $ad-bc\neq 0$. Note that $tr\ (\alpha)=a+d$ and $b\tau^2+(a-d)\tau-c=0$, where $\tau=\omega_2/\omega_1$. Since τ is an algebraic integer, we conclude that $b=1$. In view of Corollary 1, in a base $\{\omega_1',\omega_2'\}$ the α has a matrix form $(a+d,c-ad,1,0)$. To calculate $\omega\in\mathfrak{R}$ corresponding to α, we apply formula (6), which gives us

$$F:\begin{pmatrix}a+d & c-ad\\ 1 & 0\end{pmatrix}\longmapsto\begin{pmatrix}a+d & c-ad\\ -1 & 0\end{pmatrix}. \tag{7}$$

In a given base $\{\lambda_1,\lambda_2\}$ of the pseudo-lattice $\mathbb{Z}+\mathbb{Z}\theta$, one can write $\omega\lambda_1=(a+d)\lambda_1+(c-ad)\lambda_2$ and $\omega\lambda_2=-\lambda_1$. It is an easy exercise to verify that ω is a real quadratic integer with $tr\ (\omega)=a+d$; the latter coincides with the $tr\ (\alpha)$. □

Let $\omega\in\mathfrak{R}$ be an endomorphism of the pseudo-lattice $\mathbb{Z}+\mathbb{Z}\theta$ of degree $deg\ (\omega):=\omega\bar{\omega}=n$. The endomorphism maps the pseudo-lattice to a sub-lattice of index n. Any such endomorphism has a form $\mathbb{Z}+(n\theta)\mathbb{Z}$ [3], p. 131. Let us calculate ω in a base $\{1,n\theta\}$ when ω is given by the matrix $(a+d,c-ad,-1,0)$. In this case $n=c-ad$ and ω induces an automorphism $\omega^*=(a+d,1,-1,0)$ of the sublattice $\mathbb{Z}+(n\theta)\mathbb{Z}$ according to the matrix equation

$$\begin{pmatrix}a+d & n\\ -1 & 0\end{pmatrix}\begin{pmatrix}1\\ \theta\end{pmatrix}=\begin{pmatrix}a+d & 1\\ -1 & 0\end{pmatrix}\begin{pmatrix}1\\ n\theta\end{pmatrix}. \tag{8}$$

Thus, one gets a map $\rho:\mathfrak{R}\to\mathfrak{R}^*$ given by the formula $\omega=(a+d,n,-1,0)\mapsto\omega^*=(a+d,1,-1,0)$, where $\mathfrak{R}^*$ is the group of units of $\mathfrak{R}$. Since $tr\ (\omega^*)=a+d=tr\ (\omega)$ and $\omega^*=\rho(\omega)$, one gets the following:

Corollary 2. *For all $\omega\in\mathfrak{R}$, it holds that $tr\ (\omega)=tr\ (\rho(\omega))$.*

Note that $\mathfrak{R}^*=\{\pm\varepsilon^k\mid k\in\mathbb{Z}\}$, where $\varepsilon>1$ is a fundamental unit of the order $\mathfrak{R}\subseteq O_{\mathfrak{k}}$. Here $O_{\mathfrak{k}}$ means the ring of integers of a real quadratic field $\mathfrak{k}=\mathbb{Q}(\theta)$. Choosing a sign in front of ε^k, the following index map is defined as $\iota: R\xrightarrow{\mathrm{F}}\mathfrak{R}\xrightarrow{\rho}\mathfrak{R}^*\longrightarrow\mathbb{Z}$. Let $\alpha\in R$ and $deg\ (\alpha)=-n$. To calculate the $\iota(\alpha)$, recall some notation from Hasse [6], §16.5.C. Let $\mathbb{Z}/n\mathbb{Z}$ be a cyclic group of order n. For brevity, let $I=\mathbb{Z}+\mathbb{Z}\theta$ be a pseudo-lattice and $I_n=\mathbb{Z}+(n\theta)\mathbb{Z}$ be its sub-lattice of index n. The fundamental units of I and I_n are ε and ε_n, respectively. By $\mathfrak{G}_n$ one understands a subgroup of $\mathbb{Z}/n\mathbb{Z}$ of prime residue classes *mod* n. The $\mathfrak{g}_n\subset\mathfrak{G}_n$ is a subgroup of the nonzero divisors of the $\mathfrak{G}_n$. Finally, let g_n be the smallest number such that it divides $|\mathfrak{G}_n/\mathfrak{g}_n|$ and $\varepsilon^{g_n}\in I_n$. (The notation drastically simplifies in the case when $n=p$ is a prime number.)

Lemma 4. $\iota(\alpha)=g_n$.

Proof. Notice that $deg\ (\omega)=-deg\ (\alpha)=n$, where $\omega=F(\alpha)$. Then the map ρ defines I and I_n; one can now apply the calculation of [6], pp. 296-300. Namely, Theorem XIII$'$ on p. 298 yields the required result. (We kept the notation of the original.) □

Corollary 3. $\iota(\psi_{E(K)}(\mathfrak{P}))=p$.

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Proof. It is known that $deg\ (\psi_{E(K)}(\mathfrak{P})) = -p$, where $\psi_{E(K)}(\mathfrak{P}) \in R$ is the Grössencharacter. To calculate the g_n in the case when $n = p$, notice that $\mathfrak{G}_p \cong \mathbb{Z}/p\mathbb{Z}$ and $\mathfrak{g}_p$ is trivial. Thus, $|\mathfrak{G}_p/\mathfrak{g}_p| = p$ is divisible only by 1 or p. Since ε^1 is not in I_n, one concludes that $g_p = p$. The corollary follows. $\square$

Lemma 5. *$tr\ (\psi_{E(K)}(\mathfrak{P})) = tr\ (A^p)$.*

Proof. It is not hard to see that A is a hyperbolic matrix with the eigenvector $(1, \theta)$. The corresponding (Perron-Frobenius) eigenvalue is a fundamental unit $\varepsilon > 1$ of the pseudo-lattice $\mathbb{Z}+\mathbb{Z}\theta$. In other words, A is a matrix form of the algebraic number ε. It is immediate that A^p is the matrix form for the ε^p and $tr\ (A^p) = tr\ (\varepsilon^p)$. In view of Lemma 3 and Corollary 3, $tr\ (\alpha) = tr\ (F(\alpha)) = tr\ (\rho(F(\alpha)))$ for $\forall\alpha \in R$. In particular, if $\alpha = \psi_{E(K)}(\mathfrak{P})$, then, by Corollary 3, one gets $\rho(F(\psi_{E(K)}(\mathfrak{P}))) = \varepsilon^p$. Taking traces in the last equation, we obtain the conclusion of Lemma 5. $\square$

The fact $\zeta_p(\mathcal{A}^2_{RM}, z) = \zeta_p(E_{CM}, z)$ follows from Lemma 5, since the trace of the Grössencharacter coincides with such for the matrix A^p. Let E_{CM} be an elliptic curve with complex multiplication by an order in the imaginary quadratic field k and let K be the minimal field of definition of the E_{CM}.

Lemma 6. *$L(E_{CM}, s) \equiv L(\sigma_2, s)$, where $L(E_{CM}, s)$ is the Hasse-Weil L-function of E_{CM} and $L(\sigma_2, s)$ is the Artin L-function for an irreducible representation $\sigma_2 : Gal\ (K|k) \to GL_2(\mathbb{C})$.*

Proof. By the Deuring theorem (see e.g. [14], p. 175),

$$L(E_{CM}, s) = L(\psi_K, s)L(\overline{\psi}_K, s),$$

where $L(\psi_K, s)$ is the Hecke L-series attached to the Grössencharacter $\psi : \mathbb{A}^*_K \to \mathbb{C}^*$. Here $\mathbb{A}^*_K$ denotes the adele ring of the field K and the bar means a complex conjugation. Notice that since our elliptic curve has complex multiplication, the group $Gal\ (K|k)$ is abelian. One can apply Theorem 5.1, [7], which says that the Hecke L-series $L(\sigma_1 \circ \theta_{K|k}, s)$ equals the Artin L-function $L(\sigma_1, s)$, where $\psi_K = \sigma \circ \theta_{K|k}$ is the Grössencharacter and $\theta_{K|k} : \mathbb{A}^*_K \to Gal\ (K|k)$ is the canonical homomorphism. Thus, one gets $L(E_{CM}, s) \equiv L(\sigma_1, s)L(\overline{\sigma}_1, s)$, where $\overline{\sigma}_1 : Gal\ (K|k) \to \mathbb{C}$ means a (complex) conjugate representation of the Galois group. Consider the local factors of the Artin L-functions $L(\sigma_1, s)$ and $L(\overline{\sigma}_1, s)$. It is immediate that they are $(1 - \sigma_1(Fr_p)p^{-s})^{-1}$ and $(1 - \overline{\sigma}_1(Fr_p)p^{-s})^{-1}$, respectively. Let us consider a representation $\sigma_2 : Gal\ (K|k) \to GL_2(\mathbb{C})$ such that

$$\sigma_2(Fr_p) = \begin{pmatrix} \sigma_1(Fr_p) & 0 \\ 0 & \overline{\sigma}_1(Fr_p) \end{pmatrix}. \tag{9}$$

It can be verified that

$$\det{}^{-1}(I_2 - \sigma_2(Fr_p)p^{-s}) = (1 - \sigma_1(Fr_p)p^{-s})^{-1}(1 - \overline{\sigma}_1(Fr_p)p^{-s})^{-1};$$

i.e. $L(\sigma_2, s) = L(\sigma_1, s)L(\overline{\sigma}_1, s)$. Lemma 6 follows.

By Lemma 6, we conclude that $L(\mathcal{A}^2_{RM}, s) \equiv L(\sigma_2, s)$ for an irreducible representation $\sigma_2 : Gal\ (K|k) \to GL_2(\mathbb{C})$. It remains to note that $L(\sigma_2, s) = L(\sigma'_2, s)$, where $\sigma'_2 : Gal\ (K|\mathbb{Q}) \to GL_2(\mathbb{C})$ [1], §3. Case $n = 1$ of Theorem 1 follows. $\square$

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2.2. **Case** $n = 0$**.** When $n = 0$, one gets a one-dimensional (degenerate) noncommutative torus. Such an object, $\mathcal{A}_{\mathbb{Q}}$, can be obtained from the 2-dimensional torus $\mathcal{A}_\theta^2$ by forcing $\theta = p/q \in \mathbb{Q}$ to be a rational number (hence our notation). One can always assume $\theta = 0$ and, thus, $\tau(K_0(\mathcal{A}_{\mathbb{Q}})) = \mathbb{Z}$. To calculate matrix L_p^1, notice that the group of automorphisms of the $\mathbb{Z}$-module $\tau(K_0(\mathcal{A}_{\mathbb{Q}})) = \mathbb{Z}$ is trivial, i.e. is a multiplication by ± 1; hence our 1×1 (real) matrix A is either 1 or -1. Since A must be positive, we conclude that $A = 1$. However, $A = 1$ is not a prime matrix if one allows the complex entries. Indeed, for any $N > 1$ matrix $A' = \zeta_N$ gives us $A = (A')^N$, where $\zeta_N = e^{\frac{2\pi i}{N}}$ is the N-th root of unity. Therefore, $A = \zeta_N$ and $L_p^1 = tr\ (A^p) = A^p = \zeta_N^p$. A degenerate noncommutative torus, corresponding to the matrix $A = \zeta_N$, shall be written as $\mathcal{A}_{\mathbb{Q}}^N$. In turn, such a torus is the image (under the Teichmüller functor) of a zero-dimensional abelian variety, which we denote by V_0^N. Suppose that $Gal\ (K|\mathbb{Q})$ is abelian and let $\sigma : Gal\ (K|\mathbb{Q}) \to \mathbb{C}^\times$ be a homomorphism. Then, by the Artin reciprocity [5], there exists an integer N_σ and a Dirichlet character $\chi_\sigma : (\mathbb{Z}/N_\sigma\mathbb{Z})^\times \to \mathbb{C}^\times$ such that $\sigma(Fr_p) = \chi_\sigma(p)$; choose our zero-dimensional variety to be $V_0^{N_\sigma}$. In view of the notation, $L_p^1 = \zeta_{N_\sigma}^p$; on the other hand, it is verified directly that $\zeta_{N_\sigma}^p = e^{\frac{2\pi i}{N_\sigma}p} = \chi_\sigma(p)$. Thus, $L_p^1 = \chi_\sigma(p)$. To obtain a local zeta function, we substitute $a_1 = L_p^1$ into the formula (4) and obtain

$$\zeta_p(\mathcal{A}_{\mathbb{Q}}^{N_\sigma}, z) = \frac{1}{1 - \chi_\sigma(p) z}, \tag{10}$$

where $\chi_\sigma(p)$ is the Dirichlet character. Therefore, $L(\mathcal{A}_{\mathbb{Q}}^{N_\sigma}, s) \equiv L(s, \chi_\sigma)$ is the Dirichlet L-series. Such a series, by construction, coincides with the Artin L-series of the representation $\sigma : Gal\ (K|\mathbb{Q}) \to \mathbb{C}^\times$. Case $n = 0$ of Theorem 1 follows.

3. Teichmüller functors

Denote by Λ a lattice of rank $2n$. Recall that an n-dimensional (principally polarized) abelian variety, V_n, is the complex torus $\mathbb{C}^n/\Lambda$, which admits an embedding into a projective space [9].

3.1. Abelian varieties of dimension $n = 1$.

A. Basic example. Let $n = 1$ and consider the complex torus $V_1 \cong \mathbb{C}/(\mathbb{Z} + \tau\mathbb{Z})$; it always embeds (via the Weierstrass $\wp$ function) into a projective space as a non-singular elliptic curve. Let $\mathbb{H} = \{\tau = x + iy \in \mathbb{C} \mid y > 0\}$ be the upper half-plane and $\partial\mathbb{H} = \{\theta \in \mathbb{R} \mid y = 0\}$ its (topological) boundary. We identify $V_1(\tau)$ with the points of $\mathbb{H}$ and $\mathcal{A}_\theta^2$ with the points of $\partial\mathbb{H}$. Let us show that the boundary is natural. The latter means that the action of the modular group $SL_2(\mathbb{Z})$ extends to the boundary, where it coincides with the stable isomorphisms of tori. Indeed, conditions (1) are equivalent to

$$A = \begin{pmatrix} a & 0 \\ 0 & a \end{pmatrix}, \quad B = \begin{pmatrix} 0 & b \\ -b & 0 \end{pmatrix}, \quad C = \begin{pmatrix} 0 & -c \\ c & 0 \end{pmatrix}, \quad D = \begin{pmatrix} d & 0 \\ 0 & d \end{pmatrix}, \tag{11}$$

where $ad - bc = 1$, $a, b, c, d \in \mathbb{Z}$ and $\Theta' = (A\Theta + B)/(C\Theta + D) = (0, \frac{a\theta+b}{c\theta+d}, -\frac{a\theta+b}{c\theta+d}, 0)$. Therefore, $\theta' = (a\theta + b)(c\theta + d)^{-1}$ for a matrix $(a, b, c, d) \in SL_2(\mathbb{Z})$. Thus, the action of $SL_2(\mathbb{Z})$ extends to the boundary $\partial\mathbb{H}$, where it induces stable isomorphisms of the noncommutative tori.

B. The Teichmüller functor ([10])**.** There exists a continuous map $F_1 : \mathbb{H} \to \partial\mathbb{H}$, which sends isomorphic complex tori to the stably isomorphic noncommutative

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tori. An exact result is this: Let ϕ be a closed form on the torus whose trajectories define a measured foliation. According to the Hubbard-Masur theorem (applied to the complex tori), this foliation corresponds to a point $\tau \in \mathbb{H}$. The map $F_1 : \mathbb{H} \to \partial\mathbb{H}$ is defined by the formula $\tau \mapsto \theta = \int_{\gamma_2} \phi / \int_{\gamma_1} \phi$, where γ_1 and γ_2 are generators of the first homology of the torus. The following is true: (i) $\mathbb{H} = \partial\mathbb{H} \times (0, \infty)$ is a trivial fiber bundle whose projection map coincides with F_1; (ii) F_1 is a functor which sends isomorphic complex tori to the stably isomorphic noncommutative tori. We shall refer to F_1 as the *Teichmüller functor*. Recall that the complex torus $\mathbb{C}/(\mathbb{Z}+\tau\mathbb{Z})$ is said to have a complex multiplication if the endomorphism ring of the lattice $\Lambda = \mathbb{Z}+\tau\mathbb{Z}$ exceeds the trivial ring $\mathbb{Z}$. The complex multiplication happens if and only if τ is an algebraic number in an imaginary quadratic field. The following is true: $F_1(V_1^{CM}) = \mathcal{A}_{RM}^2$, where V_1^{CM} is a torus with complex multiplication.

3.2. **Abelian varieties of dimension $n \geq 1$.**

A. The Siegel upper half-space ([9]). The space

$$\mathbb{H}_n := \{\tau = (\tau_i) \in \mathbb{C}^{\frac{n(n+1)}{2}} \mid Im\ (\tau_i) > 0\}$$

of symmetric $n \times n$ matrices with complex entries is called a *Siegel upper half-space*. The points of $\mathbb{H}_n$ are one-to-one with the n-dimensional principally polarized abelian varieties. Let $Sp(2n, \mathbb{R})$ be the symplectic group. It acts on $\mathbb{H}_n$ by the linear fractional transformations $\tau \to \tau' = (a\tau + b)/(c\tau + d)$, where $(a, b, c, d) \in Sp(2n, \mathbb{R})$ and a, b, c and d are the $n \times n$ matrices with real entries. The abelian varieties V_n and V_n' are isomorphic if and only if τ and τ' belong to the same orbit of the group $Sp(2n, \mathbb{Z})$; the action is discontinuous on $\mathbb{H}_n$ [9], Ch. 2, §4. Denote by Σ_{2n} a space of the $2n$-dimensional normal noncommutative tori. The following lemma is critical.

Lemma 7. *$Sp(2n, \mathbb{R}) \subseteq O(n, n|\mathbb{R})$.*

Proof. (i) The group $O(n, n \mid \mathbb{R})$ can be defined as a subgroup of $GL_2(\mathbb{R})$ which preserves the quadratic form $f(x_1, \ldots, x_{2n}) = x_1x_{n+1} + x_2x_{n+2} + \cdots + x_nx_{2n}$ [13]. We shall denote $u_i = x_1$ for $1 \leq i \leq n$ and $v_i = x_i$ for $n+1 \leq i \leq 2n$. Consider the following skew symmetric bilinear form $q(u, v) = u_1v_{n+1} + \cdots + u_nv_{2n} - u_{n+1}v_1 - \cdots - u_{2n}v_n$, where $u, v \in \mathbb{R}^{2n}$. It is known that each linear substitution $g \in Sp(2n, \mathbb{R})$ preserves the form $q(u, v)$. Since $q(u, v) = f(x_1, \ldots, x_{2n}) - u_{n+1}v_1 - \cdots - u_{2n}v_n$, one concludes that g also preserves the form $f(x_1, \ldots, x_{2n})$, i.e. $g \in O(n, n|\mathbb{R})$. It is easy to see that the inclusion is proper except in the case $n = 1$, i.e. when $Sp(2, \mathbb{R}) \cong O(1, 1|\mathbb{R}) \cong SL_2(\mathbb{R})$. The lemma follows.

(ii) We wish to give a second proof of this important fact, which is based on the explicit formulas for the block matrices A, B, C and D. The fact that a symplectic linear transformation preserves the skew symmetric bilinear form $q(u, v)$ can be written in a matrix form:

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix}^t \begin{pmatrix} 0 & I \\ -I & 0 \end{pmatrix} \begin{pmatrix} a & b \\ c & d \end{pmatrix} = \begin{pmatrix} 0 & I \\ -I & 0 \end{pmatrix}, \tag{12}$$

where t is the transpose of a matrix. Performing the matrix multiplication, one gets the matrix identities $a^td - c^tb = I,\ a^tc - c^ta = 0 = b^td - d^tb$. Let us show that these identities imply the Rieffel-Schwarz identities (1) imposed on the matrices

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A, B, C and D. Indeed, in view of the formulas (11), the Rieffel-Schwarz identities can be written as

$$(13)\quad \begin{cases} \begin{pmatrix} a^t & 0 \\ 0 & a^t \end{pmatrix}\begin{pmatrix} d & 0 \\ 0 & d \end{pmatrix} + \begin{pmatrix} 0 & c^t \\ -c^t & 0 \end{pmatrix}\begin{pmatrix} 0 & b \\ -b & 0 \end{pmatrix} = \begin{pmatrix} I & 0 \\ 0 & I \end{pmatrix}, \\ \begin{pmatrix} a^t & 0 \\ 0 & a^t \end{pmatrix}\begin{pmatrix} 0 & -c \\ c & 0 \end{pmatrix} + \begin{pmatrix} 0 & c^t \\ -c^t & 0 \end{pmatrix}\begin{pmatrix} a & 0 \\ 0 & a \end{pmatrix} = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}, \\ \begin{pmatrix} 0 & -b^t \\ b^t & 0 \end{pmatrix}\begin{pmatrix} d & 0 \\ 0 & d \end{pmatrix} + \begin{pmatrix} d^t & 0 \\ 0 & d^t \end{pmatrix}\begin{pmatrix} 0 & b \\ -b & 0 \end{pmatrix} = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}. \end{cases}$$

A step-by-step matrix multiplication in (13) shows that the identities $a^td - c^tb = I$, $a^tc - c^ta = 0 = b^td - d^tb$ imply the identities (13). (Beware: the operation is not commutative.) Thus, any symplectic transformation satisfies the Rieffel-Schwarz identities, i.e. belongs to the group $O(n, n|\mathbb{R})$. Lemma 7 follows. □

B. The generalized Teichmüller functors. By Lemma 7, the action of $Sp(2n, \mathbb{Z})$ on the $\mathbb{H}_n$ extends to the Σ_{2n}, where it acts by stable isomorphisms of the noncommutative tori. Thus, Σ_{2n} is a natural boundary of the Siegel upper half-space $\mathbb{H}_n$. However, unless $n = 1$, the Σ_{2n} is *not* a topological boundary of $\mathbb{H}_n$. Indeed, $dim_{\mathbb{R}}(\mathbb{H}_n) = n(n+1)$ and $dim_{\mathbb{R}}(\partial\mathbb{H}_n) = n^2 + n - 1$, while $dim_{\mathbb{R}}(\Sigma_{2n}) = n$. Thus, Σ_{2n} is an n-dimensional subspace of the topological boundary of $\mathbb{H}_n$. This subspace is everywhere dense in $\partial\mathbb{H}_n$, since the $Sp(2n, \mathbb{Z})$-orbit of an element of Σ_{2n} is everywhere dense in $\partial\mathbb{H}_n$ [13]. A (conjectural) continuous map $F_n : \mathbb{H}_n \to \Sigma_{2n}$ shall be called a *generalized Teichmüller functor*. The F_n has the following properties: (i) it sends each pair of isomorphic abelian varieties to a pair of the stably isomorphic even-dimensional normal tori; (ii) the range of F_n on the abelian varieties with complex multiplication consists of the noncommutative tori with real multiplication. As explained, such a functor has been constructed only in the case $n = 1$. The difficulties in higher dimensions are due to the lack (so far) of a proper Teichmüller theory for the abelian varieties of dimension $n \geq 2$.

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