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RECOLLEMENTS OF GORENSTEIN DERIVED CATEGORIES

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Abstract. A necessary and sufficient condition for the existence of recollements of bounded Gorenstein derived categories of CM-finite Gorenstein Artin algebras is given.

1. Introduction and preliminaries

Gorenstein homological algebra has been developed to an advanced level; see for example [2], [4], [14], [11], [9], [3], [15], [5], [10], [7], [17]. It is natural to study the corresponding version of derived categories in this context. Nan Gao and Pu Zhang [12] introduce the Gorenstein derived category, which makes Gorenstein quasi-isomorphisms become isomorphisms. The Gorenstein derived category has some advantages in the relative setting. For example, the Gorenstein derived functors can be interpreted as the Hom functors of the Gorenstein derived category, the dimension ([21]) of the bounded Gorenstein derived category is an upper bound of the one of the bounded derived category, and Gorenstein derived equivalences relate to derived equivalences. In particular we show that Gorenstein derived equivalences between CM-finite Gorenstein artin algebras and between finite-dimensional algebras correspond to what we call Gorenstein tilting complexes.

Beilinson, Bernstein and Deligne [6] introduce the recollements of triangulated categories with the idea that $\mathcal{T}$ can be viewed as being "glued together" from $\mathcal{T}^{'}$ and $\mathcal{T}^{''}$. The canonical example of a recollement has $\mathcal{T}$, $\mathcal{T}^{'}$, and $\mathcal{T}^{''}$ equal to suitable derived categories of sheaves on spaces X, Z and U, where X is the union of the closed subspace Z and its open complement U. Cline, Parshall and Scott [19] use the definition of recollement of triangulated categories to obtain what they call stratification of certain derived categories. König [18] provides a link between the two concepts of tilting complexes and recollements of derived categories and gives a necessary and sufficient condition for the existence of recollements, which recently has been extended by Nicolas in his PhD thesis. Also, Angeleri, Koenig and Liu develop more general criteria for the existence of recollements of derived categories.

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This paper's aim is to provide a link between the two concepts of Gorenstein tilting complexes and recollements of bounded Gorenstein derived categories of CM-finite Gorenstein Artin algebras, and then give a necessary and sufficient criterion for the existence of the recollements situation. The main idea of the paper is from [18].

In the following we recall some definitions and results used in [11]. Throughout, A is a finite-dimensional algebra, and A-$\mathcal{P}$ is the full subcategory of projective A-modules.

An A-module G is *Gorenstein-projective* if there is an exact sequence $\cdots \longrightarrow P_1 \longrightarrow P_0 \longrightarrow P^0 \longrightarrow P^1 \longrightarrow \cdots$ of projective A-modules which stays exact after applying $\mathrm{Hom}_A(-, P)$ for any $P \in A$-$\mathcal{P}$, such that $G \cong \mathrm{Im}(P_0 \longrightarrow P^0)$ (see [11]). Let A-$\mathcal{GP}$ be the full subcategory of Gorenstein-projective A-modules and A-$\mathcal{G}proj$ be the full subcategory of finitely-generated Gorenstein-projective A-modules. Then A-$\mathcal{GP}$ is *resolving* (see [15]) in the sense of [4]: it contains all the projective objects and is closed under direct summands, extensions, and the kernels of epimorphisms. Also, it is closed under arbitrary direct sums and is a Frobenius category with projective modules as projective-injective objects.

A complex $C^\bullet$ is A-$\mathcal{GP}$*-acyclic* if $\mathrm{Hom}_{\mathcal{A}}(G, C^\bullet)$ is acyclic for any $G \in A$-$\mathcal{GP}$. It is also called *proper exact*, for example, in [3]. A chain map $f^\bullet : X^\bullet \longrightarrow Y^\bullet$ is *an* A-$\mathcal{GP}$*-quasi-isomorphism* if $\mathrm{Hom}_{\mathcal{A}}(G, f^\bullet)$ is a quasi-isomorphism for any $G \in \mathcal{GP}$; i.e., there are isomorphisms of abelian groups

$$H^n \mathrm{Hom}_{\mathcal{A}}(G, f^\bullet): \ H^n \mathrm{Hom}_{\mathcal{A}}(G, X^\bullet) \cong H^n \mathrm{Hom}_{\mathcal{A}}(G, Y^\bullet), \ \forall\, n \in \mathbb{Z}, \ \forall\, G \in \mathcal{GP}.$$

The *Gorenstein derived category* $D^*_{gp}(A\text{-Mod})$ with $* \in \{\text{blank}, -, b\}$ is defined as the Verdier quotient of the homotopy category $K^*(A\text{-Mod})$ with respect to the triangulated subcategory $K^*_{gpac}(A\text{-Mod})$ of A-$\mathcal{GP}$-acyclic complexes.

A ring R is a *Gorenstein ring* if R is two-sided noetherian and has finite injective dimension, both as left and right R-modules. If R is also an Artin algebra, then R is called a *Gorenstein Artin algebra*. A Gorenstein ring R is n*-Gorenstein* if $\mathrm{inj.dim}\ {}_R R \le n < \infty$. In this case $\mathrm{inj.dim}\ R_R \le n$ ([11, 9.1.9]).

Lemma 1.1 ([12, Corollary 3.8])**.** *Let R be a Gorenstein ring. Then there is a triangle-equivalence*

$$D^b_{gp}(R\text{-Mod}) \cong K^b(R\text{-}\mathcal{GP}).$$

An Artin algebra A is *CM-finite* if A has only finitely many isomorphism classes of indecomposable finitely generated Gorenstein-projective modules. We refer to [5] and [7] for some properties of this class of algebras.

For a CM-finite algebra A, let $G_1, \cdots, G_m$ be all the pairwise non-isomorphic indecomposable finitely generated Gorenstein-projective A-modules. Put $\mathcal{G}p(A) := (\mathrm{End}_A(\bigoplus\limits_{1\le i\le m} G_i))^{op}$. Denote by $\mathcal{G}p(A)$-*proj* the full subcategory of finitely-generated projective $\mathcal{G}p(A)$-modules.

Lemma 1.2 ([12, Theorem 3.14])**.** *Let A be a CM-finite Gorenstein Artin algebra with* $\mathrm{inj.dim}\ {}_A A = n$. *Then we have that if* $n \ge 2$, *then* $\mathrm{gl.dim}\, \mathcal{G}p(A) = n$*; and if* $n = 0, 1$, *then* $\mathrm{gl.dim}\, \mathcal{G}p(A) \le 2$.

Theorem 1.3. *Let A be a CM-finite Gorenstein Artin algebra. Then there is a triangle-equivalence*
$$D^b_{gp}(A\text{-Mod}) \cong D^b(\mathcal{G}p(A)\text{-Mod}).$$

Proof. Since A is a CM-finite Gorenstein algebra, by the Main Theorem in [8] any Gorenstein-projective module is a direct sum of finitely generated Gorenstein-projective modules. Since G is finitely generated, functor $\mathrm{Hom}_A(G,-) : A\text{-Mod} \to \mathcal{G}p(A)\text{-Mod}$ preserves arbitrary direct sums, and hence the equivalence
$$\mathrm{Hom}_A(G,-) :\ A\text{-}\mathcal{G}proj \cong \mathcal{G}p(A)\text{-proj}$$
induces an equivalence $A\text{-}\mathcal{GP} \cong \mathcal{G}p(A)\text{-}\mathcal{P}$. The latter induces an equivalence $K^b(A\text{-}\mathcal{GP}) \cong K^b(\mathcal{G}p(A)\text{-}\mathcal{P})$, which is clearly a triangle functor. By Lemma 1.2, we obtain $\mathrm{gl.dim}\,\mathcal{G}p(A) < \infty$, and thus we get a triangle-equivalence
$$K^b(\mathcal{G}p(A)\text{-}\mathcal{P}) \cong D^b(\mathcal{G}p(A)\text{-Mod}).$$
Therefore, we have a triangle-equivalence by Lemma 1.1:
$$D^b_{gp}(A\text{-Mod}) \cong D^b(\mathcal{G}p(A)\text{-Mod}).$$
$\square$

A *recollement* of triangulated categories ([6]) is a diagram of triangulated categories and exact functors
$$\mathcal{T}' \begin{smallmatrix} \xleftarrow{\ i^*\ } \\ \xrightarrow{i_*=i_!} \\ \xleftarrow[\ i^!\ ]{} \end{smallmatrix} \mathcal{T} \begin{smallmatrix} \xleftarrow{\ j_!\ } \\ \xrightarrow{j^*=j^!} \\ \xleftarrow[\ j_*\ ]{} \end{smallmatrix} \mathcal{T}''$$
satisfying:

(1) (i^*, i_*), $(i_*, i^!)$, $(j_!, j^*)$ and (j^*, j_*) are adjoint pairs.
(2) $j^* i_* = 0$.
(3) $i_*, j_!$ and j_* are full embeddings.
(4) Each object X in $\mathcal{T}$ determines distinguished triangles
$$i_* i^! X \to X \to j_* j^* X \to i_* i^! X[1]$$
and
$$j_! j^* X \to X \to i_* i^* X \to j_! j^* X[1],$$
where the arrows to X and from X are counit and unit morphisms.

2. Criterion for the existence of recollements

Lemma 2.1 ([18, Theorem 7])**.** *Let A, B and C be rings. Assume that at least one of the rings A and C has finite global dimension. Then $D^b(A\text{-Mod})$ admits recollement relative to the bounded derived categories of B and C,*
$$D^b(B\text{-Mod}) \begin{smallmatrix} \xleftarrow{\ i^*\ } \\ \xrightarrow{i_*=i_!} \\ \xleftarrow[\ i^!\ ]{} \end{smallmatrix} D^b(A\text{-Mod}) \begin{smallmatrix} \xleftarrow{\ j_!\ } \\ \xrightarrow{j^*=j^!} \\ \xleftarrow[\ j_*\ ]{} \end{smallmatrix} D^b(C\text{-Mod}),$$
if and only if there exist two partial tilting complexes $\mathcal{B}$ in $K^b(A\text{-}\mathcal{P})$ and $\mathcal{C}$ in $K^b(A\text{-}proj)$ which satisfy:

(1) $\mathrm{End}(\mathcal{B}) \cong B$.
(2) $\mathrm{End}(\mathcal{C}) \cong C$.
(3) $\mathrm{Hom}_{D^b(A\text{-Mod})}(\mathcal{C}, \mathcal{B}[n]) = 0,\ \forall n$.
(4) $\mathcal{B}^\perp \cap \mathcal{C}^\perp = \{0\}$.

Definition 2.2. A partial Gorenstein tilting complex over a ring R is a complex $T^\bullet$ which is in $K^b(R\text{-}\mathcal{GP})$ and satisfies:

(1) $\mathrm{Hom}_{D^b_{gp}(R\text{-Mod})}(T^\bullet, T^\bullet[i]) = 0$ for $i \neq 0$ and
(2) for all indexed families $\{T_i^\bullet\}_{i\in I}$ of copies of T holds:

$$\bigoplus_{i\in I} \mathrm{Hom}_{D^b_{gp}(R\text{-Mod})}(T, T_i) \cong \mathrm{Hom}_{D^b_{gp}(R\text{-Mod})}(T, \bigoplus_{i\in I} T_i).$$

Theorem 2.3. *Let A be a CM-finite Gorenstein Artin algebra. Then $D^b_{gp}(A\text{-Mod})$ admits recollement relative to the bounded Gorenstein derived categories of CM-finite Gorenstein Artin algebras B and C,*

$$\begin{tikzcd} D^b_{gp}(B\text{-Mod}) \arrow[r, "i_*=i_!" description] & D^b_{gp}(A\text{-Mod}) \arrow[l, bend right, "i^*"'] \arrow[l, bend left, "i^!"] \arrow[r, "j^*=j^!" description] & D^b_{gp}(C\text{-Mod}), \arrow[l, bend right, "j_!"'] \arrow[l, bend left, "j_*"] \end{tikzcd}$$

if and only if there exist two partial Gorenstein tilting complexes $\mathcal{B}$ in $K^b(A\text{-}\mathcal{GP})$ and $\mathcal{C}$ in $K^b(A\text{-}\mathcal{G}proj)$ which satisfy:

(1) $\mathrm{End}(\mathcal{B}) \cong \mathcal{G}p(B)$*, where B is a CM-finite Gorenstein Artin algebra.*
(2) $\mathrm{End}(\mathcal{C}) \cong \mathcal{G}p(C)$*, where C is a CM-finite Gorenstein Artin algebra.*
(3) $\mathrm{Hom}_{D^b_{gp}(A\text{-Mod})}(\mathcal{C}, \mathcal{B}[n]) = 0,\ \forall n.$
(4) $\mathcal{B}^\perp \cap \mathcal{C}^\perp = \{0\}.$

Proof. By Theorem 1.3 there exists a triangle-equivalence

$$K^b(\mathcal{G}p(A)\text{-}\mathcal{P}) \cong K^b(A\text{-}\mathcal{GP}),$$

denoted by F. Note that F induces a triangle-equivalence; also denote by F, $K^b(\mathcal{G}p(A)\text{-}proj) \cong K^b(A\text{-}\mathcal{G}proj)$. There are triangle-equivalences again by Theorem 1.3:

$$D^b(\mathcal{G}p(A)\text{-Mod}) \cong D^b_{gp}(A\text{-Mod}) \quad D^b(\mathcal{G}p(B)\text{-Mod}) \cong D^b_{gp}(B\text{-Mod}),$$
$$D^b(\mathcal{G}p(C)\text{-Mod}) \cong D^b_{gp}(C\text{-Mod}).$$

Then $D^b_{gp}(A\text{-Mod})$ admits recollement relative to the bounded Gorenstein derived categories of CM-finite Gorenstein algebras B and C if and only if $D^b(\mathcal{G}p(A)\text{-Mod})$ admits recollement relative to the bounded derived categories of $\mathcal{G}p(B)$ and $\mathcal{G}p(C)$.

Since A, B and C are CM-finite Gorenstein Artin algebras, it follows that $\mathcal{G}p(A)$, $\mathcal{G}p(B)$ and $\mathcal{G}p(C)$ are Artin algebras with finite global dimension by Lemma 1.2. By Lemma 2.1 we get that $D^b(\mathcal{G}p(A)\text{-Mod})$ admits recollement relative to the bounded derived categories of $\mathcal{G}p(B)$ and $\mathcal{G}p(C)$ if and only if there exist two partial tilting complexes $\mathcal{B}'$ in $K^b(\mathcal{G}p(A)\text{-}\mathcal{P})$ and $\mathcal{C}'$ in $K^b(\mathcal{G}p(A)\text{-}proj)$ which satisfy $\mathrm{End}(\mathcal{B}') \cong \mathcal{G}p(B)$, $\mathrm{End}(\mathcal{C}') \cong \mathcal{G}p(C)$, $\mathrm{Hom}_{D^b(\mathcal{G}p(A)\text{-Mod})}(\mathcal{C}', \mathcal{B}'[n]) = 0$, and $\mathcal{B}'^\perp \cap \mathcal{C}'^\perp = \{0\}$.

We first verify the necessary condition. Set $\mathcal{B} := F(\mathcal{B}')$ and $\mathcal{C} := F(\mathcal{C}')$. Then we easily see that $\mathcal{B} \in K^b(A\text{-}\mathcal{GP})$ and $\mathcal{C} \in K^b(A\text{-}\mathcal{G}proj)$ are two partial Gorenstein tilting complexes. Also, we have that $\mathrm{End}(\mathcal{B}) \cong \mathrm{End}(\mathcal{B}') \cong \mathcal{G}p(B)$, $\mathrm{End}(\mathcal{C}) \cong \mathrm{End}(\mathcal{C}') \cong \mathcal{G}p(C)$, $\mathrm{Hom}_{D^b_{gp}(A\text{-Mod})}(\mathcal{C}, \mathcal{B}[n]) \cong \mathrm{Hom}_{D^b(\mathcal{G}p(A)\text{-Mod})}(\mathcal{C}', \mathcal{B}'[n]) = 0$, and $\mathcal{B}^\perp \cap \mathcal{C}^\perp = \mathcal{B}'^\perp \cap \mathcal{C}'^\perp = \{0\}$.

Now we verify the sufficient condition. Set $\mathcal{B}' := F^-(\mathcal{B})$ and $\mathcal{C}' := F^-(\mathcal{C})$, where F^- is the quasi-inverse of F. Then we easily see that $\mathcal{B}' \in K^b(\mathcal{G}p(A)\text{-}\mathcal{P})$ and $\mathcal{C} \in K^b(\mathcal{G}p(A)\text{-}proj)$ are two partial tilting complexes. Also, we have that

$\mathrm{End}(\mathcal{B}') \cong \mathrm{End}(\mathcal{B}) \cong \mathcal{G}p(B)$, $\mathrm{End}(\mathcal{C}') \cong \mathrm{End}(\mathcal{C}) \cong \mathcal{G}p(C)$, $\mathrm{Hom}_{D^b(\mathcal{G}p(A)\text{-Mod})}(\mathcal{C}', \mathcal{B}'[n]) \cong \mathrm{Hom}_{D^b_{gp}(A\text{-Mod})}(\mathcal{C}, \mathcal{B}[n]) = 0$, and ${\mathcal{B}'}^{\perp} \cap {\mathcal{C}'}^{\perp} = \mathcal{B}^{\perp} \cap \mathcal{C}^{\perp} = \{0\}$. Again by Lemma 2.1 we get that $D^b(\mathcal{G}p(A)\text{-Mod})$ admits recollement relative to the bounded derived categories of $\mathcal{G}p(B)$ and $\mathcal{G}p(C)$. Thus $D^b_{gp}(A\text{-Mod})$ admits recollement relative to the bounded Gorenstein derived categories of CM-finite Gorenstein Artin algebras B and C. □

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