

ON THE EXTENSION OF h^p -CR DISTRIBUTIONS DEFINED ON ROUGH TUBES

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ABSTRACT. We consider rough tubes $X + i\mathbb{R}^m \subset \mathbb{C}^m$ and generalized CR functions in $L^\infty(X, h^p(\mathbb{R}^m))$, where $h^p(\mathbb{R}^m)$, $0 < p < \infty$, is Goldberg's semilocal Hardy space. We show that if X is arcwise connected by rectifiable arcs, then all such CR functions can be extended to the convex hull of the tube as CR functions $\in L^\infty(\text{ch}(X), h^p(\mathbb{R}^m))$. This extends previous work of the authors.

INTRODUCTION

A classical theorem of Bochner ([Bo], [BM]) states that a holomorphic function defined on a tube $T(\Omega) = \Omega + i\mathbb{R}^m$ in \mathbb{C}^m for some domain $\Omega \subset \mathbb{R}^m$ extends as a holomorphic function to the convex hull $T(\text{ch}(\Omega)) = \text{ch}(\Omega) + i\mathbb{R}^m$. Much later, CR versions of this extension theorem were proved by Kazlow in [K] and Boivin and Dwilewicz in [BD] (see also [Ko]). In these papers, the tube above is replaced by a tube CR manifold $T(X) = X + i\mathbb{R}^m$, where X is a connected embedded submanifold of \mathbb{R}^m of class C^2 or higher. It was then proved that for any CR function h , there exists a function H holomorphic on the interior of the convex hull of $T(X)$ that extends h . The study of the extension of CR functions on tube manifolds with tempered growth in the tube direction was first considered in [B2], where Boggess treated the case of L^p -CR functions, $1 \leq p \leq \infty$. Quite recently, it was realized ([HHdS] and [BH]) that such extension results actually hold for tubes $T(X) = X + i\mathbb{R}^m$ over more general sets $X \subset \mathbb{R}^m$ which are not necessarily manifolds (rough tubes). The work in [HHdS] deals with h^p -CR functions, $0 < p \leq \infty$, over rough tubes. They are functions, $F(x + iy)$ defined on $T(X)$, whose restriction to $T(\{x\}) = x + i\mathbb{R}^m$, $x \in X$, belongs to the local Hardy space $h^p(\mathbb{R}^m)$ of Goldberg [G], which for $p > 1$ coincides with the Lebesgue space $L^p(\mathbb{R}^m)$, and $\sup_{x \in X} \|F(x + i\cdot)\|_{h^p} < \infty$. The (possible) irregularity of X needs to be dealt with when defining the CR character of such an $F(x + iy)$. When X (hence also $T(X)$) is a smooth manifold, CR functions and distributions are defined as the homogeneous solutions of certain systems of linear PDE's (the CR equations). Of course, this is no longer possible when X is a more irregular set. When X is a connected manifold, the Baouendi-Treves approximation theorem ([BT], [T], [B1],

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[BCH]) shows that any CR function on $T(X)$ can be approximated by restrictions to $T(X)$ of entire functions. This motivates the following definition, which extends the classical notion of a CR function:

Definition 0.1. Assume $0 < p < \infty$. Let f be in $L^\infty(X, h^p(\mathbb{R}^m))$.¹ We say that f is a (generalized) CR function if there exists a sequence $f_j(x+iy)$ of entire functions in \mathbb{C}^m such that

- (1) $\sup_{x \in X} \|f_j(x+i\cdot)\|_{h^p} \leq C \|f\|_{L^\infty(X, h^p(\mathbb{R}^m))}$;
- (2) $\lim_{j \rightarrow \infty} \|f_j(x+i\cdot) - f(x+i\cdot)\|_{h^p} = 0, x \in X$.

One of the main results in [HHdS] is

Theorem A. Assume that $X = \Phi(N) \subset \mathbb{R}^m$ is the image of a C^1 map $\Phi : N \rightarrow \mathbb{R}^m$ that takes a connected C^1 manifold N into \mathbb{R}^m and $0 < p < \infty$. Then any generalized CR function $g \in L^\infty(X, h^p(\mathbb{R}^m))$ has a generalized CR extension $g^b \in L^\infty(\text{ch}(X), h^p(\mathbb{R}^m))$. Furthermore, the map

$$\text{Int ch}(X) \ni x \mapsto g^b(x) \in h^p(\mathbb{R}^m)$$

is continuous and satisfies

$$\|g^b\|_{L^\infty(\text{ch}(X), h^p(\mathbb{R}^m))} = \|g\|_{L^\infty(X, h^p(\mathbb{R}^m))}.$$

If the affine subspace generated by $\text{ch}(X)$ has dimension m , g^b is holomorphic on the open tube $\text{Int ch}(X) + i\mathbb{R}^m$.

In the theorem above, the interior $\text{Int ch}(X)$ is taken with respect to the affine space generated by $\text{ch}(X)$. Note that X is allowed to be a polyhedron, a self-intersecting polygonal line or, more generally, a submanifold with self-intersections, corners and cusps.

Our purpose in this paper is to present an improved version of Theorem A by relaxing the hypotheses on the set X . We will prove below that for the conclusion to hold it is enough to assume that X is a rectifiably connected set (see Section 1 for the precise definitions and statement of results).

The organization of the paper is as follows. In section 1 we state the extension result (Theorem B). In section 2 we prove that the Baouendi-Treves approximation scheme holds for generalized h^p -functions defined over tubes $T(X)$ when X is rectifiably connected (this is one of the main tools in the proof of Theorem B) and prove the extension result. Section 3 is devoted to remarks and examples; in particular, we show that Theorem B is a strict extension of Theorem A.

1. THE EXTENSION THEOREM

For a set $X \subset \mathbb{R}^m$, $T(X) \subset \mathbb{C}^m$ will denote the tube over X ; namely, $T(X) = X + i\mathbb{R}^m$.

Definition 1.1. We say that a set $X \subset \mathbb{R}^m$ is rectifiably connected or arcwise connected by rectifiable arcs if for every pair of points $x_0, x_1 \in X$, there is a continuous map $\gamma : [0, 1] \rightarrow \mathbb{R}^m$ such that $\gamma(0) = x_0$, $\gamma(1) = x_1$ and $\gamma([0, 1]) \subset X$ has a finite length.

¹The space of bounded, h^p -valued functions defined on X .

Recall that a rectifiable curve is a continuous function $\gamma : [0, 1] \rightarrow \mathbb{R}^m$ of bounded variation, and its length is defined as the supremum over all finite partitions $t_0 < t_1 < \dots < t_N = 1$ of the sums

$$\sum_{j=0}^{N-1} |\gamma(t_{j+1}) - \gamma(t_j)|.$$

From now on, we will always assume that the set $X \subset \mathbb{R}^m$ is rectifiably connected. Our main result is the following.

Theorem B. *Assume $X \subset \mathbb{R}^m$ is rectifiably connected. Any generalized CR function $g \in L^\infty(X, h^p(\mathbb{R}^m))$, $0 < p < \infty$, has a generalized CR extension $g^b \in L^\infty(\text{ch}(X), h^p(\mathbb{R}^m))$. Furthermore, the map*

$$\text{Int ch}(X) \ni x \mapsto g^b(x) \in h^p(\mathbb{R}^m)$$

is continuous and satisfies

$$\|g^b\|_{L^\infty(\text{ch}(X), h^p(\mathbb{R}^m))} = \|g\|_{L^\infty(X, h^p(\mathbb{R}^m))}.$$

If the affine subspace generated by $\text{ch}(X)$ has dimension m , g^b is holomorphic on the open tube $\text{Int ch}(X) + i\mathbb{R}^m$.

The proof will be given in the next section.

2. THE APPROXIMATION THEOREM

One of the tools used in the proof of Theorem A in [HHdS] was the Baouendi-Treves approximation theorem ([BT], [BCH], [T]), which we were able to use thanks to the hypothesis that $X = \Phi(N)$, Φ a C^1 map from a connected C^1 manifold N onto X . Hence we start by extending the Baouendi-Treves approximation formula to our situation. Assume $0 \in X$, let $f \in L^\infty(X, h^p(\mathbb{R}^m))$ be a generalized CR function and set

$$E_\tau f(z) = (\tau/\pi)^{m/2} \int_{\mathbb{R}^m} e^{\tau[z-i\eta]^2} f(0+i\eta) d\eta, \quad z \in \mathbb{C}^m.$$

Here $[z]^2 = \zeta_1^2 + \dots + \zeta_m^2$ for $\zeta = (\zeta_1, \dots, \zeta_m) \in \mathbb{C}^m$, which explains the meaning of $[z-i\eta]^2$ in the formula. For $0 < p < 1$, the integral is understood as the tempered distribution $f(0+i\eta)$ acting on the rapidly decreasing function $\eta \mapsto e^{\tau[z-i\eta]^2}$. Clearly, $E_\tau f(z)$ is an entire function of z .

Theorem C. *Assume that X is rectifiably connected and suppose that $f \in L^\infty(X, h^p(\mathbb{R}^m))$ is a generalized CR function. Then, for any sequence of numbers $\tau_j \rightarrow \infty$, the sequence of entire functions $E_{\tau_j} f(z)$ satisfies*

$$(2.1) \quad \|E_{\tau_j} f(x, \cdot)\|_{h^p} \leq C \|f(x, \cdot)\|_{h^p}, \quad x \in X,$$

$$(2.2) \quad \lim_{j \rightarrow \infty} \|f(x, \cdot) - E_{\tau_j} f(x, \cdot)\|_{h^p} = 0, \quad x \in X,$$

$$(2.3) \quad |E_{\tau_j} f(x+iy)| \leq c_j(x), \quad x, y \in \mathbb{R}^m,$$

for some continuous function $c_j(x) \in C(\mathbb{R}^m)$.

Proof. Let $f_k(z)$ be the sequence of entire functions guaranteed by Definition 0.1. Assume that $0 \in X$, choose a function $h(y) \in C_c^\infty(\mathbb{R}^m)$ satisfying $h(y) = 0$ for $|y| \geq 2$, $h(y) = 1$ for $|y| \leq 1$ and set $h_r(y) \doteq h(y/r)$. For $\tau, r > 0$, the approximation operator applied to f_k is

$$(2.4) \quad E_{\tau,r}f_k(z) = (\tau/\pi)^{m/2} \int_{\mathbb{R}^m} e^{\tau[z-i\eta]^2} f_k(0+i\eta)h_r(\eta) d\eta, \quad z \in \mathbb{C}^m.$$

Clearly, $E_{\tau,r}f_k(z)$ is an entire function of z . The modified approximation operator is

$$(2.5) \quad G_{\tau,r}f_k(z) = (\tau/\pi)^{m/2} \int_{\mathbb{R}^m} e^{-\tau|y-\eta|^2} f_k(x+i\eta)h_r(\eta) d\eta, \quad (x, y) \in X \times \mathbb{R}^m,$$

and the remainder operator is

$$(2.6) \quad R_{\tau,r}f_k(z) = G_{\tau,r}f_k(z) - E_{\tau,r}f_k(z), \quad x+iy \in X \times \mathbb{R}^m.$$

For $x \in \mathbb{R}^m$, a convenient formula for the remainder operator expresses it as a path integral of a one-form (see [BCH, p. 64])

$$R_{\tau,r}f_k(z) = \int_{[0,x]} \sum_{\ell=1}^m r_\ell(x, y, \xi, \tau, r, k) d\xi_\ell,$$

where $[0, x]$ denotes the straight segment in \mathbb{R}^m that joins 0 to x and the coefficients of the one-forms are given by

$$r_\ell(x, y, \xi, \tau, r, k) = (\tau/\pi)^{m/2} \int_{\mathbb{R}^m} e^{\tau[x-\xi+i(y-\eta)]^2} f_k(\xi+i\eta)L_\ell h_r(\eta)(-i)^m d\zeta.$$

Here L_ℓ denotes the vector field $L_\ell = -i\partial/\partial\eta_\ell$, and $d\zeta = d\zeta(\xi, \eta)$ is the m -form

$$\begin{aligned} d\zeta(\xi, \eta) &= d\zeta_1 \wedge \cdots \wedge d\zeta_m \\ &= d(\xi_1 + i\eta_1) \wedge \cdots \wedge d(\xi_m + i\eta_m). \end{aligned}$$

Now fix $x \in X$ and consider a rectifiable curve $\gamma_x \subset X$ joining the origin in \mathbb{R}^m to x . We have the alternative description

$$R_{\tau,r}f_k(z) = \int_{\gamma_x} \sum_{\ell=1}^m r_\ell(x, y, \xi, \tau, r, k) d\xi_\ell,$$

because the one-form $\sum_{\ell=1}^m r_\ell(x, y, \xi, \tau, r, j) d\xi_\ell$ is exact. Taking into consideration that $e^{\tau[x-\xi+i(y-\eta)]^2} L_\ell h_r(\eta) \rightarrow 0$ in $\mathcal{S}(\mathbb{R}_\eta^m)$ as $r \rightarrow \infty$ and that the collection of functions $\{y \mapsto f_k(x+iy) : x \in X\}$ is a bounded subset of $\mathcal{S}'(\mathbb{R}^m)$, we obtain, letting $r \rightarrow \infty$,

$$E_\tau f_k(x, y) = G_\tau f_k(x, y), \quad x+iy \in X \times \mathbb{R}^m,$$

with

$$G_\tau f_k(x, y) = (\tau/\pi)^{m/2} \int_{\mathbb{R}^m} e^{-\tau|y-\eta|^2} f_k(x+i\eta) d\eta.$$

Next, if we let $k \rightarrow \infty$ and take account of (2) in Definition 0.1, we conclude that

$$(2.7) \quad E_\tau f(x, y) = (\tau/\pi)^{m/2} \int_{\mathbb{R}^m} e^{-\tau|y-\eta|^2} f(x+i\eta) d\eta.$$

Now fix some sequence $\tau_j \rightarrow \infty$. Since the expression (2.7) for $E_{\tau_j}f(x, y)$ is given by convolution of $\eta \mapsto f(x+i\eta)$ with the Gaussian, properties (2.1) and (2.2) are well known and easy to prove. Finally, the proof of (2.3) uses the original definition of $E_\tau f(z)$ and is based on the following observations:

- $\eta \mapsto e_{z,\tau_j}(\eta) \doteq e^{\tau_j[z-i\eta]^2}$ is in $\mathcal{S}(\mathbb{R}^m)$ for fixed $z \in \mathbb{C}^m$ and $\tau_j > 0$ and depends holomorphically on z ;
- for $R, \tau_j > 0$ fixed, the collection of functions

$$X(R, j) \doteq \{e_{z,\tau_j}(\eta) : |\Re z| \leq R\}$$

is a bounded subset of $(h^p(\mathbb{R}^m))^*$.

To check the latter claim, note that the norm in $h^p(\mathbb{R}^m)^*$ is invariant under translation, so $\|e_{x+iy,\tau_j}\|_{h^{p*}} = \|e_{x,\tau_j}\|_{h^{p*}}$, $x, y \in \mathbb{R}^n$, and the conclusion follows from the trivial estimate $|e_{x,\tau_j}(\eta)| \leq e^{\tau_j R^2 - \tau_j |\eta|^2}$ and similar bounds for the derivatives $D_\eta^\alpha e_{x,\tau_j}(\eta)$, $\alpha \in \mathbb{N}^m$. Hence,

$$\sup_{|x| \leq R} \sup_{y \in \mathbb{R}^m} |E_{\tau_j} f(x + iy)| < \infty$$

for any $R > 0$, and continuous functions $c_j(x)$ such that (2.3) holds are easily constructed. \square

We now need to recall a result that was stated and proved as Theorem 3.1 in [HHdS].

Theorem 2.1. *Let $X \subset \mathbb{R}^m$ be a set, $0 < p < \infty$, and suppose that $f \in L^\infty(X, h^p(\mathbb{R}^m))$ is a generalized CR function. Assume that there is a sequence $(h_j(x + iy))$ of entire functions in \mathbb{C}^m that satisfy:*

$$(2.8) \quad \|h_j(x, \cdot)\|_{h^p} \leq C \|f(x, \cdot)\|_{h^p}, \quad x \in X.$$

$$(2.9) \quad \lim_{j \rightarrow \infty} \|f(x, \cdot) - h_j(x, \cdot)\|_{h^p} = 0, \quad x \in X.$$

$$(2.10) \quad |h_j(x + iy)| \leq c_j(x), \quad x, y \in \mathbb{R}^m,$$

for some continuous function $c_j(x) \in C(\mathbb{R}^m)$. Then the sequence $y \mapsto h_j(x + iy)$ converges in $h^p(\mathbb{R}^m)$ for all $x \in \text{ch}(X)$ and the limit yields an $h^p(\mathbb{R}^m)$ -valued function $f^b(x)$, $x \in \text{ch}(X)$, that is nontangentially continuous on $\text{ch}(X)$. Furthermore,

$$\|f^b\|_{L^\infty(\text{ch}(X), h^p(\mathbb{R}^m))} = \|f\|_{L^\infty(X, h^p(\mathbb{R}^m))}.$$

Thus, an application of Theorem 2.1 using the approximating sequence $h_j = E_{\tau_j} f$ given by Theorem C proves Theorem B.

3. REMARKS AND EXAMPLES

Remark 3.1. In [HHdS, Theorem 3.1] the set X was assumed to be measurable. However, the proof also holds without changes if X is not measurable, provided the gauge $\|f\|_{L^\infty(X, h^p(\mathbb{R}^m))}$ is understood as $\sup_{x \in X} \|f(x + i \cdot)\|_{h^p}$ rather than an essential supremum. It is easy to give examples of sets X that are rectifiably arcwise connected and not measurable.

Remark 3.2. One of the difficulties in dealing with rough tubes is the absence of CR vector fields which in the smooth setup are used to define CR functions and are frequently useful tools. In the rough setup, the definition of CR functions depends on a rather weak approximation property by entire functions (Definition 0.1). Similarly, the proof of the Baouendi-Treves approximation formula (Theorem C), which in the smooth setup depends on Stokes' theorem, exploits that weak approximation property (together with some type of connectedness of X) to derive the stronger approximation property that allows the extension in a situation where integration by parts is not available.

Remark 3.3. Our proof of Theorem B gives more than stated, since Theorem 2.1 shows that the extension is nontangentially continuous on $\text{ch}(X)$. For the notion of nontangential continuity used here, which allows a less restrictive way of approaching the boundary than the standard one, see [HHdS].

Example 3.1. Let $\Phi : \mathbb{R} \rightarrow \mathbb{R}^3$ be an immersion such that $X = \Phi(\mathbb{R})$ is dense in a 2-torus $\mathbb{T}_2 \subset \mathbb{R}^3$. Then $Z(x, t) = x + i\Phi(t)$ defines a globally integrable tube structure (in the sense of the theory of Locally Integrable Structures) on $\mathbb{R} \times \mathbb{R}^3$ generated by a single vector field. This structure is isomorphic to the CR structure on $T(X) = \mathbb{R}^3 + iX \subset \mathbb{C}^3$ with base $X = \Phi(\mathbb{R})$, which is a self-winding orbit of a real vector field in \mathbb{R}^3 with a single minimal invariant set equal to \mathbb{T}^2 . Every generalized h^p -CR function f on $T(X)$ has a holomorphic extension to $\text{Int ch}(X) + i\mathbb{R}^3 = \text{Int ch}(\mathbb{T}^2) + i\mathbb{R}^3$.

The existence of a holomorphic extension in the example above follows from Theorem A. We now give an example of a set X that can be handled by Theorem B but not by Theorem A. In other words, we exhibit a set X which is rectifiably connected but is not the C^1 image of a paracompact manifold.

Example 3.2. Let S denote the set $(\mathbb{R} \setminus \mathbb{Q}) \cap (0, 1)$ of irrational numbers in $(0, 1)$ and consider the comb space,

$$CS = ([0, 1] \times \{0\}) \cup (S \times [0, 1]).$$

The comb space CS is clearly rectifiably connected; in fact, given any two points $p, q \in CS$ there is a polygonal path joining p and q with length ≤ 3 .

Assume by contradiction that there is a C^1 manifold Σ such that $CS = \Phi(\Sigma)$, $\Phi \in C^1$. Endow Σ with a Riemannian metric and consider a sequence of relatively compact open sets $G_j \subset \Sigma$ such that

$$\Sigma = \bigcup_{j=1}^{\infty} G_j, \quad G_j \subset \overline{G_j} \subset G_{j+1}, \quad j = 1, 2, \dots$$

Let Y be the set $\{(x, 1) : x \in S\}$. Note that the cardinality of each one of the sets $\Phi^{-1}(Y) \cap G_j$ must satisfy

$$(3.1) \quad \aleph(\Phi^{-1}(Y) \cap G_j) < \infty, \quad j = 1, 2, \dots;$$

otherwise there would be a sequence (σ_k) in $\Phi^{-1}(Y) \cap G_j$ such that $\sigma_j \neq \sigma_k$ for $j \neq k$ and $\sigma_k \rightarrow \sigma \in \overline{G_j}$. This would imply that the length of the geodesic distance in Σ between σ and σ_k converges to zero as $k \rightarrow \infty$ and for k large we could find arbitrarily short curves γ_k joining σ_k to σ , so the curves $\Phi \circ \gamma_k$ would join $\Phi(\sigma_k) = (x_k, 1) \in Y$ to $\Phi(\sigma) = (x, 1) \in Y$ and their length would tend to zero as $k \rightarrow \infty$. Since the length of a curve joining $(x, 1)$ to $(x_k, 1)$ in CS is always greater than or equal to 2 if $x_k \neq x$, we get a contradiction. Now, (3.1) implies that $(\mathbb{R} \setminus \mathbb{Q}) \cap (0, 1)$ is countable. This contradiction proves that CS cannot be the C^1 image of a paracompact manifold.

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