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EQUIVARIANT K-THEORY AND THE CHERN CHARACTER FOR DISCRETE GROUPS

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ABSTRACT. Let X be a compact Hausdorff space, let Γ be a discrete group that acts continuously on X from the right, define $\widetilde{X} = \{(x, \gamma) \in X \times \Gamma : x \cdot \gamma = x\}$, and let Γ act on $\widetilde{X}$ via the formula $(x, \gamma) \cdot \alpha = (x \cdot \alpha, \alpha^{-1}\gamma\alpha)$. Results of P. Baum and A. Connes, along with facts about the Chern character, imply that $K^i_\Gamma(X)$ and $K^i(\widetilde{X}/\Gamma)$ are isomorphic up to torsion for $i = 0, 1$. In this paper, we present an example where the groups $K^i_\Gamma(X)$ and $K^i(\widetilde{X}/\Gamma)$ are not isomorphic.

Let Γ be a finite discrete group acting continuously on a compact Hausdorff space X from the right. Define $\widetilde{X} = \{(x, \gamma) \in X \times \Gamma : x \cdot \gamma = x\}$ and endow $\widetilde{X}$ with the subspace topology that it inherits from $X \times \Gamma$. The group Γ acts on $\widetilde{X}$ via the formula $(x, \gamma) \cdot \alpha = (x \cdot \alpha, \alpha^1\gamma\alpha)$, and we can consider the orbit space $\widetilde{X}/\Gamma$. Theorem 1.19 in [1] states that there exist isomorphisms

$$K^0_\Gamma(X) \otimes \mathbb{C} \cong \sum_{j=0}^{\infty} H^{2j}(\widetilde{X}/\Gamma; \mathbb{C}),$$

$$K^1_\Gamma(X) \otimes \mathbb{C} \cong \sum_{j=0}^{\infty} H^{2j+1}(\widetilde{X}/\Gamma; \mathbb{C}),$$

where $H^*(\widetilde{X}/\Gamma; \mathbb{C})$ denotes the Čech cohomology of $\widetilde{X}/\Gamma$ with complex coefficients. We also have the Chern character isomorphisms

$$K^0(\widetilde{X}/\Gamma) \otimes \mathbb{C} \cong \sum_{j=0}^{\infty} H^{2j}(\widetilde{X}/\Gamma; \mathbb{C}),$$

$$K^1(\widetilde{X}/\Gamma) \otimes \mathbb{C} \cong \sum_{j=0}^{\infty} H^{2j+1}(\widetilde{X}/\Gamma; \mathbb{C}).$$

Therefore $K^i_\Gamma(X) \otimes \mathbb{C} \cong K^i(\widetilde{X}/\Gamma) \otimes \mathbb{C}$ for $i = 0, 1$. In fact, a careful reading of [1] shows that we can replace $\mathbb{C}$ by $\mathbb{Q}(\omega)$, where n is the order of Γ and where $\omega = \exp(2\pi i/n)$. In any event, the groups $K^i_\Gamma(X)$ and $K^i(\widetilde{X}/\Gamma)$ are isomorphic up

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to torsion. In this paper, we present an example where these groups do not have isomorphic torsion subgroups.

Our example is Example B in [4]. Consider the unit 3-sphere S^3 in $\mathbb{R}^4$ and define $\alpha : S^3 \longrightarrow S^3$ by $\alpha(x, y, z, t) = (-x, -y, -z, t)$. The map α defines a $\mathbb{Z}_2$ action on S^3. From [2], we know that the Γ-equivariant K-theory groups of a compact Hausdorff space X are isomorphic to the operator algebra K-theory groups of $C(X) \rtimes \Gamma$. Combining this fact with the computations in [4] we have $K^0_{\mathbb{Z}_2}(S^3) \cong \mathbb{Z}^3$ and $K^1_{\mathbb{Z}_2}(S^3) \cong 0$.

Observe that $\widetilde{S^3}$ is homeomorphic to the disjoint union of S^3 and the fixed point set F of the action of α, and hence $\widetilde{S^3}/\mathbb{Z}_2$ is homeomorphic to the disjoint union of $S^3/\mathbb{Z}_2$ and F. In our example, F is the two-point set $\{(0,0,0,1),(0,0,0,-1)\}$, so $K^0(F) \cong \mathbb{Z}^2$ and $K^1(F) \cong 0$.

To compute the K-theory of $S^3/\mathbb{Z}_2$, define closed sets

$$
\begin{aligned}
A &= \{(x, y, z, t) \in S^3 : t \geq 0\},\\
B &= \{(x, y, z, t) \in S^3 : t \leq 0\}.
\end{aligned}
$$

Then $(A/\mathbb{Z}_2) \cup (B/\mathbb{Z}_2) = S^3/\mathbb{Z}_2$ and $(A/\mathbb{Z}_2) \cap (B/\mathbb{Z}_2) \cong S^2/\mathbb{Z}_2 \cong \mathbb{R}P^2$. Applying the Mayer-Vietoris sequence for reduced K-theory ([3], Exercise 3.2), we have the six-term exact sequence

$$
\begin{tikzcd}
\widetilde{K}^0(S^3/\mathbb{Z}_2) \arrow[r] & \widetilde{K}^0(A/\mathbb{Z}_2) \oplus \widetilde{K}^0(B/\mathbb{Z}_2) \arrow[r] & \widetilde{K}^0(\mathbb{R}P^2) \arrow[d] \\
\widetilde{K}^1(\mathbb{R}P^2) \arrow[u] & \widetilde{K}^1(A/\mathbb{Z}_2) \oplus \widetilde{K}^1(B/\mathbb{Z}_2) \arrow[l] & \widetilde{K}^1(S^3/\mathbb{Z}_2). \arrow[l]
\end{tikzcd}
$$

Both $A/\mathbb{Z}_2$ and $B/\mathbb{Z}_2$ are homeomorphic to the cone over $\mathbb{R}P^2$, which has trivial reduced K-theory groups, and so the vertical maps in the six-term exact sequence are isomorphisms. Therefore

$$
\begin{aligned}
K^0(S^3/\mathbb{Z}_2) &\cong \widetilde{K}^0(S^3/\mathbb{Z}_2) \oplus \mathbb{Z} \cong \widetilde{K}^1(\mathbb{R}P^2) \oplus \mathbb{Z} \cong \mathbb{Z},\\
K^1(S^3/\mathbb{Z}_2) &\cong \widetilde{K}^1(S^3/\mathbb{Z}_2) \cong \widetilde{K}^0(\mathbb{R}P^2) \cong \mathbb{Z}_2,
\end{aligned}
$$

which yield

$$
\begin{aligned}
K^0(\widetilde{S^3}/\mathbb{Z}_2) &\cong K^0(S^3/\mathbb{Z}_2) \oplus K^0(F) \cong \mathbb{Z} \oplus \mathbb{Z} \oplus \mathbb{Z},\\
K^1(\widetilde{S^3}/\mathbb{Z}_2) &\cong K^1(S^3/\mathbb{Z}_2) \oplus K^1(F) \cong \mathbb{Z}_2.
\end{aligned}
$$

Thus $K^1_{\mathbb{Z}_2}(S^3)$ and $K^1(\widetilde{S^3}/\mathbb{Z}_2)$ are isomorphic up to torsion, but the groups $K^1_{\mathbb{Z}_2}(S^3)$ and $K^1(\widetilde{S^3}/\mathbb{Z}_2)$ are not isomorphic.

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