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WEIGHTED PALEY–WIENER SPACES SHARING A MAJORANT-WEIGHT

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ABSTRACT. We point out a need to slightly modify the statement of Theorem 2 in Yurii Lyubarskii and Kristian Seip's work *Weighted Paley–Wiener spaces*. This last theorem lists all weighted Paley–Wiener spaces (in reduced form) sharing a prescribed majorant-weight. Attention is called to a part of its proof that requires an additional argument. Such an argument, based on a new characterization of Beurling's lower uniform density, is then presented.

1. INTRODUCTION

In the explorative paper [1], Lyubarskii and Seip introduced a family of de Branges spaces subject to a natural axiom: in these spaces, the norm of a function is comparable to its L^2-norm against $M(x)^{-2}\,\mathrm{d}x$, where $M(x)$ denotes the norm of the reproducing kernel at $x \in \mathbb{R}$ (see Section 2). In such circumstances, $M(x)$ is said to be a *majorant-weight*, while the de Branges space is said to be a *weighted Paley–Wiener space*.

From a deep study of the Hermite–Biehler function associated with weighted Paley–Wiener spaces (Theorem 1 in [1]), they showed that the majorant-weight of a weighted Paley–Wiener space is always comparable to a function of the form $\mathrm{e}^{g(x)}\mathrm{e}^{\omega_m(x)}$, where $g(z)$ is real-entire (that is, real on the real line and entire), $m(x)$ is comparable to a constant, and $\omega_m(z)$ is the potential of $m(x)\,\mathrm{d}x$ (see Section 2).

Furthermore, they proved that any weighted Paley–Wiener space is of the form $\mathrm{e}^g PW(m)$ for such a g and an m, where

$$PW(m) = \{f \text{ entire}\ ;\ \|f\mathrm{e}^{-\omega_m}\|_2 < \infty,\ |f(z)|\mathrm{e}^{-\omega_m(z)} \leq C_\varepsilon \mathrm{e}^{\varepsilon|z|}\}.$$

They then aimed to list all weighted Paley–Wiener spaces of the form $PW(\cdot)$ whose majorant-weight is comparable to a prescribed $\mathrm{e}^{\omega_m(x)}$. They obtained such a list (Theorem 2 in [1]), which consists exactly of the following spaces,

$$PW_{-b}(m) = \{f \text{ entire}\ ;\ \|f\mathrm{e}^{-\omega_m}\|_2 < \infty,\ |f(z)|\mathrm{e}^{-\omega_m(z)+\pi b|\Im z|} \leq C_\varepsilon \mathrm{e}^{\varepsilon|z|}\},$$

where b is any real number inferior to the lower uniform density of m. This last is defined as

$$D_m = \lim_{R\to\infty} \inf_{x\in\mathbb{R}} \frac{1}{2R}\int_{-R}^{R} m(x+t)\,\mathrm{d}t.$$

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However, as we shall discuss later, a close examination of their proof reveals its incompleteness, and new ideas seem necessary for completing their work.

The present paper aims to remedy the situation. Section 2 provides the reader with the main definitions used in the sequel. In Section 3.1, we shall question the original argument that D_m is a majorant of b and then provide our own proof, based on a new characterization of the lower uniform density. Finally, in Section 3.2, we shall show that D_m is the least majorant of b.

2. Definitions

In the sequel, given two nonnegative functions f and g, $f \lesssim g$ indicates that $f \leq Cg$ for a positive constant C, and $f \simeq g$ indicates that f is comparable to g (that is, $f \lesssim g$ and $g \lesssim f$).

A Hilbert space $\mathcal{H}$ of entire functions is a *de Branges space* [2] if it satisfies the following axioms:

(1) *The linear functional* $\mathcal{H} \to \mathbb{C},\ f \mapsto f(z_0)$ *is bounded for all* $z_0 \in \mathbb{C}$.
(2) *If* $f(z) \in \mathcal{H}$, *then* $f^*(z) = \overline{f(\overline{z})}$ *also belongs to* $\mathcal{H}$ *and has the same norm as* $f(z)$.
(3) *If* $f(z) \in \mathcal{H}$ *and* $f(z_0) = 0$, *then* $f(z)\dfrac{z-\overline{z_0}}{z-z_0}$ *also belongs to* $\mathcal{H}$ *and has the same norm as* $f(z)$.

By the first axiom, $\mathcal{H}$ admits a *reproducing kernel*, that is, a function $k_w(z)$ of the variables $w, z \in \mathbb{C}$ such that $k_w \in \mathcal{H}$ for all $w \in \mathbb{C}$ and

$$\langle f, k_w \rangle_{\mathcal{H}} = f(w) \quad \text{for all } f \in \mathcal{H}.$$

The *majorant* of $\mathcal{H}$ at $z \in \mathbb{C}$ is then defined as

$$M(z) = \|k_z\|_{\mathcal{H}} = \sup_{\|f\|_{\mathcal{H}}=1} |f(z)|.$$

Let $M(x)$ be the restriction of M to the real axis. Following Lyubarskii and Seip, we shall say that $M(x)$ is a *majorant-weight* if

(1) $M(x) > 0$ for all $x \in \mathbb{R}$;
(2) $\|f\|_{\mathcal{H}} \simeq \|f/M\|_2$ for all $f \in \mathcal{H}$.

Then, the corresponding $\mathcal{H}$ is called a *weighted Paley–Wiener space*.

A *Hermite–Biehler function* E is an entire function satisfying $|E(z)| > |E(\overline{z})|$ for all $z \in \mathbb{C}^+$. Such a function may be factorized as

$$E(z) = Cz^m \mathrm{e}^{h(z)} \mathrm{e}^{-\mathrm{i}\alpha z} \prod_{\gamma \in \Gamma} (1 - z/\gamma) \mathrm{e}^{z \Re(1/\gamma)}, \tag{2.1}$$

where $C \in \mathbb{C}$, $h(z)$ is real-entire, $\alpha \geq 0$, and Γ is a family of nonzero elements lying in the closed lower half-plane (with possible repetitions). Conversely, given such a C, $h(z)$, α, and Γ, if the right-hand side in (2.1) defines an entire function, then it is in the Hermite–Biehler class (provided that $\Gamma \not\subseteq \mathbb{R}$ or $\alpha \neq 0$).

In the case where E does not have real zeroes, its restriction to the real axis may be written

$$E(x) = |E(x)| \mathrm{e}^{-\mathrm{i}\varphi(x)},$$

where the *phase*, $\varphi(x)$, is real-analytic and well-defined (up to the addition of $2k\pi$). The factorization (2.1) then implies

$$\varphi'(x) = \alpha + \sum_{\xi - i\eta \in \Gamma} \frac{\eta}{(x-\xi)^2 + \eta^2}. \tag{2.2}$$

From an arbitrary Hermite–Biehler function E, one may build a prototypical example of a de Branges space, namely

$$\mathcal{H}(E) = \left\{ f \text{ entire} \, ; \, \|f/E\|_2 < \infty, \; |f^\sharp(z)/E(z)| \le C_\varepsilon e^{\varepsilon |z|} \text{ for } \Im z \ge 0 \right\},$$

for $f^\sharp$ running over $\{f, f^*\}$. It is equipped with the norm $\|f\|_{\mathcal{H}(E)} = \|f/E\|_2$. In fact, a theorem of de Branges (Theorem 23 in [2]) shows that *every de Branges space is isometrically equal to a space of the form* $\mathcal{H}(E)$, where E is not unique in general.

The reproducing kernel in $\mathcal{H}(E)$ is given by

$$k_w(z) = \frac{E^*(z)\overline{E^*(w)} - E(z)\overline{E(w)}}{2\pi i(z - \overline{w})}.$$

In particular, if E does not have real zeroes,

$$M(x) = \sqrt{k_x(x)} = \frac{1}{\sqrt{\pi}} \sqrt{\varphi'(x)} |E(x)|$$

for all $x \in \mathbb{R}$.

Example 2.1. Let φ be the phase of a Hermite–Biehler function E without real zero. If $\varphi'(x) \simeq 1$, then $\mathcal{H}(E)$ is obviously a weighted PW-space. The converse statement however does not hold in general (Remark 3 on the last page of [1]).

Lyubarskii and Seip made a bridge between Hermite–Biehler functions and a certain kind of potentials, namely, potentials of measures of the form $m(x)\,dx$ for $m(x)$ measurable, positive, and $\simeq 1$. Such a potential cannot be defined as $\int_{-\infty}^{\infty} \log|1 - z/t| m(t)\,dt$. In fact, this last integral does not exist due to the dominating term in the expansion

$$\log|1 - z/t| = -x/t - \sum_{n=2}^{\infty} (1/n) \Re(z^n)/t^n$$

for $|t|$ large, where $z = x + iy$. It suggests defining

$$\omega_m(z) = \int_{-\infty}^{\infty} \log^* |1 - z/t| m(t)\,dt$$

where $\log^*|1 - z/t| = \log|1 - z/t| + \chi(t)x/t, \;\; \chi(t) = 1 - \chi_{[-1,1]}(t)$.

We first show that $\omega_m(z)$ is well-defined, indeed, that the above integral is absolutely convergent. The previous expansion gives, for $|t|$ large,

$$\begin{aligned} |\log^*|1 - z/t|\,| &\le \sum_{n=2}^{\infty} (1/n)|z/t|^n \\ &\le |z/t|^2 (1/2 - \log(1 - |z/t|)\,). \end{aligned} \tag{2.3}$$

Since $m(x) \simeq 1$, it suffices to show that $\int_R^\infty -(1/t^2)\log(1 - |z|/t)\,dt < \infty$ for R large. This last relation follows from the substitution $u = 1 - |z|/t$. Therefore, $\int_{-\infty}^{\infty} |\log^*|1 - z/t|\,|\,dt < \infty$.

The inequality (2.3) and the dominated convergence theorem also yield that ω_m is continuous. The dominated convergence theorem then implies that for $z \notin \mathbb{R}$

$$\partial_y \omega_m(z) = \pi P_m(z), \tag{2.4}$$

where

$$P_m(z) = \frac{1}{\pi} \int_{-\infty}^{\infty} \frac{y}{(x-t)^2 + y^2} m(t)\,\mathrm{d}t$$

is the Poisson transform of $m(x)\,\mathrm{d}x$. Similarly,

$$\partial_x \omega_m(z) = \int_{-\infty}^{\infty} \left(\frac{x-t}{(x-t)^2 + y^2} + \frac{\chi(t)}{t} \right) m(t)\,\mathrm{d}t$$

for $z \notin \mathbb{R}$. Finally, a straight adaptation of the classical argument (Theorem 3.7.4 in [3]) gives $\Delta\omega_m = 2\pi m(x)\,\mathrm{d}x\,\mathrm{d}\delta_0(y)$ in the sense of distribution, where δ_0 denotes the 1-dimensional Dirac measure at 0.

Example 2.2. For $m(x) = 1$ and $z \notin \mathbb{R}$, $\partial_y \omega_1(z) = \pi\,\mathrm{sgn}(y)$, while $\partial_x \omega_1(z) = 0$. Hence, $\omega_1(z) = \pi|y| + C$. By continuity, this last relation applies for all $z \in \mathbb{C}$. Since $\omega_1(0) = 0$, we deduce $\omega_1(z) = \pi|y|$.

The aforementioned link between Hermite–Biehler functions and potentials of the form ω_m is given by the following *multiplier lemma*:

Proposition 2.3. *Let $m(x) \simeq 1$ be a measurable function. There exists a Hermite–Biehler function E_m which satisfies*

$$|E_m(z)| \simeq \mathrm{e}^{\omega_m(z)} \quad \textit{when } \Im z \geq 0$$

and whose zeroes are simple and of the form $\xi_k - \mathrm{i}$, where $\xi_{k+1} - \xi_k \simeq 1$, $\xi_k \in \mathbb{R}$.

Let φ be the phase of E_m. By (2.2),

$$\varphi'(x) = \sum_k \frac{1}{(x-\xi_k)^2 + 1}.$$

The condition $\xi_{k+1} - \xi_k \simeq 1$ then implies $\varphi'(x) \simeq 1$. Therefore $\mathcal{H}(E_m)$ is a weighted PW-space. By the multiplier lemma, $|E_m|$ may be replaced with e^{ω_m} in the definition of $\mathcal{H}(E_m)$. Since $\omega_m(\overline{z}) = \omega_m(z)$, it follows that $\mathcal{H}(E_m)$ is equal with equivalent norms to the space

$$PW(m) = \{f \text{ entire ; } \|f\mathrm{e}^{-\omega_m}\|_2 < \infty,\ |f(z)|\mathrm{e}^{-\omega_m(z)} < C_\varepsilon \mathrm{e}^{\varepsilon|z|} \text{ for } z \in \mathbb{C}\},$$

equipped with the norm $\|f\|_{PW(m)} = \|f\mathrm{e}^{-\omega_m}\|_2$.

In particular, for any measurable $m(x) \simeq 1$, $PW(m)$ is a weighted PW-space whose majorant-weight is comparable to $|E_m| \simeq \mathrm{e}^{\omega_m(x)}$. Consequently, given a real-entire g, $\mathrm{e}^g PW(m)$ is also a weighted PW-space; its majorant-weight is comparable to $\mathrm{e}^g \mathrm{e}^{\omega_m}$. The converse statement constitutes the remarkable achievement in [1]: Lyubarskii and Seip proved that all weighted PW-spaces have a representation $\mathrm{e}^g PW(m)$, where g is real-entire and $m \simeq 1$ is measurable.

Example 2.4. We have seen that $\omega_1(z) = \pi|y|$. Consequently, $PW(1)$ is the classical Paley–Wiener space, L^2_π.

3. Spaces sharing a given majorant-weight

In [1], Lyubarskii and Seip investigated the following question: which weighted PW-spaces share a prescribed majorant-weight? They gave special attention to spaces of the form $\mathrm{e}^{az}PW(m)$, $a \in \mathbb{R}$, which we will call *linearly reduced* PW-spaces. They obtained an answer for these last spaces, involving the following object: for m measurable and $\simeq 1$ and for $\tau \in \mathbb{R}$,

$$PW_\tau(m) = \{f \text{ entire };\ \|f\mathrm{e}^{-\omega_m}\|_2 < \infty,\ |f(z)|\mathrm{e}^{-\omega_m(z)} \le C_\varepsilon \mathrm{e}^{\varepsilon|z|}\mathrm{e}^{\pi\tau|\Im z|}\}.$$

Notice that $PW_\tau(m) = PW(m+\tau)$ if $m+\tau \simeq 1$, but this last relation is not assumed.

Proposition 3.1. *Suppose* $\mathrm{e}^{a_0x}\mathrm{e}^{\omega_{m_0}(x)} \simeq \mathrm{e}^{ax}\mathrm{e}^{\omega_m(x)}$ *on the real axis, where* a, a_0 *are in* $\mathbb{R}$ *and* m, m_0 *are measurable and* $\simeq 1$*. Then, there exists a real number* b *such that*

$$\mathrm{e}^{a_0x}\mathrm{e}^{\omega_{m_0}(z)} \simeq \mathrm{e}^{ax}\mathrm{e}^{\omega_m(z)}\mathrm{e}^{-\pi b|y|}$$

on the whole complex plane, where $z = x + \mathrm{i}y$*.*

Proof. By hypothesis $|\omega_{m_0-m}(x) + (a_0-a)x|$ is bounded, and hence there exists a $C > 0$ such that

$$\begin{aligned}|\omega_{m_0-m}(z) + (a_0-a)x| &\le |\omega_{m_0-m}(z) - \omega_{m_0-m}(x)| + |\omega_{m_0-m}(x) + (a_0-a)x| \\ &\le C(|y|+1).\end{aligned}$$

In particular, $Cy + \omega_{m_0-m}(z) + (a_0-a)x$ is bounded below on the upper half-plane and hence admits a Poisson representation

$$Dy + \frac{y}{\pi}\int_{-\infty}^{\infty} \frac{\omega_{m_0-m}(t) + (a_0-a)t}{(x-t)^2+y^2}\,\mathrm{d}t \qquad (y>0).$$

Observe that the last term in the previous expression is bounded. Consequently, letting $b = (C-D)/\pi$,

$$|\pi by + \omega_{m_0-m}(z) + (a_0-a)x| \lesssim 1 \qquad (\Im z \ge 0),$$

that is, $\mathrm{e}^{a_0x}\mathrm{e}^{\omega_{m_0}(z)} \simeq \mathrm{e}^{ax}\mathrm{e}^{\omega_m(z)}\mathrm{e}^{-b\pi\Im z}$ when $\Im z \ge 0$. The result follows. □

Corollary 3.2. *Each linearly reduced weighted PW-space whose majorant-weight is comparable to* $\mathrm{e}^{ax}\mathrm{e}^{\omega_m(x)}$ *is of the form* $\mathrm{e}^{az}PW_{-b}(m)$ *for a certain* $b \in \mathbb{R}$*.*

Remark 3.3. In their original paper Lyubarskii and Seip stated their result not in terms of linearly reduced PW-spaces, but in terms of linearly reduced majorant-weights (that is, weights of the form $\mathrm{e}^{ax}\mathrm{e}^{\omega_m(x)}$). Their statement is too general: clearly, $\exp(\exp(-z^2))PW(1)$ is a weighted PW-space whose majorant-weight is $\simeq 1$, but it is not equal to $PW_{-b}(1) = L^2_{\pi(1-b)}$ for any b.

Linearly reduced PW-spaces of majorant $\mathrm{e}^{ax}\mathrm{e}^{\omega_m}$ ($a \in \mathbb{R}$) thus take the form $\mathrm{e}^{az}PW_{-b}(m)$, but for which possible b? Clearly each $b < \inf_{x\in\mathbb{R}} m(x)$ is possible, since then $PW_{-b}(m) = PW(m-b)$ with $m - b \simeq 1$. However, this majoration cannot be optimal in general, since it is easy to change the infimum of m without altering e^{ω_m} (using for instance Proposition 2.3). Lyubarskii and Seip stated that the optimal majoration is $b < D_m$, where

$$D_m = \lim_{R\to\infty} \inf_{x\in\mathbb{R}} \frac{1}{2R}\int_{-R}^{R} m(x+t)\,\mathrm{d}t$$

is the uniform lower density of m. Notice that this last limit exists by Fekete's lemma, since $\inf_x \int_{-R}^{R} m(x+t)\,\mathrm{d}t = \inf_x \int_0^{2R} m(x+t)\,\mathrm{d}t$ is superadditive.

3.1. **The lower density is a majorant.** For showing that $b < D_m$ is necessary, Lyubarskii and Seip argued by contradiction: they assumed that given an $\varepsilon > 0$, for all sufficiently large R there exists an x_R satisfying

$$\int_{-R}^{R} (m(x_R+s)-b)\,\mathrm{d}s \le \varepsilon R. \tag{3.1}$$

For convenience they set $x_R = 0$. Let $\mathrm{e}^{a_0 z} PW(m_0)$ be a representation of $\mathrm{e}^{az} PW_{-b}(m)$ as a weighted Paley–Wiener space. By an elegant argument based on Green's formula they proved

$$\left| \int_{-R}^{R} (m_0(t) - m(t) + b)(R^2 - t)\,\mathrm{d}t \right| \lesssim R^2.$$

They claimed however that (3.1) would imply

$$2\int_0^R t \int_{-t}^{t} (m_0(s) - m(s) + b)\,\mathrm{d}s\,\mathrm{d}t \ge (\inf m_0)\frac{4}{3}R^3 - \varepsilon R^3,$$

a contradiction. Unfortunately the use of the estimate

$$2\int_0^R t \int_{-t}^{t} (m(s) - b)\,\mathrm{d}s\,\mathrm{d}t \le \varepsilon R^3$$

is not explicitly justified. A conscientious reader may get puzzled: one cannot for instance restrict the domain of integration to large t and invoke (3.1) with t instead of R, since $x_t \ne x_R$.

We prefer to present another proof, based on a new characterization of the uniform lower density:

Proposition 3.4. *If $m \simeq 1$ is measurable, then*

$$D_m = \lim_{R\to\infty} \inf_{x\in\mathbb{R}} \frac{1}{R}\int_0^R \frac{1}{2\rho}\int_{-\rho}^{\rho} m(x+t)\,\mathrm{d}t\,\mathrm{d}\rho.$$

Proof. For $R > 0$ and $x \in \mathbb{R}$, let us write

$$m_R(x) = \frac{1}{2R}\int_{-R}^{R} m(x+t)\,\mathrm{d}t \quad \text{and} \quad A_R(x) = \frac{1}{R}\int_0^R m_\rho(x)\,\mathrm{d}\rho.$$

In this notation we want to prove $D_m = \lim_{R\to\infty} \inf_x A_R(x)$, given that $D_m = \lim_{R\to\infty} \inf_x m_R(x)$.

Given $\varepsilon > 0$, let R_0 be such that $|\inf_x m_\rho(x) - D_m| < \varepsilon$ when $\rho \ge R_0$. Then, for R large

$$\inf_x A_R(x) \ge \frac{1}{R}\Big(\int_0^{R_0} + \int_{R_0}^{R}\Big) \inf_x m_\rho(x)\,\mathrm{d}\rho \ge \frac{R_0 \inf m}{R} + \frac{R - R_0}{R}(D_m - \varepsilon).$$

Letting $R \to \infty$ along an appropriate sequence, we conclude

$$\liminf_{R\to\infty} \inf_x A_R(x) \ge D_m.$$

Let us derive the converse inequality. Let $0 < \varepsilon < 1/4$ be given, and define $\eta = \varepsilon/\log(1/2\varepsilon)$. There exists an R_0 depending on ε such that

$$|\inf_x m_\rho(x) - D_m| < \eta \quad \text{whenever } \rho \geq R_0.$$

Let $R \geq R_0$ be arbitrarily fixed, and let $S = R/\varepsilon$. By the previous relation there exists an x^* such that $m_S(x^*) < D_m + 2\eta$. Consequently, for $\rho < S - 2R$

$$\frac{S-\rho}{2S} m_{\frac{S-\rho}{2}}(x^* - \frac{S+\rho}{2}) + \frac{\rho}{S} m_\rho(x^*) + \frac{S-\rho}{2S} m_{\frac{S-\rho}{2}}(x^* + \frac{S+\rho}{2}) < D_m + 2\eta$$

(since the left-hand side is equal to $m_S(x^*)$), while

$$m_{\frac{S-\rho}{2}}(x^* \pm \frac{S+\rho}{2}) \geq \inf_x m_{\frac{S-\rho}{2}}(x) > D_m - \eta.$$

It follows that $m_\rho(x^*) < D_m + \dfrac{3\eta S}{\rho}$ for such ρ. Therefore, for $\rho \in [R, S - 2R]$

$$A_\rho(x^*) = \frac{1}{\rho}(\int_0^R + \int_R^\rho) m_r(x^*)\,\mathrm{d}r \leq \frac{R \sup m}{\rho} + \frac{3\eta S}{\rho} \log\left(\frac{\rho}{R}\right) + D_m.$$

Letting $\rho = S/2 = R/2\varepsilon$, the definition of η yields

$$\inf_{x\in\mathbb{R}} A_{R/2\varepsilon}(x) \leq A_{R/2\varepsilon}(x^*) \leq D_m + C\varepsilon,$$

where $C = 2\sup m + 6$. This last relation holds for all $R \geq R_0$. In other words

$$\sup_{R\geq R_0/2\varepsilon} \inf_{x\in\mathbb{R}} A_R(x) \leq D_m + C\varepsilon.$$

Therefore, $\limsup_{R\to\infty} \inf_x A_R(x) \leq D_m + C\varepsilon$. Since $\varepsilon > 0$ is arbitrarily small, we conclude

$$\limsup_{R\to\infty} \inf_{x\in\mathbb{R}} A_R(x) \leq D_m,$$

as desired. □

The following identity is also useful:

Lemma 3.5. *For m measurable and bounded and $R > 0$,*

$$\frac{1}{\pi}\int_0^\pi \omega_m(R\mathrm{e}^{\mathrm{i}\theta})\,\mathrm{d}\theta = \int_0^R \frac{1}{\rho}\int_{-\rho}^\rho m(t)\,\mathrm{d}t\,\mathrm{d}\rho.$$

Proof. Fubini's theorem and the relation $\int_0^\pi \cos\theta\,\mathrm{d}\theta = 0$ give

$$\frac{1}{\pi}\int_0^\pi \omega_m(R\mathrm{e}^{\mathrm{i}\theta})\,\mathrm{d}\theta = \frac{1}{\pi}\int_{-\infty}^\infty m(t)\int_0^\pi \log\left|1 - \frac{R\mathrm{e}^{\mathrm{i}\theta}}{t}\right|\,\mathrm{d}\theta\,\mathrm{d}t.$$

Moreover, Jensen's formula gives

$$\frac{1}{\pi}\int_0^\pi \log\left|1 - \frac{R\mathrm{e}^{\mathrm{i}\theta}}{t}\right|\,\mathrm{d}\theta = \frac{1}{2\pi}\int_{-\pi}^\pi \log\left|1 - \frac{R\mathrm{e}^{\mathrm{i}\theta}}{t}\right|\,\mathrm{d}\theta = \chi_{[-R,R]}(t)\log(R/|t|).$$

Therefore,

$$\frac{1}{\pi}\int_0^\pi \omega_m(R\mathrm{e}^{\mathrm{i}\theta})\,\mathrm{d}\theta = \int_{-R}^R \log(R/|t|) m(t)\,\mathrm{d}t = \int_0^R \frac{1}{\rho}\int_{-\rho}^\rho m(t)\,\mathrm{d}t\,\mathrm{d}\rho,$$

by Fubini's theorem again. □

Proposition 3.6. *In Corollary* 3.2*, $b < D_m$.*

Proof. Let $\mathrm{e}^{a_0 z}PW(m_0) = \mathrm{e}^{az}PW_{-b}(m)$ be the linearly reduced space in question, and suppose by contradiction that $b \geq D_m$. By our characterization of D_m, for any large R there exists an x_R such that

$$\frac{1}{R}\int_0^R \frac{1}{2\rho}\int_{-\rho}^{\rho}(m(x_R+t)-b)\,\mathrm{d}t\,\mathrm{d}\rho \leq \frac{\inf m_0}{2}.$$

In particular, for R large

$$\frac{1}{R}\int_0^R \frac{1}{2\rho}\int_{-\rho}^{\rho}(m_0(x_R+t)-m(x_R+t)+b)\,\mathrm{d}t\,\mathrm{d}\rho \geq \frac{\inf m_0}{2}.$$

The last lemma then implies

$$\frac{1}{2\pi R}\int_0^\pi (\omega_{m_0(x_R+\cdot)}(R\mathrm{e}^{\mathrm{i}\theta}) - \omega_{m(x_R+\cdot)}(R\mathrm{e}^{\mathrm{i}\theta}) + \pi b R\sin\theta)\,\mathrm{d}\theta \geq \frac{\inf m_0}{2}.$$

Notice that in general

$$\omega_{M(X+\cdot)}(z) = \omega_M(z+X) - \omega_M(X) + \Re z\int_{-\infty}^{\infty}\left(\frac{\chi(t-X)}{t-X} - \frac{\chi(t)}{t}\right)M(t)\,\mathrm{d}t,$$

and hence

$$\int_0^\pi \omega_{M(X+\cdot)}(R\mathrm{e}^{\mathrm{i}\theta})\,\mathrm{d}\theta = \int_0^\pi (\omega_M(R\mathrm{e}^{\mathrm{i}\theta}+X) - \omega_M(X))\,\mathrm{d}\theta.$$

Therefore,

$$\text{(3.2)}\qquad \frac{1}{2\pi R}\int_0^\pi (\omega_{m_0-m}(R\mathrm{e}^{\mathrm{i}\theta}+x_R) + \pi b R\sin\theta - \omega_{m_0-m}(x_R))\,\mathrm{d}\theta \geq \frac{\inf m_0}{2}.$$

However, Proposition 3.1 implies that

$$|\omega_{m_0-m}(R\mathrm{e}^{\mathrm{i}\theta}+x_R) + \pi b R\sin\theta + (a_0-a)(R\cos\theta + x_R)| \lesssim 1,$$

while $|\omega_{m_0-m}(x_R) + (a_0-a)x_R| \lesssim 1$. Since $\int_0^\pi R\cos\theta\,\mathrm{d}\theta = 0$, the integral in the relation (3.2) is bounded, a contradiction. □

3.2. **The lower density is the least majorant.** For showing that any $b < D_m$ is suitable, Lyubarskii and Seip replaced m with a smoothing of m of the form

$$m_R(x) = \frac{1}{2R}\int_{-R}^{R} m(x+t)\,\mathrm{d}t.$$

They justified this replacement by the relation $|\omega_m(z) - \omega_{m_R}(z)| \lesssim 1$, which is essentially right (after addition of a linear term αx). In fact, $\omega_{m_R}(z) - \omega_m(z)$ is equal to

$$\begin{aligned}
&\frac{1}{2R}\int_{-R}^{R}\left(\int_{-\infty}^{\infty}\log^*|1-z/t|m(t+s)\,\mathrm{d}t - \int_{-\infty}^{\infty}\log^*|1-z/t|m(t)\,\mathrm{d}t\right)\mathrm{d}s\\
&= \frac{1}{2R}\int_{-R}^{R}\int_{-\infty}^{\infty}\left(\log\left|1-\frac{z}{t-s}\right| + \frac{\chi(t-s)}{t-s}\,x - \log\left|1-\frac{z}{t}\right| - \frac{\chi(t)}{t}\,x\right)m(t)\,\mathrm{d}t\,\mathrm{d}s\\
&= F(z) + \alpha x,
\end{aligned}$$

where

$$F(z) = \frac{1}{2R}\int_{-R}^{R}\int_{-\infty}^{\infty} \log\frac{|1-z/(t-s)|}{|1-z/t|}\, m(t)\,\mathrm{d}t\,\mathrm{d}s \qquad \text{and}$$

$$\alpha = \frac{1}{2R}\int_{-R}^{R}\int_{-\infty}^{\infty} \left(\frac{\chi(t-s)}{t-s} - \frac{\chi(t)}{t}\right) m(t)\,\mathrm{d}t\,\mathrm{d}s.$$

Notice that both α and $\omega_{m_R-m}(z)$ are absolutely convergent, forcing $F(z)$ to be such.

Let us prove that $F(z)$ is bounded. In fact, $\displaystyle\int_{-\infty}^{\infty} \log\frac{|1-x/(t-s)|}{|1-x/t|} m(t)\,\mathrm{d}t$ is equal to

$$\omega_{m(x+\cdot)}(s) - \omega_m(s) + s\int_{-\infty}^{\infty}\left(\frac{\chi(t)}{t} - \frac{\chi(t-x)}{t-x}\right) m(t)\,\mathrm{d}t. \tag{3.3}$$

Clearly $|\omega_{m(x+\cdot)}(s)| \lesssim \int_{-\infty}^{\infty} |\log^* |1-(s/t)||\,\mathrm{d}t < \infty$ uniformly in $s \in [-R,R]$, and similarly for $|\omega_m(s)|$. Moreover, the last term in (3.3) disappears when averaging over $s \in [-R,R]$. Hence, $|F(z)| \lesssim 1$.

In total, $\mathrm{e}^{\omega_m(z)}$ is comparable to $\mathrm{e}^{\omega_{m_R}}(z)\mathrm{e}^{-\alpha x}$, and hence $PW_{-b}(m)$ equals $\mathrm{e}^{-\alpha z}PW(m_R - b)$, which is a weighted PW-space for all $b < \inf_x m_R(x)$, eventually for all $b < D_m$.

Joining this result with Corollary 3.2 and Proposition 3.6, we have completed the proof of the following theorem:

Theorem 3.7. *Let $m \simeq 1$ be measurable and $a \in \mathbb{R}$. The family of linearly reduced weighted Paley–Wiener spaces whose majorant-weight is comparable to $\mathrm{e}^{ax}\mathrm{e}^{\omega_m(x)}$ consists of all $\mathrm{e}^{az}PW_{-b}(m)$ with $b < D_m$.*

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