

LOW DEGREE REPRESENTATIONS OF SIMPLE LIE GROUPS

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ABSTRACT. We show that the number of irreducible representations of degree $\leq n$ of any simple compact Lie group G is $\leq n$. If the rank of G is large, the bound is much smaller. A consequence is that the number of conjugacy classes of closed maximal subgroups for a simple compact or complex Lie group of rank r is $O(r)$.

1. INTRODUCTION

We say that a compact Lie group is *simple* if it is connected, has finite center and is a simple group modulo its center. This paper is mostly concerned with upper bounds for $R_n(G)$, the number of characters of G of degree $\leq n$. It is convenient to define the *Witten zeta-function*

$$\zeta_G(s) = \sum_{\chi} \chi(1)^{-s},$$

where the sum is taken over all irreducible characters of G . By [LL], the abscissa of convergence of this Dirichlet series is $2/h$, where h denotes the Coxeter number of G . This implies that

$$R_n(G) = O(n^{2/h+\epsilon})$$

for all $\epsilon > 0$. One can turn this around and try to bound $R_n(G)$ for fixed n and varying G . In this direction, Wallach [W2] proved that $R_n(G) \leq 7n$.

Our first result gives the optimal bound for this problem.

Theorem 1. *If G is a simple compact Lie group and n is a positive integer,*

$$R_n(G) \leq n$$

with equality only if $n = 1$, $G = \mathrm{SU}(2)$, or $G = \mathrm{SU}(3)$ and $n = 3$.

If we exclude groups of low Coxeter number, we can improve on Theorem 1. A formulation in terms of the Witten zeta-function is as follows.

Theorem 2. *If G_1, G_2, \dots is a sequence of pairwise non-isomorphic simple compact Lie groups, and $s > 0$, then*

$$\lim_{n \rightarrow \infty} \zeta_{G_n}(s) = 1.$$

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This can be regarded as an analogue for compact Lie groups of the main theorem of [LS]. An immediate consequence of Theorem 2 is

Corollary 3. *For every $\epsilon > 0$ there exists N such that if G is a simple compact Lie group of dimension $\geq N$, then $R_n(G) \leq n^\epsilon$ for all $n \geq 1$.*

As a further application, we prove the following theorem.

Theorem 4. *For all $\epsilon > 0$, the number of isomorphism classes of pairs (G, V) , where G is a compact semisimple group and V is a faithful, irreducible representation of dimension n , is $O(n^\epsilon)$.*

If G is either a connected algebraic group or a compact Lie group, let $\mathcal{M}(G)$ denote the set of conjugacy classes of maximal closed subgroups of G (i.e. maximal subgroups among closed proper subgroups). The previous result can be used to give bounds for $|\mathcal{M}(G)|$ in characteristic 0. Considering parabolic subgroups of algebraic groups shows that $|\mathcal{M}(G)|$ is greater than the rank of G . Using the previous result, we can prove:

Theorem 5. *Let G be a simple algebraic group over an algebraically closed field of characteristic 0 or a compact Lie group of rank r . Then $|\mathcal{M}(G)| = O(r)$.*

With some effort, one could get a more precise upper bound in the previous result. The previous result uses a result of Häsä [H] which does depend upon the classification of finite simple groups. The result for positive dimensional maximal closed subgroups does not depend on this. See [GLT] for a weaker version of the previous theorem in positive characteristic.

The last result of the paper is about dimensions of fixed spaces of elements in compact groups. The first author and Malle [GM] showed that if G is an irreducible subgroup of $\mathrm{GL}(n, k)$, $n > 1$ (for any field k), then there is an element $g \in G$ with fixed space of dimension at most $n/3$ (the example $\mathrm{SO}(3)$ shows that this is best possible). This answered a conjecture of Peter Neumann. We show that as n increases we can do much better for Lie groups. Let $w(g) = w_V(g)$ denote the largest dimension of any eigenspace of g acting on V .

Theorem 6. *Let $\epsilon > 0$. There exists a positive integer $N = N(\epsilon)$ such that if G is a simple compact Lie group and V is a finite dimensional irreducible G -module with $\dim V > N$, then for a dense open subset X of G , $g \in X$ implies that $w_V(g) < \epsilon \dim V$.*

This immediately implies that the same result holds for algebraic groups over algebraically closed fields of characteristic 0.

2. COUNTING REPRESENTATIONS

We begin by noting that if G is a simple compact Lie group, every representation of G can be regarded as a representation of its universal covering group \tilde{G} . Thus,

$$R_n(\tilde{G}) \geq R_n(G)$$

for all n , and

$$\zeta_{\tilde{G}}(s) \geq \zeta_G(s)$$

for all $s > 0$ for which the left hand side is defined. Consequently, it suffices to prove the first two theorems for simply connected simple compact Lie groups, and we will assume henceforth that $G = \tilde{G}$. This is immediate for Theorem 2; for Theorem 1,

we note that if G is a non-trivial quotient of $SU(2)$ (resp. $SU(3)$), $R_n(G) < n$ for $n \geq 2$ (resp. for $n = 3$). We therefore consider only simply connected groups henceforth.

The irreducible characters χ of G are indexed by dominant weights of G . Let r be the rank of G , and $\varpi_1, \dots, \varpi_r$ the fundamental weights. Then the dominant weights are the non-negative linear combinations

$$\lambda = (c_1 - 1)\varpi_1 + \dots + (c_r - 1)\varpi_r,$$

where the c_i are positive integers. The half sum ρ of positive roots equals the sum of the fundamental weights [B, VI, §1, Prop. 29], so

$$c_1\varpi_1 + \dots + c_r\varpi_r = \lambda + \rho.$$

Weyl's dimension formula for the character χ_λ of highest weight λ asserts that

$$\chi_\lambda(1) = \prod_{\alpha \in \Phi^+} \frac{\alpha^\vee(\lambda + \rho)}{\alpha^\vee(\rho)},$$

where Φ^+ denotes the set of positive roots of G , and α^\vee denotes the dual root of α . We can express this in terms of the c_i as follows. If $\alpha_1^\vee, \dots, \alpha_r^\vee$ are the simple positive roots of the dual root system Φ^\vee , and $\phi_1^\vee, \dots, \phi_u^\vee$ are all the positive roots in Φ^\vee , then every root $\beta_i^\vee \in \Phi^\vee$ can be written uniquely as

$$\beta_i^\vee = \sum_{j=1}^r a_{i,j} \alpha_j^\vee,$$

where the $a_{i,j}$ are non-negative integers. As $\alpha_i^\vee(\varpi_j) = \delta_{ij}$ [B, VI, §1.10], we have

$$\chi_\lambda(1) = \prod_{i=1}^u \frac{\sum_{j=1}^r a_{i,j} c_j}{\sum_{j=1}^r a_{i,j}} = \prod_{i=1}^u \sum_{j=1}^r w_{i,j} c_j,$$

where the

$$w_{i,j} = \frac{a_{i,j}}{\sum_{k=1}^r a_{i,k}}$$

are non-negative and sum to 1.

By the weighted arithmetic-geometric mean inequality,

$$\sum_{j=1}^r w_{i,j} c_j \geq \prod_{j=1}^r c_j^{w_{i,j}}.$$

Multiplying over $i = 1, 2, \dots, u$, we obtain

$$(1) \quad \chi_\lambda(1) \geq \prod_{j=1}^r c_j^{v_j},$$

where

$$(2) \quad v_j = v_j(\Phi^\vee) = \sum_{i=1}^u w_{i,j}.$$

Defining

$$Z_\Phi(s) = \prod_{i=1}^r \zeta(v_j(\Phi) s),$$

where $\zeta(s)$ denotes the Riemann zeta-function, we obtain

$$\begin{aligned}
 \zeta_G(s) &= \sum_{c_1=1}^{\infty} \cdots \sum_{c_r=1}^{\infty} \chi_{(c_1-1)\varpi_1+\cdots+(c_r-1)\varpi_r}(1)^{-s} \\
 &\leq \sum_{c_1=1}^{\infty} \cdots \sum_{c_r=1}^{\infty} c_1^{-v_1s} \cdots c_r^{-v_rs} = Z_{\Phi^\vee}(s)
 \end{aligned}
 \tag{3}$$

for all s for which the right hand side converges.

Our next task is to estimate values of $Z_{\Phi}(s)$ for the various root systems Φ .

Lemma 7. *If Φ is of type A_r and the simple roots α_i are numbered as usual, then*

$$v_j = \sum_{i=1}^j \sum_{k=j}^r \frac{1}{1+k-i}.$$

Proof. The roots of Φ are precisely the sums of consecutive simple roots,

$$\alpha_i + \alpha_{i+1} + \cdots + \alpha_k.$$

Each such root contributes $\frac{1}{1+k-i}$ to the sum (2) if $i \leq j \leq k$ and 0 otherwise. \square

Lemma 8. *If Φ is of type A and $j \leq \frac{r+1}{2}$, then*

$$v_j = v_{r+1-j} \geq j \max(1, \log r - \log j).$$

Proof. The equality $v_j = v_{r+1-j}$ is clear from Lemma 7. If $1 \leq m \leq n \leq j$, then $1 \leq 1+n-m \leq j \leq j+n-m \leq r$. Setting $i = 1+n-m$ and $k = j+n-m$, we obtain

$$v_j \geq \sum_{i=1}^j \sum_{k=j}^r \frac{1}{1+k-i} \geq \sum_{n=1}^j \sum_{m=1}^n \frac{1}{n} = j.$$

On the other hand,

$$v_j \geq \sum_{i=1}^j \sum_{k=j}^r \frac{1}{1+k-i} \geq j \sum_{k=j}^r \frac{1}{k} \geq j \int_j^r \frac{dx}{x}. \quad \square$$

Lemma 9. *For $s \geq 3$,*

$$\log \zeta(s) \leq 2^{1-s}.$$

Proof. As x^{-s} is convex for $x > 0$,

$$\zeta(s) < 1^{-s} + 2^{-s} + \int_{5/2}^{\infty} x^{-s} dx = 1 + 2^{-s} + \frac{(5/2)^{1-s}}{s-1}.$$

As $s \geq 3$,

$$(5/2)(5/2)^{-s} \leq (5/2)(5/4)^{-3} 2^{-s} < \frac{32}{25} \cdot 2^{-s},$$

so

$$\zeta(s) < 1 + 2^{-s} + \frac{2^{1-s}}{2} = 1 + 2 \cdot 2^{-s},$$

which implies the lemma. \square

Proposition 10. *If Φ is of type A_r with $r \geq 9$, of type D_r with $r \geq 5$, or of type E_r , then*

$$\zeta_G(1) < 13/7.$$

Proof. Table 1 gives a direct machine computation of the values of $Z_\Phi(1) = Z_{\Phi^\vee}(1)$ to four decimal places.

We begin with type A_r . Table 1 shows that this inequality is true for $9 \leq r \leq 20$. We therefore assume $r \geq 21$. By (3), Lemma 8, and Lemma 9,

$$\begin{aligned} \zeta_G(1) &\leq \prod_{i=1}^6 \zeta(v_i)^2 \prod_{i=7}^{(r+1)/2} \zeta(v_i)^2 \leq \prod_{i=1}^6 \zeta(i \log(21/i))^2 \prod_{i=7}^{(r+1)/2} \zeta(v_i)^2 \\ &\leq \prod_{i=1}^6 \zeta(i \log(21/i))^2 \prod_{i=7}^{\infty} e^{2^{2-i}} = \prod_{i=1}^6 \zeta(i \log(21/i))^2 e^{1/16} < 13/7. \end{aligned}$$

Next, we consider type D_r , using the standard numbering of simple roots [B, Planche IV]. If $1 \leq i \leq r - 2$ and $1 \leq j \leq r$, there is a unique path from the i th to the j th vertex in the Dynkin diagram; its length is $|j - i|$ unless $j = r$, in which case it is $j - 1 - i = |j - i| - 1$. By [B, VI, §1, Prop. 19, Cor. 3], the sum of the simple roots corresponding to the vertices in this path is again a positive root. It follows that for $1 \leq i \leq r - 2$,

$$v_i(D_r) \geq v_i(A_r).$$

If $1 \leq p < q \leq r - 1$, then

$$(5) \quad \sum_{i=p}^r \alpha_i + \sum_{j=q}^{r-2} \alpha_j$$

is a positive root, and so is

$$(6) \quad \beta + \sum_{i=p}^{r-2} \alpha_i,$$

for $\beta \in \{\alpha_{r-1}, \alpha_r\}$ and $1 \leq p \leq r - 1$. For $1 \leq i \leq r - 1$, we count positive roots which are sums of i distinct positive roots, including α_r . There are $\lceil i/2 \rceil - 1$ of type (5), with $p = r - (i - k) < r - k = q$ for $1 \leq k < i/2$, and there is one of type (6), with $p = r - i$. Thus,

$$v_{r-1}(D_r) = v_r(D_r) \geq \sum_{i=1}^{r-1} \frac{\lceil i/2 \rceil}{i} \geq \frac{r-1}{2}.$$

If $r \geq 11$, then

$$v_{r-1}(D_r) > v_{r-1}(A_r) > v_r(A_r).$$

Therefore, in this range, the proposition is implied by the result for type A_r . For $5 \leq r \leq 10$, it follows from Table 1.

Likewise, for E_6, E_7 , and E_8 , it follows from Table 1. □

We can now prove Theorem 1.

Proof. If Φ is among the root systems covered by Proposition 10, then the same is true for its dual, and it follows that $\zeta_G(1) < 13/7$. If, for some $n \geq 7$, we have $R_n(G) \geq n$, then

$$\zeta_G(1) > 1 + (n - 1) \cdot n^{-1} \geq 13/7.$$

Thus, $n \leq 6$. However, $\dim G \geq 45 > 6^2$ for all the cases covered by Proposition 10, so G has no non-trivial representations of degree ≤ 6 .

TABLE 1. $Z_\Phi(1)$

A_9	1.8558	D_5	1.7269
A_{10}	1.7336	D_6	1.4609
A_{11}	1.6400	D_7	1.3244
A_{12}	1.5667	D_8	1.2462
A_{13}	1.5083	D_9	1.1978
A_{14}	1.4610	D_{10}	1.1160
A_{15}	1.4221	E_6	1.2522
A_{16}	1.3896	E_7	1.0836
A_{17}	1.3621	E_8	1.0178
A_{18}	1.3386		
A_{19}	1.3182		
A_{20}	1.3004		

TABLE 2. $Z_\Phi(3/4)^4$

A_2	29547000
A_3	194100
A_4	27475
A_5	7055.8
A_6	2414.5
A_7	1001.9
A_8	481.56
D_4	994.94

Again, if $R_n(G) \geq n$, we have $\zeta_G(3/4) \geq n^{1/4}$, and therefore

$$n \leq \zeta_G(3/4)^4 \leq Z_{\Phi^\vee}(3/4)^4.$$

For each simply laced root system of rank ≥ 2 not covered by Proposition 10, Table 2 indicates the value of $Z_\Phi(3/4)^4 = Z_{\Phi^\vee}(3/4)^4$. An exhaustive machine search for representations of degree less than or equal to $Z_\Phi(3/4)^4$ yields precisely one non-trivial case where $R_n(G) \geq n$, namely the equality $R_3(\mathrm{SU}(3)) = 3$. Theorem 1 follows for all G with simply laced root systems.

The remaining irreducible root systems all have the same underlying graph as A_r , and we use the same numbering convention for the simple roots. By [B, VI, §1, Prop. 19, Cor. 3], $\alpha_i + \alpha_{i+1} + \cdots + \alpha_j$ is a positive root for all $i \leq j$. Therefore, the dimension of the irreducible representation of G associated to the r -tuple (c_1, \dots, c_r) is at least as great as the dimension of the irreducible representation of $\mathrm{SU}(r+1)$ associated to the same r -tuple. This proves $R_n(G) \leq n$ for root systems of type B, C, F, and G as well. Moreover, in the case $r = 2$, the pairs (1, 2) and (2, 1) give representations of B_2 and G_2 of dimension strictly greater than 3. Thus, the non-simply laced root systems give no additional cases for which $R_n(G) = n > 1$. \square

Now we prove Theorem 2.

Proof. Since we are taking the limit as $n \rightarrow \infty$, we can exclude the exceptional groups and groups of type D_r , $r \leq 10$. Then, if G is of rank r and $s > 2/(r + 1)$,

$$1 \leq \zeta_G(s) \leq Z_{\Phi^\vee}(s) \leq Z_{A_r}(s).$$

Thus, it suffices to prove that for all $s > 0$,

$$(7) \quad \lim_{r \rightarrow \infty} Z_{A_r}(s) = 1.$$

By (4),

$$Z_{A_r}(s) < \prod_{i=1}^{(r+1)/2} \zeta\left(s \sum_{i=1}^j \sum_{k=j}^r \frac{1}{1+k-i}\right)^2 < \prod_{j=1}^{\infty} \zeta(s \max(j, j \log r/j))^2.$$

If we assume $r > (3/s)e^{3/s}$, for each positive integer j , either $j > 3/s$ or $r/j > e^{3/s}$, so by Lemma 9,

$$Z_{A_r}(s) < \exp\left(4 \sum_{j=1}^{\infty} 2^{-s \max(j, j \log r/j)}\right).$$

If $N > \exp(1/s)$ and $r > N^2$,

$$\sum_{j=1}^{\infty} 2^{-s \max(j, j \log r/j)} \leq \sum_{j=1}^N 2^{-sj \log N} + \sum_{j=N+1}^{\infty} 2^{-sj} \leq 2^{1-s \log N} + \frac{2^{-sN}}{1-2^{-s}}.$$

If $s > 0$ is fixed and $N \rightarrow \infty$, the right hand side goes to 0. This implies (7). \square

Lemma 11. *Let $\epsilon > 0$, and let a_1, a_2, \dots be a sequence of non-negative real numbers whose growth is $O(n^\epsilon)$. Then*

$$b_n := \sum_{\substack{d_1 \cdots d_k = n \\ 1 < d_1 \leq \dots \leq d_k}} a_{d_1} \cdots a_{d_k}$$

also has growth $O(n^\epsilon)$.

Proof. If

$$f(s) := 1^{-s} + \sum_{n=2}^{\infty} a_n n^{-s}$$

converges for all $s > 0$, then it converges absolutely for $s > 0$, so

$$\prod_{n=2}^{\infty} \frac{1}{1 - a_n n^{-s}} = 1 + \sum_{n=2}^{\infty} b_n n^{-s}$$

converges for all $s > 0$. If $a_n = O(n^\epsilon)$ for all $\epsilon > 0$, then $f(s)$ converges for all $s > 0$, so $b_n = O(n^\epsilon)$ for all $\epsilon > 0$. \square

Lemma 12. *For all positive integers n there exist integers R and m and a finite set $\Lambda \subset \mathbb{N}^m$ such that if G is a simply connected, simple, compact Lie group of rank $r \geq R$ and $\chi_\lambda(1) \leq r^n$, then either G is of type A_r and*

$$(8) \quad \lambda = a_1 \varpi_1 + \cdots + a_m \varpi_m + b_m \varpi_{r+1-m} + \cdots + b_1 \varpi_r$$

for some $(a_1, \dots, a_m), (b_1, \dots, b_m) \in \Lambda$, or G is of type B_r, C_r , or D_r , and

$$(9) \quad \lambda = a_1 \varpi_1 + \cdots + a_m \varpi_m$$

for some $(a_1, \dots, a_m) \in \Lambda$.

Proof. For $1 \leq i \leq r$,

$$v_i(\Phi) \geq 1 + 1/2 + \cdots + 1/r - 1 > \log r.$$

By (1), this implies that $c_i \leq m/\log 2$ for all i . Our goal is therefore to prove that $c_i = 1$ except when i is bounded above or, in the case of A_r , when $r - i$ is bounded above. For A_r , this is an easy consequence of Lemma 7. For the remaining root systems, we need only prove that for every constant C , $v_i(\Phi) \leq C \log r$ implies an upper bound on i . For B_r , by [B, Planche II], for $1 \leq p < q \leq r$,

$$\sum_{i=p}^{q-1} \alpha_i + 2 \sum_{j=q}^r \alpha_j \in \Phi^+.$$

Thus, the sum for $v_i(B_r)$ contains at least $i(2r - 1 - i)/2$ terms, each of which is at least $1/(2r - 1)$. Thus,

$$v_i(B_r) \geq \max(v_i(A_r), \frac{i(r - 1)}{2r - 1}),$$

so $v_i(B_r) \leq C \log r$ implies an upper bound on i independent of r . For C_r , by [B, Planche III], for $1 \leq p < q \leq r$,

$$\alpha_r + \sum_{i=p}^{q-1} \alpha_i + \sum_{j=q}^{r-1} \alpha_j \in \Phi^+.$$

As in the B_r case, $v_i(C_r)$ is the sum of at least $i(2r - 1 - i)/2$ terms, each of which is at least $1/(2r - 1)$, so

$$v_i(C_r) \geq \max(v_i(A_r), \frac{i(r - 1)}{2r - 1}),$$

which implies an upper bound on i , independent of r . By (5),

$$v_i(D_r) \geq \max(v_i(A_r), \frac{i(r - 1)}{2r - 3}),$$

for $1 \leq i \leq r - 2$, and

$$v_{r-1}(D_r) = v_r(D_r) \geq \frac{(r - 1)(r - 2)}{2r - 3},$$

which again implies an upper bound on i , independent of r . □

We now prove Theorem 4.

Proof. If G is a semisimple compact Lie group with universal covering group \tilde{G} and V is a faithful irreducible representation of G , then we can regard V as an almost faithful irreducible representation of \tilde{G} . Writing \tilde{G} as a product of its simple factors $G_1 \times \cdots \times G_k$, we can factor V as $V_1 \boxtimes \cdots \boxtimes V_k$, where each V_i is a non-trivial representation. We can order the factors G_i so that $\dim V_1 \leq \dim V_2 \leq \cdots \leq \dim V_k$. By Lemma 11, it suffices to prove that for each n , the number of pairs (G, V) where G is a simply connected, simple, compact Lie group and V is a representation of dimension n is $O(n^\epsilon)$ for all $\epsilon > 0$.

By Theorem 2, for every $\epsilon > 0$, there exists a finite set Σ of groups such that for every simply connected, simple, compact Lie group G not isomorphic to any element of Σ , we have $\zeta_G(\epsilon/2) \leq 2$. For any such G , $R_n(G) \leq 2n^{\epsilon/2}$. If we limit ourselves to groups G of rank $\leq n^{\epsilon/2}$ which do not lie in Σ , the total contribution is bounded by $2n^\epsilon$.

Next we show that for each $G \in \Sigma$ there are $O(n^\epsilon)$ representations of G of degree n . Suppose that G has rank r . The representations of G are indexed by $(c_1, \dots, c_r) \in \mathbb{N}^r$. It suffices to show that for each $j \in \{1, \dots, r\}$, the number of representations with $c_j > 0$ and degree n is $O(n^\epsilon)$. Applying [B, VI, §1, Prop. 19, Cor. 3], for each $k \in \{1, \dots, r\}$, the sum of all the simple roots corresponding to vertices on the path from j to k is a positive root β_k . Clearly, β_1, \dots, β_r span the root system. Ordering the positive roots of G so that β_1, \dots, β_r come first, we deduce from Weyl's dimension formula that

$$\prod_{i=1}^r \sum_{j=1}^r a_{i,j} c_j \mid C_G n,$$

where

$$C_G = \prod_{i=1}^u \sum_{j=1}^r a_{i,j}.$$

Moreover, the linear independence of β_1, \dots, β_r implies that the r -tuple

$$(d_1, \dots, d_r) = \left(\sum_{j=1}^r a_{1,j} c_j, \dots, \sum_{j=1}^r a_{r,j} c_j \right)$$

determines (c_1, \dots, c_r) . We define

$$d_{r+1} = \frac{C_G n}{d_1 \cdots d_r}$$

and then let e_1, \dots, e_s , $s \leq r + 1$, denote the sequence obtained by ordering all elements in the set $\{d_1, \dots, d_{r+1}\}$ which are greater than 1. Thus

$$\sum_{d_1 \cdots d_r \mid C_G n} 1 \leq (r + 1)! \sum_{\substack{e_1 \cdots e_s = C_G n \\ 1 < e_1 \leq e_2 \leq \cdots \leq e_s}} 1.$$

By Lemma 11, this sum is $O((C_G n)^\epsilon)$ and therefore $O(n^\epsilon)$.

This leaves only groups of rank $\geq n^{\epsilon/2}$, and taking n sufficiently large, we may assume that only classical groups need be considered. Setting $m = 2/\epsilon$ in Lemma 12, there exists a finite set Λ , so that if G is of type A_r , λ is given by (8), and if G is of type B_r , C_r , or D_r , it is given by (9). We claim that for each element $(a_1, \dots, a_m) \in \Lambda$ and each family B, C, D, the dimension of the representation is a strictly monotone function of r ; in case A, the analogous statement is true for each pair $(a_1, \dots, a_m), (b_1, \dots, b_m) \in \Lambda$. Perhaps the easiest way to see this is to note that the branching rules for classical groups [K, 9.14, 9.16, 9.18] imply that for $X \in \{A, B, C, D\}$, the restriction of the representation of X_r associated to (a_1, \dots, a_m) to X_{r-1} contains the representation of X_{r-1} associated to (a_1, \dots, a_m) as a proper subrepresentation. We conclude that the groups of rank $\geq n^{\epsilon/2}$ contribute at most a constant term $|\Lambda|^2 + 3|\Lambda|$ to the total number of pairs (G, V) . \square

3. MAXIMAL SUBGROUPS OF CLASSICAL GROUPS

Let k be an algebraically closed field of characteristic $p \geq 0$. Let G be a simple classical group over k (i.e. $G = \text{SL}(n), n > 1, \text{Sp}(2n), n > 1, \text{SO}(2n), n > 3, \text{SO}(2n + 1), n > 2, p \neq 2$). Table 3 gives Aschbacher's description of proper subgroups in the case of algebraic groups. See [A, KL, G, GT, LSe]. Let V denote the natural module for G . If H is a proper closed subgroup of G , then H belongs to

TABLE 3. Aschbacher Classes

\mathcal{C}_1	H preserves a totally singular or non-degenerate subspace of V
\mathcal{C}_2	H preserves an additive decomposition of V
\mathcal{C}_4	V is tensor decomposable for H
\mathcal{C}_6	H normalizes a subgroup of symplectic type
\mathcal{C}_7	V is tensor induced for H
\mathcal{C}_8	V is the natural module for a classical subgroup H
\mathcal{S}_1	H is the normalizer of a quasisimple finite group S
\mathcal{S}_2	H is the normalizer of a simple algebraic subgroup S

one of the classes $\mathcal{C}_i(G)$ or $\mathcal{S}_i(G)$ (note that for the last two classes, we may also assume that S acts irreducibly or we would be in one of the other cases).

Note that we have kept the same numbering as in Aschbacher's theorem for finite groups but certain cases do not arise in the algebraic group case.

It is easy to see, as in the proof [LPS, Lemmas 2.1, 2.4], that:

Lemma 13. *Let G be a classical group over k with n being the dimension of the natural module V . The number of conjugacy classes of maximal subgroups of G in $\bigcup_{H \in \mathcal{C}_i}$ is at most $n + 3 \log n + 3$.*

We need to deal with the maximal subgroups in \mathcal{S}_1 and \mathcal{S}_2 . Unfortunately in positive characteristic, one has much less information available about dimensions of irreducible representations. See [GLT] for a weaker version of these results. Now we assume that we are in characteristic 0.

By Theorem 4, it follows that the number of conjugacy classes of maximal subgroups in $\mathcal{S}_2(G)$ is $O(n^\epsilon)$.

The result for \mathcal{S}_1 we need has recently been obtained by Häsä [H]. This is the only result in the paper that requires the classification of finite simple groups.

Lemma 14. *The number of conjugacy classes of maximal subgroups in $\mathcal{S}_1(G)$ is at most $O(n)$.*

Now Theorem 5 for algebraic groups follows, for if the rank of G is fixed, there is nothing to prove. So we may assume that G is a classical group in sufficiently large dimension, and the remarks above immediately give the desired bound.

It is now quite easy to obtain the result for compact groups. A nice observation of Nolan Wallach shows that the Cartan decomposition for complex algebraic groups [W1, Lemma 2.1.5] gives:

Lemma 15. *Let G be a compact real Lie group and $G_{\mathbb{C}}$ its complexification. If A, B are subgroups of G and are conjugate in $G_{\mathbb{C}}$, then they are conjugate in G .*

It follows that the number of conjugacy classes of maximal closed subgroups of a compact real Lie group is the number of conjugacy classes of maximal closed reductive subgroups of $G_{\mathbb{C}}$. Assume that G (and so also $G_{\mathbb{C}}$) is connected. Suppose that M is a maximal reductive subgroup of $G_{\mathbb{C}}$ but is not maximal. Then $M < X$, where X has a unipotent radical. By the Borel-Tits lemma, this implies that $M < P$, where P is a parabolic subgroup with unipotent radical Q . Since M is reductive, $M \cap Q = 1$. Let L be a Levi subgroup of Q . Set $H = MQ \cap L$. Then $HQ = MQ$. Since $H^1(M, V) = 0$ for any H -invariant section of A , it follows that H and M are conjugate. Thus, by maximality, M is conjugate to L . Since M

is maximal reductive, it follows that P is a maximal parabolic. This shows that the number of conjugacy classes of maximal closed subgroups of G is at most the number of conjugacy classes of maximal closed subgroups of $G_{\mathbb{C}}$, whence the result. Note that it is possible that G has strictly fewer conjugacy classes of maximal closed subgroups than $G_{\mathbb{C}}$, since Levi subgroups need not be maximal reductive subgroups and the Levi subgroups of non-conjugate parabolic subgroups may be conjugate. See [GW] for more details.

4. DIMENSIONS OF FIXED SPACES

In this section, we prove Theorem 6. Let G be a simple compact Lie group and V a non-trivial irreducible finite dimensional module for G . The minimum value of $w_V(g)$ as g ranges over G is the maximal dimension of the weight spaces of V , and this value is achieved on a dense open subvariety of G . Therefore, it suffices to exhibit just one g satisfying the asserted inequality.

We require two lemmas. The first is due to Seitz [S] (and the result holds in all characteristics and for finite Chevalley groups as well).

Lemma 16. *Let T be a maximal torus of G . Then the dimension of any weight space for T is at most $1 + (\dim V)/h$, where h is the Coxeter number of G .*

This proves the theorem if the rank is increasing. So it suffices to prove the theorem for a fixed G . Daniel Goldstein [Go] proved Theorem 6 in this case. We give a different proof (Goldstein’s proof gives somewhat better bounds).

Lemma 17. *Let G be a simple compact Lie group. Let V be an irreducible finite dimensional module for G . Let $g \in G$ be a regular element of prime order p . Then*

$$\limsup_{\dim V \rightarrow \infty} \frac{w_V(g)}{\dim V} = \frac{1}{p}.$$

Proof. Let χ denote the character of V . By the Weyl character formula,

$$|\chi(g^k)| \leq \frac{|W|}{|1 - e^{2\pi i/p}|^{|\Phi^+|}}$$

for $1 \leq k \leq p - 1$, where Φ^+ is the set of positive roots of the root system of G and W is the Weyl group. Let λ be any irreducible character for $L := \langle g \rangle$. If $m(\lambda, \chi)$ is the multiplicity of λ in the restriction of χ to L , then

$$(10) \quad pm(\lambda, \chi) = \sum_{k=0}^{p-1} \chi(g^k)\lambda(g^{-k}) \leq \chi(1) + (p - 1) \frac{|W|}{|1 - e^{2\pi i/p}|^{|\Phi^+|}}.$$

Note that in its action on V , g has only p eigenvalues, so

$$w_V(g) = \max_{\lambda} \{m(\lambda, \chi)\} \geq \chi(1)/p.$$

Combining this with (10), the result follows. □

Since for any sufficiently large prime p there are regular elements of order p in G , we see that there is some element $g \in G$ with $w(g) < \epsilon \dim V$ as long as $\dim V$ is sufficiently large. This completes the proof.

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