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EXOTIC EMBEDDINGS OF OPEN SUBSETS OF AFFINE SPACE

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ABSTRACT. We show that every Zariski open affine proper subset $X \subset \mathbb{A}^n$ has infinitely many non-equivalent embeddings into $\mathbb{A}^{n+1}$. Moreover, we give some examples of non-equivalent embeddings of such X in higher codimension.

1. INTRODUCTION

Let k be an algebraically closed field. Let X be an affine variety. We say that X has a unique embedding in $\mathbb{A}^n$ if for every two polynomial embeddings $\phi, \psi : X \to \mathbb{A}^n$ there exists a polynomial automorphism $\Sigma : \mathbb{A}^n \to \mathbb{A}^n$ such that $\phi = \Sigma \circ \psi$. In particular if $X \subset \mathbb{A}^n$ is a subvariety, this means that X has no exotic embeddings into $\mathbb{A}^n$ (an exotic embedding is an embedding not equivalent by automorphism to the standard one given by the inclusion $i : X \subset \mathbb{A}^n$). There exist smooth affine subvarieties of $\mathbb{A}^n$ without a unique embedding in $\mathbb{A}^n$ (see [3], [5], [9], [12], [15]). However a smooth subvariety $X \subset \mathbb{A}^n$ of sufficiently large codimension has a unique embedding in $\mathbb{A}^n$, which was established in [4], [7], [10], [11] and [16].

In a small codimension the problem whether a given variety has a unique embedding in $\mathbb{A}^n$ seems to be very difficult, and it is completely open: in fact we know only a few hypersurfaces which do not have exotic embeddings (see [2], [8], [17]). Therefore it is important to find examples of subvarieties which have small codimension in $\mathbb{A}^n$ and which have or do not have exotic embeddings.

Here we restrict our attention to the embeddings of Zariski open affine subsets of $\mathbb{A}^n$. It is interesting that a Zariski open proper affine subset of $\mathbb{A}^n$ does not have a unique embedding in $\mathbb{A}^{n+1}$:

Theorem 1.1. *Every Zariski open affine proper subset $X \subset \mathbb{A}^n$ has infinitely many non-equivalent embeddings into $\mathbb{A}^{n+1}$.*

Here "a proper subset X" means $X \neq \emptyset$ and $X \neq \mathbb{A}^n$. Let us recall that the (closed) subvariety $X \subset \mathbb{A}^n$ has the extension of automorphism property if every polynomial automorphism $\phi : X \to X$ has an extension to a polynomial automorphism of the whole of $\mathbb{A}^n$. Of course if X has no exotic embeddings, then it has

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the extension of automorphism property. There are algebraic hypersurfaces in $\mathbb{A}^n$ without the extension of automorphism property (see [3]). We show:

Theorem 1.2. *Let X be a Zariski open affine proper subset $X \subset \mathbb{A}^n$. If X is sufficiently general, then we can embed X into $\mathbb{A}^{n+1}$ in such a way that X has the extension of automorphism property.*

Finally we show:

Theorem 1.3. *For every even number $n \geq 4$ there exists a Zariski open affine subset $X \subset \mathbb{A}^{2n+2}$ and two embeddings $\phi, \psi : X \to \mathbb{A}^{3n+1}$ such that $\phi(X)$ is a complete intersection and $\psi(X)$ is not a complete intersection. Consequently, these embeddings are not equivalent.*

2. Exotic embeddings in codimension one

We start with some simple but important invariants of a polynomial (for another type of invariant, see [15]):

Definition 2.1. Let $f \in k[x_1, ..., x_n]$ be a polynomial. Denote by $\rho(f)$ the number of irreducible factors in f; i.e., if

$$f = c\prod_{i=1}^{n} f_i^{\alpha_i},$$

where the polynomials f_i are irreducible, then $\rho(f) = \sum_{i=1}^{n} \alpha_i$. Now let

$$\mu(f) = \max_{t\in k}\{\rho(f - t)\}.$$

Remark 2.1. It is easy to see that $\rho(f) \leq \deg f$. This implies $\mu(f) \leq \deg f$. In particular the number $\mu(f)$ is well defined.

The following proposition is obvious.

Proposition 2.1. *The number $\mu(f)$ is an invariant of the group of polynomial automorphisms of $k[x_1, ..., x_n]$.*

Definition 2.2. Let X be an affine variety. Let $\phi_1, \phi_2 : X \to \mathbb{A}^n$ be embeddings. We say that ϕ_1 is equivalent to ϕ_2 if there exists a polynomial automorphism $\Sigma : \mathbb{A}^n \to \mathbb{A}^n$ such that

$$\phi_2 = \phi_1 \circ \Sigma.$$

More generally, if X, X' are subvarieties of $\mathbb{A}^n$, we say that they are equivalent if there exists a polynomial automorphism $\Sigma : \mathbb{A}^n \to \mathbb{A}^n$ such that $X' = \Sigma(X)$.

Of course, if ϕ_1, ϕ_2 are equivalent embeddings and $X_i = \phi_i(X)$, then X_1 is equivalent to X_2. Let $X \subset \mathbb{A}^n$ be a hypersurface and let $f \in I(X)$ be a generator of the ideal $I(X)$ of X. The number $\mu(f)$ depends on X only. Indeed if f' is another generator, then $f' = cf$ for some $c \in k^*$. Consequently $\mu(f) = \mu(f')$, because $\rho(cf - t) = \rho(f - t/c)$. Hence we can state the following:

Definition 2.3. Let $X \subset \mathbb{A}^n$ be a hypersurface. We denote $\mu(X) = \mu(f)$, where f is any generator of $I(X)$.

Now we show that equivalent hypersurfaces have the same numbers μ:

Proposition 2.2. *Let $X, X' \subset \mathbb{A}^n$ be hypersurfaces. If X is equivalent to X', then $\mu(X) = \mu(X')$.*

Proof. Let $I(X) = f$ and $I(X) = f'$. By the assumption we have $(f') = (f \circ \Phi)$ as ideals for some automorphism $\Phi : \mathbb{A}^n \to \mathbb{A}^n$. This implies that the polynomial $f \circ \Phi$ is a generator of $I(X')$ and consequently $\mu(X') = \mu(f \circ \Phi) = \mu(f) = \mu(X)$. □

Now we are ready to prove:

Theorem 2.1. *Every Zariski open affine proper subset $X \subset \mathbb{A}^n$ has infinitely many non-equivalent embeddings into $\mathbb{A}^{n+1}$.*

Proof. Let $X = \mathbb{A}^n \setminus \{x : h(x) = 0\}$ be a Zariski open subset of $\mathbb{A}^n$. Define the embedding $\phi_j : X \ni x \mapsto (x, 1/h(x)^j) \in k^{n+1}$, where $j = 1, 2, \ldots$. Put $X_i = \phi_i(X)$. The hypersurface X_i has the equation $X_i = \{(x, z) \in \mathbb{A}^n \times k : h(x)^i z = 1\}$. Note that the polynomial $h(x)^\alpha z - 1 - t$ is irreducible for $t \neq -1$; hence $\rho(h(x)^\alpha z - 1 - t) = 1$ for $t \neq -1$. Moreover, for $t = -1$ we have $\rho(h(x)^\alpha z - 1 - t) = \rho(h(x)^\alpha z) = \alpha\rho(h) + 1$. Consequently $\mu(X_i) = i\rho(h) + 1$. Thus $\mu(X_i) \neq \mu(X_j)$ for $i \neq j$, which implies that X_i is not equivalent to X_j (see Proposition 2.2). □

A natural question is whether a "sufficiently" general affine submanifold of $\mathbb{A}^n$ has a unique embedding into $\mathbb{A}^n$. Of course there are different notions of generality. One of them is "a manifold of hyperbolic type", i.e. a manifold which has the maximal logarithmic Kodaira dimension. Our theorem shows that such submanifolds in general do not have a unique embedding into $\mathbb{A}^n$:

Corollary 2.1. *For every $n \geq 2$ there are hypersurfaces $X \subset \mathbb{A}^n$ of maximal logarithmic Kodaira dimension, which have infinitely many non-equivalent embeddings into $\mathbb{A}^n$.*

Proof. Let $H \subset \mathbb{A}^{n-1}$ be a general hypersurface of degree greater than or equal to $n + 1$. Then $X := \mathbb{A}^{n-1} \setminus H$ is of logarithmic Kodaira dimension $n - 1$ and does not have a unique embedding in $\mathbb{A}^n$ by Theorem 2.1. □

Let k_0 be the simple field contained in k (i.e., $k_0 = \mathbb{Q}$ if char $k = 0$ or $k_0 = \mathbb{F}_p$ if char $k = p$). The following conjecture is still completely open:

Conjecture. Let $n \geq 2$ and let

$$X = \{x \in \mathbb{A}^n : \sum_{|\alpha| \leq d} a_\alpha x^\alpha = 0\},$$

where $\alpha = (\alpha_1, \ldots, \alpha_d)$ and $|\alpha| = \sum_{i=1}^n \alpha_i$. If $d > n$ and all a_{α_i} are algebraically independent over k_0, then X has a unique embedding in $\mathbb{A}^n$.

3. Extension of automorphisms

Let us recall that a variety X is called ruled if it is birationally equivalent to a cylinder $W \times \mathbb{P}^1$. We have the following lemma of Abhyankar ([1]):

Lemma 3.1. *Let $f : X \to Y$ be a birational morphism of algebraic varieties. If X is normal and Y is smooth, then every exceptional divisor E of f is ruled.*

Definition 3.1. Let X be an affine smooth variety. We say that X is ruled at infinity if there exists a smooth completion $\overline{X}$ of X such that all the components of the hypersurface $\overline{X} \setminus X$ are ruled.

From the Abhyankar Lemma we can deduce that if X is ruled at infinity and Y is any normal completion of X, then the hypersurface $Y \setminus X$ has ruled components. We have the following generalization of Theorem 52 from [6] (see also [19]):

Theorem 3.1. *Let X be a smooth affine variety, which is ruled at infinity. Let $H \subset X$ be a hypersurface without ruled components. Then every automorphism of $X \setminus H$ is induced by an automorphism of the whole of X, i.e., $Aut(X \setminus H) \subset Aut(X)$.*

Proof. Let $\overline{X}$ be a smooth completion of X and let $V = \overline{X} \setminus X$. By the assumption, every irreducible component of V is ruled. Let $\phi \in Aut(X \setminus H)$. Then ϕ induces the birational map $f : X \to X$ as well as the birational map $\overline{\phi} : \overline{X} \to \overline{X}$. Let Γ be the normalization of the closure of the $graph(\phi)$. We have the following commutative diagram:

$$\begin{tikzcd} & \Gamma \arrow[dl, "\pi_1"'] \arrow[dr, "\pi_2"] & \\ \overline{X} \arrow[rr, dashed, "\overline{\phi}"] & & \overline{X} \end{tikzcd}$$

Let $H = \bigcup_{i=1}^{r} H_i$ be a decomposition of H into irreducible components and let $\overline{H_i}$ be the closure of H_i in $\overline{X}$. By the Abhyankar Lemma we see that the set $\pi_1^{-1}(V)$ is ruled. Since the set $R := \pi_1^{-1}(V)$ is ruled, then the set $\pi_2(R)$ cannot contain any of the sets $\overline{H_j}$; i.e., $\pi_2(\pi_1^{-1}(V)) \cap \overline{H_j} \neq \overline{H_j}$.

Moreover, among components of the $\pi_1^{-1}(\overline{H_i})$ only the proper preimage $\overline{H_i}'$ of $\overline{H_i}$ can be non-ruled. Since π_2 is a surjection we have that for every $\overline{H_j}$ there exists $\overline{H_i}'$ such that $\pi_2(\overline{H_i}') = \overline{H_j}$. This implies in particular that all proper preimages $\overline{H_i}'$ are hypersurfaces and the mapping $\pi_1 : \overline{H_i}' \to \overline{H_i}$ is birational. Consequently the mapping f is defined at a generic point of H_i.

Now one can easily see that the set S of all poles and points of indeterminancy of the map f has codimension ≥ 2 in X. This implies that f is in fact regular (we can always extend a regular mapping of affine normal varieties by subsets of codimension ≥ 2). By symmetry f^{-1} is regular, too. □

Corollary 3.1. *Let $H \subset \mathbb{A}^n$ be a hypersurface without ruled components. Then every automorphism of $\mathbb{A}^n \setminus H$ is induced by an automorphism of the whole of $\mathbb{A}^n$, i.e., $Aut(\mathbb{A}^n \setminus H) \subset Aut(\mathbb{A}^n)$.*

Let $X \subset \mathbb{A}^n$ be a Zariski open affine subset. We say that X is general if $X = \mathbb{A}^n \setminus H$, where H is a hypersurface without ruled components. The automorphism group of a general open subset of $\mathbb{A}^n$ is usually finite; however, it may be large:

Proposition 3.1. *For every finite group G there is an integer n and a general open subset $X \subset \mathbb{A}^n$ such that $G \subset Aut(X)$.*

Proof. Let $n = Card(G)$ so that we can embed the group G in the group of linear automorphisms of $\mathbb{A}^n = \mathbb{P}^n \setminus H$. Let $Z' \subset \mathbb{P}^n$ be a smooth hypersurface of degree $d > n+1$. Then the variety Z' is not separable uniruled because $H^0(Z', \mathcal{O}(K_{Z'})) \neq 0$. In particular it is not ruled. Let $Z = Z' \setminus H$. Now it is enough to take

$$X = \mathbb{A}^n \setminus \bigcup_{g \in G} g(Z). \qquad \square$$

Remark 3.1. In a similar way we can prove that for any $n \geq 2$ and any number $m > 0$ there is a general open subset $X \subset \mathbb{A}^n$ such that $Card(Aut(X)) > m$.

At the end of this section we prove:

Theorem 3.2. *Let $X = \{x \in \mathbb{A}^n : h(x) \neq 0\}$ be a general open subset of $\mathbb{A}^n$. For an integer $\alpha \geq 1$ let $\phi_\alpha : X \ni x \mapsto (x, 1/h(x)^\alpha) \in k^{n+1}$ and $X'_\alpha = \phi_\alpha(X)$. Then X'_α has the extension of automorphism property.*

Proof. Let $f : X'_\alpha \to X'_\alpha$ be an automorphism. Then f induces an automorphism $\overline{f}$ of X. By Corollary 3.1, $\overline{f}$ extends to an automorphism of $\mathbb{A}^n$; hence we can assume that $\overline{f} \in Aut(\mathbb{A}^n)$. Since $\overline{f}$ transforms the hypersurface $H := \{x \in \mathbb{A}^n : h(x) = 0\}$ onto itself we have $h \circ \overline{f} = ch$, where $c \in k^*$ is a constant. Further $f(x, 1/h(x)^\alpha) = (\overline{f}(x), 1/(h(\overline{f}(x))^\alpha) = (\overline{f}(x), 1/c^\alpha h(x)^\alpha)$. In the coordinates (x, z) in $k^{n+1} = k^n \times k$ we obtain

$$f(x, z) = (\overline{f}(x), c^{-\alpha} z)$$

for $(x, z) \in X'_\alpha$. $\square$

Corollary 3.2. *There are submanifolds of $\mathbb{A}^n$ which have the extension of automorphisms property but which do not have a unique embedding into $\mathbb{A}^n$.*

4. Exotic embeddings in higher codimension

For a field k, we let X be an affine variety over k and let $R = k[X]$ be a ring of regular functions on X. Let us recall some basic facts about algebraic vector bundles over X, which we identify with finitely generated projective $R-$modules. We say that the algebraic vector bundle $\mathbf{E}$ is stably trivial if

$$\mathbf{E} \oplus \mathbf{E_t} = \mathbf{E_s},$$

for some trivial vector bundles $\mathbf{E_t}$ and $\mathbf{E_s}$. Recall that a sequence $(f_1, ..., f_l) \in R^l$ is called a unimodular row if $(f_1, ..., f_l) = R$ (as an ideal). If the field k is algebraically closed, this is equivalent to the fact that $f_1, ..., f_l$ have no common zero on X. If the sequence $(f_1, ..., f_l)$ is unimodular, then we have the following epimorphism of trivial vector bundles:

$$F : \mathbf{E}^l \ni (v_1, ..., v_l) \mapsto \sum_{i=1}^{l} f_i v_i \in \mathbf{E}^1.$$

We say that the vector bundle $\mathbf{E}(f_1, ..., f_l) := ker\ F$ is given by the unimodular row $(f_1, ..., f_l)$. Of course the bundle $\mathbf{E}(f_1, ..., f_l)$ is stably trivial of rank $l - 1$.

In this section we modify our method from [9] to construct an example of a Zariski open subset of $\mathbb{A}^{2n+2}$ which admits at least two different embeddings into $\mathbb{A}^{3n+1}$. We start with the example of Raynaud (see [14], [18]):

Example 4.1. Let $n \geq 3$ and

$$R = \frac{k[x_1, ..., x_n, y_1, ..., y_n]}{(\sum_{i=1}^n x_i y_i - 1)}.$$

Then the stably free submodule of R^n given by the unimodular row $(x_1, ..., x_n)$ is not free.

Proposition 4.1. *Let $n \geq 3$ be an integer and let $h = \sum_{i=1}^n x_i y_i \in k[x_1, \ldots, x_n; y_1, \ldots, y_n]$. Define*

$$Y_{2n}(h) = \{(x, y) \in k^{2n} : \; h(x, y) \neq 0\}.$$

Then the vector bundle $\mathbf{E}(x_1, ..., x_n)$ given on Y_{2n} by the unimodular row $(x_1, ..., x_n)$ is non-trivial.

Proof. Let $Z = \{(x, y) \in k^{2n} : h(x, y) = 1\}$. By the Raynaud example we know that the row $(x_1, \ldots, x_n)$ which is unimodular on Z gives on Z a non-trivial vector bundle $\mathbf{F}_1$. However this row, when considered on the whole of Y_{2n}, is also unimodular; consequently it gives a stably trivial vector bundle $\mathbf{F}_2 := \mathbf{E}(x_1, ..., x_n)$. Now it is enough to note that $Z \subset Y_{2n}$ and

$$\mathbf{F}_2|_Z = \mathbf{F}_1. \qquad \square$$

Let $n \geq 4$ be an even number and consider the open subvariety $Y_{2n} \subset k^{2n}$. Let $\mathbf{A}$ be the stably trivial vector bundle given by the row $(x_1, ..., x_n)$ and let $\langle y_1, ..., y_n \rangle \subset \mathbf{E}^n$ denote the line bundle determined by the vector $\overline{y} := (y_1, ..., y_n)$. It is easy to see that

$$\mathbf{E}^n = \mathbf{A} \oplus \langle y_1, ..., y_n \rangle.$$

Moreover, since n is an even number we have that the line bundle $\langle x_2, -x_1, ..., x_n, -x_{n-1} \rangle$ determined by the vector $\overline{x} = (x_2, -x_1, ..., x_n, -x_{n-1})$ is contained in the bundle $\mathbf{A}$. Consequently there is a stably trivial (but non-trivial) bundle $\mathbf{B} \subset \mathbf{A}$ such that

$$\mathbf{A} = \mathbf{B} \oplus \langle \overline{x} \rangle.$$

In particular we have

$$\mathbf{B} = \mathbf{E}^n / (\langle \overline{x} \rangle, \langle \overline{y} \rangle).$$

Consider the embedding

$$\phi : Y_{2n} \ni (x, y) \to (x, y, 1/h(x, y)) \in k^{2n+1}.$$

Let $Y = \phi(Y_{2n})$. Thus Y is a smooth hypersurface in $\mathbb{A}^{2n+1}$. In particular the normal bundle $\mathbf{N}_{\mathbb{A}^{2n+1}/Y}$ is trivial; in fact, $\mathbf{N}_{\mathbb{A}^{2n+1}/Y} = \mathbf{E}^1$. On Y we have coordinates x, y and $z = 1/h(x, y)$. Now we are ready to prove the main result of this section:

Theorem 4.1. *For every even number $n \geq 4$ there exists a Zariski open affine subset $X \subset \mathbb{A}^{2n+2}$ and two embeddings $\phi, \psi : X \to \mathbb{A}^{3n+1}$ such that $\phi(X)$ is a complete intersection and $\psi(X)$ is not a complete intersection. Consequently, these embeddings are not equivalent.*

Proof. Let Y be as above. Consider the embeddings

$$\phi : Y \times k^2 \ni ((x,y,z),(s,t)) \to (x,y,z,s,t,0,...,0) \in k^{3n+1}$$

and

$$\psi : Y \times k^2 \ni ((x,y,z),(s,t)) \to (x,y,z,y_1 s + x_2 t, \\ y_2 s - x_1 t, ..., y_{n-1} s + x_n t, y_n s - x_{n-1} t) \in k^{3n+1}.$$

By a direct computation of the differential map $d\psi$, we see that the normal bundle $\mathbf{N}(k^{3n+1}/\psi(Y \times k^2))$ restricted to the submanifold $Y \times \{0\}$ is equal to

$$\mathbf{N}(k^{2n+1}/Y) \oplus \mathbf{E}^n/(\langle \overline{x} \rangle, \langle \overline{y} \rangle) = \mathbf{E_1} \oplus \mathbf{B} = \mathbf{A}.$$

This means that this normal bundle is non-trivial along $Y \times \{0\}$. In particular the normal bundle $\mathbf{N}(k^{3n+1}/\psi(Y \times k^2))$ is non-trivial, too.

However, it is easy to see that the normal bundle $\mathbf{N}(k^{3n+1}/\phi(Y \times k^2))$ is trivial; in fact, the manifold $\phi_1(Y \times k^2)$ is a complete intersection. This means that ϕ and ψ are not equivalent. Now it is enough to take $X = Y \times k^2$. $\square$

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