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A NOTE ON THE HITCHIN-THORPE INEQUALITY AND RICCI FLOW ON 4-MANIFOLDS

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Abstract. In this short paper, we prove a Hitchin-Thorpe type inequality for closed 4-manifolds with non-positive Yamabe invariant and admitting long time solutions of the normalized Ricci flow equation with bounded scalar curvature.

1. Introduction

A Riemannian metric g on a smooth manifold M is called an Einstein metric if

$$Ric_g = cg,$$

where Ric_g is the Ricci tensor and c is a constant. If a closed oriented 4-manifold M admits an Einstein metric g, there is an inequality, the Hitchin-Thorpe inequality, for the Euler number $\chi(M)$ and the signature $\tau(M)$ of M:

$$2\chi(M) - 3|\tau(M)| \geq 0 \tag{1.1}$$

(cf. [10] or Theorem 6.35 in [3]). This inequality serves as a topological obstruction for the existence of Einstein metrics on 4-manifolds; i.e. if (1.1) is not satisfied, then M would not admit any Einstein metric.

The Ricci flow was introduced by Hamilton in [8] to find Einstein metrics on a given manifold, which is the following evolution equation for a smooth family of metrics $g(t), t \in [0, T)$:

$$\frac{\partial}{\partial t} g(t) = -2Ric_t. \tag{1.2}$$

The normalized Ricci flow is

$$\frac{\partial}{\partial t} g(t) = -2Ric_t + \frac{2r(t)}{n} g(t), \tag{1.3}$$

where $r(t) = \frac{\int_M R_t dv_{g(t)}}{\mathrm{Vol}_{g(t)}(M)}$ denotes the average scalar curvature and R_t denotes the scalar curvature of $g(t)$. The normalized Ricci flow is just a transformation of (1.2) by rescaling the space and time such that the volume preserves to be a constant along the flow. If (1.3) admits a long time solution $g(t), t \in [0, \infty)$, and $g(t)$ converges to a Riemannian metric g_∞ on M in some suitable sense, when $t \to \infty$,

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then g_∞ is an Einstein metric or a Ricci-soliton. However, (1.3) may not admit any long time solution, and even if there is such a solution, $g(t)$ may not converge to a Riemannian metric on M. It is expected that the inequality (1.1) is also a topological obstruction for the existence of long time solutions of (1.3) on 4-manifolds, at least for 4-manifolds with a non-positive Yamabe invariant. In [6], an analog inequality of (1.1) was obtained for long time solutions of (1.3) under some hypothesis. In this paper, we will continue to study the Hitchin-Thorpe type inequality for 4-manifolds admitting long time solutions of (1.3).

For a closed Riemannian n-manifold (M, g) and a function $f \in C^\infty(M)$, set

$$\mathcal{F}(g, f) = \int_M (R_g + |\nabla f|^2) e^{-f} dv_g.$$

The Perelman λ-functional is defined by

$$\lambda_M(g) = \inf_f \{\mathcal{F}(g, f) | \int_M e^{-f} dv_g = 1\},$$

which is the lowest eigenvalue of the operator $-4\triangle + R_g$. Perelman [15] has established the monotonicity property of $\overline{\lambda}_M(g(t)) = \lambda_M(g(t))\mathrm{Vol}_{g(t)}(M)^{\frac{2}{n}}$ along the Ricci flow. A diffeomorphism invariant $\overline{\lambda}_M$ of M is defined (cf. [16], [12]) by

$$\overline{\lambda}_M = \sup_{g \in \mathcal{M}} \overline{\lambda}_M(g),$$

where $\mathcal{M}$ is the set of Riemannian metrics on M. By [1] and [2], $\overline{\lambda}_M$ is equal to the classical Yamabe invariant whenever $\overline{\lambda}_M \leq 0$ or the Yamabe invariant is non-positive.

In [6] it was proved that if $g(t), t \in [0, \infty)$, is a solution to (1.3) with $|R(g(t))| < C$ for a constant C independent of t on a closed oriented 4-manifold M with $\overline{\lambda}_M < 0$, then

$$2\chi(M) - 3|\tau(M)| \geq \frac{1}{96\pi^2} \overline{\lambda}_M^2 > 0.$$

Based on this inequality and the Seiberg-Witten theory, Ishida [11] recently showed that the existence of a long time non-singular solution really depends on the smooth structure of the underlying manifold. The goal of the present paper is to relax the assumption $\overline{\lambda}_M < 0$ to $\overline{\lambda}_M \leq 0$.

Theorem 1.1. *Let M be a closed oriented 4-manifold with $\overline{\lambda}_M \leq 0$. If M admits a long time solution $g(t), t \in [0, \infty)$, of (1.3) with scalar curvature $|R_t| < C$ for a constant C independent of t, then*

$$2\chi(M) - 3|\tau(M)| \geq \frac{1}{96\pi^2} \overline{\lambda}_M^2.$$

The hypothesis of $\overline{\lambda}_M \leq 0$ holds for many cases, for example, complex projective surfaces with non-negative Kodaira dimension by [14], which include K3-surfaces, projective surfaces of general type and some surfaces of elliptic type, etc. Furthermore, Theorem 2 in [13] shows that if M is a closed oriented 4-manifold with a monopole class $c_1(\mathfrak{c})$ that is not a torsion class and satisfies $c_1^2(\mathfrak{c}) \geq 0$, then

$$\overline{\lambda}_M \leq -\sqrt{32\pi^2 c_1^2(\mathfrak{c})} \leq 0$$

(see also [5] for the case of $c_1^2(\mathfrak{c}) > 0$). In [6], a Miyaoka-Yau type inequality was proved for 4-manifolds with a monopole class $c_1(\mathfrak{c})$ such that $c_1^2(\mathfrak{c}) > 0$ and admitting long time solutions of (1.3) with bounded scalar curvature.

The assumption of bounded scalar curvature is a technique assumption, and we hope that it can be removed in future study. However, it can be verified for some cases. In [4], it was shown that a K3-surface M admits a long time solution $g(t)$, $t \in [0, \infty)$, of (1.3) and that $g(t)$ converges to a Ricci-flat Kähler-Einstein metric on M. Thus the scalar curvature of $g(t)$ is uniformly bounded. Since $\overline{\lambda}_M = 0$ (cf. [14]), Theorem 1.1 can apply to this case. If M is a complex minimal projective surface of elliptic type with Kodaira dimension 1 and no singular fibers, then $\overline{\lambda}_M = 0$ by [14] and M admits long time solutions $\tilde{g}(\tilde{t})$, $\tilde{t} \in [0, \infty)$, of the following Kähler-Ricci flow equation:

$$\frac{\partial}{\partial \tilde{t}} \tilde{g}(\tilde{t}) = -Ric_{\tilde{t}} - \tilde{g}(\tilde{t}) \tag{1.4}$$

by [19] and [20]. By Corollary 1.1 in [18], the scalar curvature of $\tilde{g}(\tilde{t})$ is uniformly bounded along (1.4). A straightforward calculation would show that a transformation of $\tilde{g}(\tilde{t})$, $\tilde{t} \in [0, \infty)$, by rescaling the metric and time, gives a long time solution of (1.3) with bounded scalar curvature. Thus the assumption of Theorem 1.1 is satisfied. If M is a complex minimal projective surface of general type, which satisfies $\overline{\lambda}_M < 0$ by [14], then M admits long time solutions of (1.3) with bounded scalar curvature by [4], [19], [20], [23] and the same arguments as above. In [21], an alternative proof of the Miyaoka-Yau inequality for minimal projective manifolds of general type was obtained by using the result in [23] and a similar Ricci flow argument.

An analog Hitchin-Thorpe type inequality for non-compact 4-manifolds admitting non-singular solutions of (1.3) is obtained in [7]. We assume $\mathrm{Vol}_{g(t)}(M) \equiv 1$ in this paper for convenience. We shall prove Theorem 1.1 in the next section.

2. Proof of Theorem 1.1

The goal of this section is to prove Theorem 1.1, which essentially depends on the following estimate for the volume along Ricci flow in [22].

Lemma 2.1 (Lemma 3.1 in [22]). *Let $\bar{g}(\bar{t}), \bar{t} \in [0, T)$, be a solution to the Ricci flow equation* (1.2)*, i.e.*

$$\frac{\partial}{\partial \bar{t}} \bar{g}(\bar{t}) = -2Ric_{\bar{t}}$$

on a closed manifold M. If $\lambda_M(\bar{g}(\bar{t})) \leq 0$, for all $\bar{t}$, then there exist constants $c_1, c_2 > 0$ depending only on $\bar{g}(0)$ such that, for all $\bar{t} \geq 0$,

$$\mathrm{Vol}_{\bar{g}(\bar{t})}(M) \geq c_1 e^{-c_2 \bar{t}}.$$

Because of the importance of this lemma, we present the sketch of the proof here for the reader's convenience.

Sketch of the proof. First, we recall some basics about the μ functional introduced by Perelman [15]. Given a closed Riemannian manifold (M, g) and a function $f \in C^\infty(M)$ and a constant $\tau > 0$, define

$$\mathcal{W}(g, f, \tau) = \int_M [\tau(R_g + |\nabla f|^2) + f - n](4\pi\tau)^{-n/2} e^{-f} dv_g$$

and then set

$$\mu(g,\tau)=\inf\{\mathcal{W}(g,f,\tau)|\int_M(4\pi\tau)^{-n/2}e^{-f}dv_g=1\}. \tag{2.1}$$

By a result of Rothaus [17], for each $\tau>0$, there is a smooth minimizer f such that $\mu(g,\tau)=\mathcal{W}(g,f,\tau)$. In [22], some bounds of the μ-functional were obtained. First, Lemma 2.1 in [22] shows that there is a lower bound for $\tau>\frac{n}{8}$,

$$\mu(g,\tau)\geq\lambda_M(g)\tau-\frac{n}{2}\ln(4\pi\tau)-n-\frac{n}{8}(\lambda_M(g)-\inf R_g)-n\ln C_s, \tag{2.2}$$

where C_s denotes the Sobolev constant for g, i.e. $\|\phi\|_{L^{\frac{2n}{n-2}}(g)}\leq C_s\|\phi\|_{H^{1,2}(g)}$ for all $\phi\in C^\infty(M)$. Second, Corollary 2.3 in [22] says that if $\lambda_M(g)\leq 0$, then

$$\mu(g,\tau)\leq\ln\mathrm{Vol}_g(M)-\frac{n}{2}\ln(4\pi\tau)-n+1. \tag{2.3}$$

In [15], Perelman proved the monotonicity of the μ-functional along the Ricci flow:

Theorem 2.2 ([15])**.** *Let $\bar g(\bar t)$ be a solution to the Ricci flow equation* (1.2) *on a closed manifold M. Denote $\tau(\bar t)=A-\bar t$ for some constant $A>0$. Then $\mu(\bar g(\bar t),\tau(\bar t))$ is non-decreasing whenever it makes sense.*

Note that $\lambda_M(\bar g(\bar t))\leq 0$. Substituting $\tau=\frac{n}{8}$ into (2.3) and then using Theorem 2.2 and (2.2), we have

$$\begin{aligned}
\mathrm{Vol}_{\bar g(\bar t)}(M) &\geq \exp(\mu(\bar g(\bar t),\frac{n}{8})+\frac{n}{2}\ln(\frac{n}{2}\pi)+n-1)\\
&\geq \exp(\mu(\bar g(0),\frac{n}{8}+\bar t)+\frac{n}{2}\ln(\frac{n}{2}\pi)+n-1)\\
&\geq \exp(\lambda_M(\bar g(0))\bar t-\frac{n}{2}\ln(1+\frac{8}{n}\bar t)+\frac{n}{8}\inf R_0-n\ln C_s(\bar g(0))-1)\\
&\geq \exp((\lambda_M(\bar g(0))-4)\bar t+\frac{n}{8}\inf R_0-n\ln C_s(\bar g(0))-1). \qquad (2.4)
\end{aligned}$$

We obtain the conclusion. ☐

Now we can prove:

Lemma 2.3. *Let M be a closed n-manifold with $\overline{\lambda}_M\leq 0$ and let $g(t),t\in[0,\infty)$, be a long time solution of* (1.3) *with scalar curvature $|R_t|<C$ for a constant C independent of t. We have*

$$\liminf_{t\to\infty}r(t)=\liminf_{t\to\infty}\int_M R_t dv_{g(t)}\leq 0.$$

Proof. If it is not true, there is a constant $\delta>0$ such that, for $t\gg 1$, $r(t)>\delta$. By a translation on t, we assume that $r(t)>\delta$ for all $t>0$.

Let $\bar g(\bar t)=\sigma(t)g(t),\bar t\in[0,T)$, be the corresponding Ricci flow solution, i.e. $\frac{\partial}{\partial t}\bar g(\bar t)=-2Ric_{\bar t}$ with $\bar g(0)=g(0)$, which implies $\sigma(t)=\exp(-\frac{2}{n}\int_0^t r(s)ds)$ and $\bar t=\int_0^t\sigma(s)ds$. Thus

$$T=\int_0^\infty\sigma(s)ds=\int_0^\infty\exp(-\int_0^s\frac{2}{n}r(u)du)ds<\frac{n}{2\delta}.$$

Now we can compute

$$
\begin{aligned}
(T-\bar{t})Vol_{\bar{g}(\bar{t})}(M)^{-\frac{2}{n}} &= \sigma(t)^{-1}(T-\bar{t}) \\
&= \exp(\int_0^t \frac{2}{n}r(u)du)\cdot\int_t^\infty \exp(-\int_0^s \frac{2}{n}r(u)du)ds \\
&\geq \int_t^\infty \exp(-\int_t^s \frac{2}{n}r(u)du)ds \\
&\geq \int_t^\infty \exp(-\frac{2}{n}C(s-t))ds \\
&\geq \frac{n}{2C}, \qquad (2.5)
\end{aligned}
$$

by $r(t) \leq \sup|R_t| \leq C$ for a constant $C > 0$ independent of t. Thus there exists $C_1 < \infty$ such that

$$
Vol_{\bar{g}(\bar{t})}(M) \leq C_1(T-\bar{t})^{n/2}, \qquad (2.6)
$$

which implies that

$$
\lim_{\bar{t}\to T} \mathrm{Vol}_{\bar{g}(\bar{t})}(M) = 0.
$$

However, since $\overline{\lambda}_M \leq 0$, we have $\lambda_M(\bar{g}(\bar{t})) \leq 0$ and

$$
\mathrm{Vol}_{\bar{g}(\bar{t})}(M) \geq c_1 e^{-c_2\bar{t}} \geq c_1 e^{-c_2 T}
$$

for two constants $c_1 > 0$ and $c_2 > 0$ by Lemma 2.1, which is a contradiction. □

Before proving Theorem 1.1, we recall the evolution equations for volume forms and scalar curvatures along the normalized Ricci flow (1.3):

$$
\frac{\partial}{\partial t}dv_{g(t)} = -(R_t - r(t))dv_{g(t)} \quad \text{and} \qquad (2.7)
$$

$$
\frac{\partial}{\partial t}R_t = \triangle_t R_t + 2|Ric_t|^2 + \frac{2}{n}R_t(R_t - r(t)), \qquad (2.8)
$$

where $Ric_t = Ric_t - \frac{R_t}{n}g(t)$ denotes the Einstein tensor (cf. [9]).

Proof of Theorem 1.1. Note that

$$
\breve{R}_t = \inf_M R_t \leq \lambda_M(g(t)) \leq \overline{\lambda}_M \leq 0.
$$

If $\breve{R}_t \leq -c < 0$ for a constant $c > 0$, we obtain the conclusion by Lemma 2.7 and Lemma 3.1 in [6].

From the maximal principal,

$$
\frac{\partial}{\partial t}\breve{R}_t \geq \frac{1}{2}\breve{R}_t(\breve{R}_t - r(t)) \geq 0,
$$

and thus $\breve{R}_t$ is non-decreasing. Therefore, the only case left to prove is that of

$$
\lim_{t\longrightarrow\infty} \breve{R}_t = 0.
$$

By Lemma 2.3 and $\breve{R}_t \leq r(t)$,

$$
\liminf_{t\to\infty} r(t) = 0,
$$

which implies that $\overline{\lambda}_M = 0$.

First, we assume that there is a sequence $t'_k \longrightarrow \infty$ such that $r(t'_k) > \epsilon$ for a constant $\epsilon > 0$ independent of k. Since $\liminf_{t\to\infty} r(t) = 0$, there are $t_k \in (t'_k, t'_{k+1})$ such that, for $k \gg 1$,

$$r(t_k) = \inf_{(t'_k, t'_{k+1})} r(t) \longrightarrow 0, \qquad \frac{dr}{dt}(t_k) = 0.$$

Now, we assume $\lim_{t\to\infty} r(t) = 0$. If $|\frac{dr}{dt}(t)| > \delta > 0$ for a constant δ independent of t when $t \gg 1$, $|r(t) - r(0)| > \delta t$, which is a contradiction. Thus there is a sequence $t_k \longrightarrow \infty$ such that

$$\lim_{t_k \longrightarrow \infty} r(t_k) = 0, \qquad \lim_{t_k \longrightarrow \infty} \frac{dr}{dt}(t_k) = 0.$$

In both cases, we have

$$\begin{aligned}
0 &= \lim_{k\to\infty} \frac{dr}{dt}(t_k) \\
&= \lim_{k\to\infty} \int_M (2|Ric^o_{t_k}|^2 - \frac{1}{2} R_{t_k}(R_{t_k} - r(t_k))) dv_{g(t_k)} \\
&\geq \lim_{k\to\infty} \int_M 2|Ric^o_{t_k}|^2 dv_{g(t_k)} - \lim_{k\to\infty} C \int_M |R_{t_k} - r(t_k)| dv_{g(t_k)} \\
&\geq \lim_{k\to\infty} \int_M 2|Ric^o_{t_k}|^2 dv_{g(t_k)} - \lim_{k\to\infty} C \int_M (R_{t_k} + r(t_k) - 2\check{R}_{t_k}) dv_{g(t_k)} \\
&= \lim_{k\to\infty} \int_M 2|Ric^o_{t_k}|^2 dv_{g(t_k)} - \lim_{k\to\infty} 2C(r(t_k) - \check{R}_{t_k}) \\
&= \lim_{k\to\infty} \int_M 2|Ric^o_{t_k}|^2 dv_{g(t_k)}
\end{aligned}$$

by (2.7), (2.8), and the assumption $|R_t| < C$ for a constant C independent of t.

The Chern-Gauss-Bonnet formula and the Hirzebruch signature theorem (cf. [3]) show that, for any metric g on M,

$$\chi(M) = \frac{1}{8\pi^2} \int_M (\frac{R_g^2}{24} + |W_g^+|^2 + |W_g^-|^2 - \frac{1}{2}|Ric_g|^2) dv_g \quad \text{ and }$$

$$\tau(M) = \frac{1}{12\pi^2} \int_M (|W_g^+|^2 - |W_g^-|^2) dv_g,$$

where W_g^+ and W_g^- are the self-dual and anti-self-dual Weyl tensors respectively. Thus

$$\begin{aligned}
2\chi(M) - 3|\tau(M)| &\geq \liminf_{k\longrightarrow\infty} \frac{1}{4\pi^2} \int_M (\frac{1}{24} R_{t_k}^2 - \frac{1}{2}|Ric_{t_k}|^2) dv_{g(t_k)} \\
&= \liminf_{k\longrightarrow\infty} \frac{1}{4\pi^2} \int_M \frac{1}{24} R_{t_k}^2 dv_{g(t_k)} \geq 0.
\end{aligned}$$

Since $\overline{\lambda}_M = 0$ in this case, we obtain the conclusion. □

References

[1] K. Akutagawa, M. Ishida, and C. LeBrun, *Perelman's invariant, Ricci flow, and the Yamabe invariants of smooth manifolds,* Arch. Math. 88 (2007), 71-76. MR2289603 (2007j:53070)

[2] M. T. Anderson, *Remarks on Perelman's papers*, preprint, available at `http://www.math.sunysb.edu/~anderson/perelman.pdf`.

[3] A. L. Besse, *Einstein manifolds*, Ergebnisse der Math. Springer-Verlag, Berlin-New York, 1987. MR867684 (88f:53087)

[4] H. D. Cao, *Deformation of Kähler metrics on compact Kähler manifolds*, Invent. Math. 81 (1985), 359-372. MR799272 (87d:58051)

[5] F. Fang, Y. Zhang, *Perelman's λ-functional and the Seiberg-Witten equations*, Frontiers of Mathematics in China, 2, no. 2 (2007), 191-210. MR2299748 (2008e:53051)

[6] F. Fang, Y. Zhang and Z. L. Zhang, *Non-singular solutions to the normalized Ricci flow equation*, Math. Ann. 340 (2008), 647-674. MR2357999 (2009a:53112)

[7] F. Fang, Y. Zhang and Z. L. Zhang, *Non-singular solutions of normalized Ricci flow on noncompact manifolds of finite volume*, Journal of Geometric Analysis, 20, no. 3 (2010), 592-608. MR2610891

[8] R. Hamilton, *Three-manifolds with positive Ricci curvature*, J. Diff. Geom. 17 (1982), 255-306. MR664497 (84a:53050)

[9] R. Hamilton, *Non-singular solutions of the Ricci flow on three-manifolds*, Commun. Anal. Geom. 7 (1999), 695-729. MR1714939 (2000g:53034)

[10] N. J. Hitchin, *On compact four-dimensional Einstein manifolds*, J. Differential Geom. 9 (1974), 435-442. MR0350657 (50:3149)

[11] M. Ishida, *The normalized Ricci flow on four-manifolds and exotic smooth structure*, arXiv:math.0807.2169.

[12] B. Kleiner and J. Lott, *Notes on Perelman's papers*, Geom. Topol. 12 (2008), 2587-2855. MR2460872 (2010h:53098)

[13] D. Kotschick, *Monopole classes and Perelman's invariant of four-manifolds,* arXiv:math. DG/0608504.

[14] C. LeBrun, *Kodaira dimension and the Yamabe problem,* Comm. Anal. Geom. 7 (1999), 133-156. MR1674105 (99m:58056)

[15] G. Perelman, *The entropy formula for the Ricci flow and its geometric applications*, arXiv:math.DG/0211159.

[16] G. Perelman, *Ricci flow with surgery on three-manifolds*, arXiv:math.DG/0303109.

[17] O. S. Rothaus, *Logarithmic Sobolev inequalities and the spectrum of Schrödinger operators*, J. Funct. Anal. 42 (1981), 110-120. MR620582 (83f:58080b)

[18] J. Song, G. Tian, *The Kähler-Ricci flow on surfaces of positive Kodaira dimension*, Inventiones Mathematicae 170, no. 3 (2007), 609-653. MR2357504 (2008m:32044)

[19] H. Tsuji, *Existence and degeneration of Kähler-Einstein metrics on minimal algebraic varieties of general type*, Math. Ann. 281 (1988), 123-133. MR944606 (89e:53075)

[20] G. Tian, Z. Zhang, *On the Kähler-Ricci flow on projective manifolds of general type*, Chinese Annals of Mathematics B 27 (2006), 179-192. MR2243679 (2007c:32029)

[21] Y. Zhang, *Miyaoka-Yau inequality for minimal projective manifolds of general type*, Proc. Amer. Math. Soc. 137 (2009), 2749-2754. MR2497488 (2010b:32036)

[22] Z. L. Zhang, *Compact blow-up limits of finite time singularities of Ricci flow are shrinking Ricci solitons*, C. R. Acad. Sci. Paris, Ser. I 345 (2007), 503-506. MR2375111 (2009a:53117)

[23] Z. Zhang, *Scalar curvature bound for Kähler-Ricci flows over minimal manifolds of general type*, International Mathematics Research Notices (2009), 3901-3912. MR2544732 (2010j:32038)

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