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EXISTENCE OF AT LEAST TWO PERIODIC SOLUTIONS OF THE FORCED RELATIVISTIC PENDULUM

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ABSTRACT. Using Szulkin's critical point theory, we prove that the relativistic forced pendulum with periodic boundary value conditions

$$\left(\frac{u'}{\sqrt{1-u'^2}}\right)' + \mu \sin u = h(t), \quad u(0)-u(T)=0=u'(0)-u'(T),$$

has at least two solutions not differing by a multiple of 2π for any continuous function $h : [0,T] \to \mathbb{R}$ with $\int_0^T h(t)dt = 0$ and any $\mu \neq 0$. The existence of at least one solution has been recently proved by Brezis and Mawhin.

1. Introduction and the main result

It is well known that the classical forced pendulum with periodic boundary value conditions

$$u'' + \mu \sin u = h(t), \quad u(0)-u(T)=0=u'(0)-u'(T),$$

has at least two solutions not differing by a multiple of 2π for any continuous function $h : [0,T] \to \mathbb{R}$ with $\int_0^T h(t)dt = 0$ and any $\mu \neq 0$. The existence of at least one solution was proved by Hamel [9] and rediscovered independently by Dancer [7] and Willem [15]. Then, the existence of a second solution has been proved by Mawhin and Willem [11] using mountain pass arguments.

Motivated by those results, Brezis and Mawhin prove in [6] that the relativistic forced pendulum with periodic boundary value conditions

$$\left(\frac{u'}{\sqrt{1-u'^2}}\right)' + \mu \sin u = h(t), \quad u(0)-u(T)=0=u'(0)-u'(T), \tag{1.1}$$

has at least one solution for any forcing term h with mean value zero and any $\mu \neq 0$. The above problem is reduced to finding a minimum for the corresponding action integral over a closed convex subset of the space of T-periodic Lipschitz functions, and then to show, using variational inequalities techniques, that such a minimum solves the problem.

In this paper we show that (1.1) has at least two solutions not differing by a multiple of 2π. Actually, we consider as in [2, 6] the more general periodic boundary value problem

$$(\phi(u'))' = f(t,u) + h(t), \quad u(0)-u(T)=0=u'(0)-u'(T), \tag{1.2}$$

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where ϕ satisfies the hypothesis

$$(H_\Phi)\qquad \begin{array}{l} \textit{there exists } \Phi : [-a, a] \to \mathbb{R} \textit{ such that } \Phi(0) = 0,\ \Phi \textit{ is continuous,} \\ \textit{of class } C^1 \textit{ on } (-a, a), \textit{ with } \phi := \Phi' : (-a, a) \to \mathbb{R} \\ \textit{an increasing homeomorphism such that } \phi(0) = 0; \end{array}$$

$f : [0, T] \times \mathbb{R} \to \mathbb{R}$ is a continuous function with its primitive

$$F(t, x) = \int_0^x f(t, \xi) d\xi, \qquad ((t, x) \in [0, T] \times \mathbb{R})$$

satisfying the hypothesis

$$(H_F)\qquad \begin{array}{l} \text{there exists } \omega > 0 \text{ such that} \\ F(t, x) = F(t, x + \omega) \text{ for all } (t, x) \in [0, T] \times \mathbb{R}; \end{array}$$

and finally the forcing term $h : [0, T] \to \mathbb{R}$ is supposed to be continuous and satisfies

$$(H_h)\qquad \int_0^T h(t) dt = 0.$$

Of course, by a solution of (1.2) we mean a function $u \in C^1[0, T]$ with $||u'||_\infty < a$, $\phi(u') \in C^1[0, T]$ and (1.2) is satisfied.

Our main result is the following one.

Theorem 1.1. *If the hypotheses* (H_Φ), (H_F) *and* (H_h) *are satisfied, then* (1.2) *has at least two solutions not differing by a multiple of* ω.

Taking in (1.2), $\phi(s) = \frac{s}{\sqrt{1-s^2}}$ so that $\Phi(s) = 1 - \sqrt{1 - s^2}$, and $f(t, x) = -\mu \sin x$ so that $F(t, x) = \mu(\cos x - 1)$ and $\omega = 2\pi$, one has the following:

Corollary 1.2. *Problem* (1.1) *has at least two solutions not differing by a multiple of* 2π *for any forcing term* h *satisfying* (H_h) *and any* $\mu \neq 0$.

Our approach is variational and is based upon Szulkin's critical point theory [14] and some results given in [2]. The corresponding result for the one-dimensional curvature operator has been recently proved by Obersnel and Omari [12] using also Szulkin's critical point theory.

We point out that the approach of Mawhin and Willem [11] has an abstract formulation given by Pucci and Serrin in [13] and then the Pucci-Serrin's variant of the Mountain Pass Lemma has been generalized by Ghoussoub and Preiss in [8]. For Szulkin type functionals, the Ghoussoub-Preiss result is proved by Marano and Motreanu [10] assuming also the reflexivity of the space. In our case, we work in the space of continuous functions defined on a compact interval, which is not reflexive, and in order to avoid this difficulty we use a truncation strategy coming from the upper and lower solutions method.

2. Auxiliary results and notation

In this section we state some results from [2], which are the main tools in the proof of Theorem 1.1.

Let $g : [0, T] \times \mathbb{R} \to \mathbb{R}$ be a continuous function with its primitive defined by

$$G(t, x) = \int_0^x g(t, \xi) d\xi, \qquad ((t, x) \in [0, T] \times \mathbb{R}),$$

and consider the periodic boundary value problem

$$(\phi(u'))' = g(t,u), \quad u(0) - u(T) = 0 = u'(0) - u'(T). \tag{2.1}$$

We set $C := C[0,T]$, $L^\infty := L^\infty(0,T)$ and $W^{1,\infty} := W^{1,\infty}(0,T)$. The usual norm $\|\cdot\|_\infty$ is considered on C and L^∞, whereas in $W^{1,\infty}$ we consider the usual norm $\|u\|_{W^{1,\infty}} = \|u\|_\infty + \|u'\|_\infty$.

We decompose any $u \in C$ as follows:

$$u = \overline{u} + \widetilde{u}, \quad \overline{u} = \frac{1}{T}\int_0^T u(t)dt \quad \text{and} \quad \int_0^T \widetilde{u}(t)dt = 0.$$

Note that one has

$$||\widetilde{v}||_\infty \le T\|v'\|_\infty \quad \text{for all} \quad v \in W^{1,\infty}. \tag{2.2}$$

Let

$$K := \{v \in W^{1,\infty} \ : \ \|v'\|_\infty \le a, \quad v(0) = v(T)\}$$

and $\Psi : C \to (-\infty, +\infty]$ be defined by

$$\Psi(v) = \begin{cases} \int_0^T \Phi(v'), & \text{if } v \in K, \\ \\ +\infty, & \text{otherwise.} \end{cases}$$

Obviously, Ψ is proper and convex. On the other hand, as shown in [6] (see also [2]), Ψ is lower semicontinuous on C.

Next, let $\mathcal{G} : C \to \mathbb{R}$ be given by

$$\mathcal{G}(u) = \int_0^T G(t,u)\,dt, \quad u \in C.$$

A standard reasoning shows that $\mathcal{G}$ is of class C^1 on C and its derivative is given by

$$\langle \mathcal{G}'(u), v\rangle = \int_0^T g(t,u)v\,dt, \quad u, v \in C.$$

Following [2], we consider the energy functional associated to (2.1) given by

$$I : C \to (-\infty, +\infty], \qquad I = \Psi + \mathcal{G}.$$

Then I has the structure required by Szulkin's critical point theory [14]. Accordingly, a function $u \in C$ is *a critical point* of I if $u \in K$ and

$$\Psi(v) - \Psi(u) + \langle \mathcal{G}'(u), v - u\rangle \ge 0 \quad \text{for all } v \in C.$$

It is shown in [2] that if u is a critical point of I, then u is a solution of (2.1).

On the other hand, $\{u_n\} \subset K$ is a *(PS)-sequence* if $I(u_n) \to c \in \mathbb{R}$ and

$$\int_0^T [\Phi(v') - \Phi(u_n') + g(t,u_n)(v - u_n)]\,dt \ge -\varepsilon_n\|v - u_n\|_\infty \text{ for all } v \in K,$$

where $\varepsilon_n \to 0_+$. According to [14], the functional I is said to satisfy the *(PS)-condition* if any (PS)-sequence has a convergent subsequence in C. Note also that if $\{u_n\}$ is a (PS)-sequence, then, from [2] one has that

- the sequence $\{\int_0^T G(t,u_n)\,dt\}$ is bounded;
- if $\{\overline{u}_n\}$ is bounded, then $\{u_n\}$ has a convergent subsequence in C.

The next lemma is a direct consequence of [4, Theorem 3].

Lemma 2.1. *Let us assume that* (2.1) *has two solutions* α, β *such that* $\alpha(t) \le \beta(t)$ *for all* $t \in [0,T]$. *Let* $\gamma : [0,T] \times \mathbb{R} \to \mathbb{R}$ *be the continuous function defined by*

$$\gamma(t,x) = \begin{cases} \beta(t), & \text{if } x > \beta(t), \\ x, & \text{if } \alpha(t) \le x \le \beta(t), \\ \alpha(t), & \text{if } x < \alpha(t). \end{cases}$$

Consider the modified problem

$$(\phi(u'))' = g(t,\gamma(t,u)) + u - \gamma(t,u), \quad u(0) - u(T) = 0 = u'(0) - u'(T). \tag{2.3}$$

If u *is a solution of* (2.3), *then*

$$\alpha(t) \le u(t) \le \beta(t) \quad \textit{for all} \quad t \in [0,T],$$

and u *is a solution of* (2.1).

3. Proof of the main result

First of all, using the corresponding result for the periodic case of Corollary 1 in [2] one has that the energy functional I associated to (1.2) is bounded from below and there exists $u_0 \in K$ a minimizer for I, which is also a solution of (1.2). On the other hand, from (H_F) it follows that

$$I(u) = I(u + j\omega) \quad \text{for all} \quad u \in C, j \in \mathbb{Z}.$$

So, taking j sufficiently large, we can assume that u_0 is strictly positive, and one has that $u_1 := u_0 + \omega$ is a minimizer of I and also a solution of (1.2).

We associate to (1.2) the corresponding modified problem

$$\begin{aligned} &(\phi(u'))' = f(t,\gamma(t,u)) + h(t) + u - \gamma(t,u), \\ &u(0) - u(T) = 0 = u'(0) - u'(T), \end{aligned} \tag{3.1}$$

where in this case $\gamma : [0,T] \times \mathbb{R} \to \mathbb{R}$ is given by

$$\gamma(t,x) = \begin{cases} u_1(t), & \text{if } x > u_1(t), \\ x, & \text{if } u_0(t) \le x \le u_1(t), \\ u_0(t), & \text{if } x < u_0(t). \end{cases}$$

So, if u is a solution of (3.1), then by Lemma 2.1,

$$u_0(t) \le u(t) \le u_1(t) \quad \text{for all} \quad t \in [0,T] \tag{3.2}$$

and u is a solution of (1.2).

Next, let $J : C \to (-\infty, \infty]$ be the energy functional associated to the modified problem (3.1). So,

$$J(u) = \int_0^T \Phi(u') + \int_0^T A(t,u)dt \quad \text{for all} \quad u \in K,$$

where $A : [0,T] \times \mathbb{R} \to \mathbb{R}$ is given by

$$A(t,x) = \int_0^x f(t,\gamma(t,\xi))d\xi + xh(t) + \frac{x^2}{2} - \int_0^x \gamma(t,\xi)d\xi,$$

for all $(t,x) \in [0,T] \times \mathbb{R}$.

Let us note that if u is a critical point of J, then u is a solution of (3.1); hence u satisfies (3.2) and u is also a solution of (1.2).

Lemma 3.1. *The following hold true.*

(i) $J(u_0) = J(u_1)$.
(ii) $\lim_{|x|\to\infty} A(t,x) = +\infty$ *uniformly in* $t \in [0,T]$.
(iii) *The functional J is bounded from below and satisfies the (PS)-condition.*

Proof. (i) From (H_F) and the definition of γ we infer that

$$A(t,u_0(t)) = u_0(t)f(t,u_0(t)) + u_0(t)h(t) - \frac{u_0^2(t)}{2}$$

and

$$A(t,u_1(t)) = u_0(t)f(t,u_0(t)) + u_1(t)h(t) - \frac{u_0^2(t)}{2},$$

for all $t \in [0,T]$. On the other hand, using (H_h) we deduce that

$$\int_0^T u_0(t)h(t)dt = \int_0^T u_1(t)h(t)dt.$$

Hence

$$\int_0^T A(t,u_0(t))dt = \int_0^T A(t,u_1(t))dt,$$

which together with

$$u_0' = u_1'$$

implies that (i) holds true.

(ii) Using that γ is bounded, it follows that there exists $c_1 > 0$ such that

$$A(t,x) \geq \frac{x^2}{2} - c_1|x| \quad \text{for all} \quad (t,x) \in [0,T] \times \mathbb{R},$$

implying that (ii) holds true.

(iii) From (ii) we deduce immediately that J is bounded from below.

Now let $\{u_n\}$ be a (PS)-sequence. It follows that the sequence $\{\int_0^T A(t,u_n)dt\}$ is bounded. This, together with (2.2) and (ii), implies that $\{\overline{u}_n\}$ is bounded. Again by (2.2) and the fact that $\{u_n\} \subset K$, we have that $\{u_n\}$ is bounded in $W^{1,\infty}$. By the compact embedding of $W^{1,\infty}$ into C (see for example [5]), it follows that $\{u_n\}$ has a convergent subsequence in C and J satisfies the (PS)-condition. □

End of the proof of the main result. We conclude the proof by using an argument inspired by [12]. Using Lemma 3.1(iii) and Theorem 1.7 from [14], we deduce that there exists u_2, a critical point of J, such that

$$J(u_2) = \inf_C J.$$

We have two cases.

Case 1. If $u_2 \neq u_0$ and $u_2 \neq u_1$, then, using the fact that u_2 satisfies (3.2), it follows that u_2 is a solution of (1.2) such that $u_2 - u_0$ is not a multiple of ω.

Case 2. If $u_2 = u_0$ or $u_2 = u_1$, then using Lemma 3.1(i), it follows that u_0 and u_1 are also minimizers of J. Hence, using Lemma 3.1(iii) and [14, Corollary 3.3], we infer that there exists u_3, a critical point of J different from u_0 and u_1. Because u_3 is a critical point of J, one has that u_3 satisfies (3.2), and therefore u_3 is a solution of (1.2) such that $u_3 - u_0$ is not a multiple of ω. □

4. Final remarks about the Neumann problem

Let us consider the Neumann problem

$$[r^{N-1}\phi(u')]' = r^{N-1}[f(r,u)+h(r)], \quad u'(R_1)=0=u'(R_2), \tag{4.1}$$

where $0 \leq R_1 < R_2$, $N \geq 1$ is an integer and ϕ, f and h satisfy hypotheses (H_Φ), (H_F) and (H_h). Then, using the same strategy as in the periodic case, without any change and the corresponding results from [2] and [1], one has that (4.1) has at least two solutions not differing by a multiple of ω. The existence of at least one solution has been proved in [3, 2].

In particular, the Neumann problem

$$\operatorname{div}\left(\frac{\nabla v}{\sqrt{1-|\nabla v|^2}}\right) + \mu \sin u = h(|x|) \quad \text{in} \quad \mathcal{A}, \quad \frac{\partial v}{\partial \nu} = 0 \quad \text{on} \quad \partial\mathcal{A},$$

where $\mathcal{A} = \{x \in \mathbb{R}^N : R_1 \leq |x| \leq R_2\}$, has at least two classical radial solutions not differing by a multiple of ω, for any $\mu \neq 0$ and any $h \in C$ such that

$$\int_{\mathcal{A}} h(|x|)\,dx = 0.$$

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