PROCEEDINGS OF THE
AMERICAN MATHEMATICAL SOCIETY
Volume 140, Number 8, August 2012, Pages 2731–2738
S 0002-9939(2011)11112-2
Article electronically published on November 30, 2011

LIOUVILLE TYPE THEOREMS FOR NONLINEAR ELLIPTIC EQUATIONS ON THE WHOLE SPACE $\mathbb{R}^N$

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(Communicated by Walter Craig)

Abstract. The aim of this paper is to study the properties of the solutions of $\Delta_p u + f_1(u) - f_2(u) = 0$ in all $\mathbb{R}^N$. We obtain Liouville type boundedness for the solutions. We show that $|u| \leq (\frac{\alpha}{\beta})^{\frac{1}{m-q+1}}$ on $\mathbb{R}^N$, under the assumptions $f_1(u) \leq \alpha u^{p-1}$ and $f_2(u) \geq \beta u^m$, for some $0 < \alpha \leq \beta$ and $m > q-1 \geq p-1 > 0$. If u does not change sign, we prove that u is constant.

1. Introduction

This paper deals with the quasilinear elliptic equation

$$\Delta_p u + f(u) = 0 \qquad \text{on } \mathbb{R}^N, \ \ N \geq 3, \tag{1.1}$$

where $\Delta_p u = \operatorname{div} |\nabla u|^{p-2} \nabla u$, $p > 1$, and f is a locally Lipschitz continuous function defined by $f = f_1 - f_2$. We assume that

$$\forall u \geq 0, f_1(u) \leq \alpha u^{q-1}, \quad f_2(u) \geq \beta u^m, \quad f(0) = 0, \tag{1.2}$$

for some

$$0 < \alpha \leq \beta \quad \text{and} \quad m \geq q-1 \geq p-1 > 0.$$

Equation (1.1) arises in many nonlinear phenomena, for instance, in the theory of quasi-regular and quasi-conformal mappings (see [15], [16], [18]) and in mathematical modeling of non-Newtonian fluids; see [1], [7], [13] and [14] for a discussion of the physical background. The equation also has a large and well-known theoretical literature, some of which will be particularly discussed below.

Another example included in this study when $p = 2$ is the Ginzburg-Landau equation

$$\Delta u + u(1 - |u|^2) = 0. \tag{1.3}$$

Liouville type results were proved for (1.3) by Brezis in [4], Hervé in [12], and Damascelli in [6]. See also Caffarelli, Garofalo and Segala in [5] and the references therein.

Received by the editors October 22, 2010 and, in revised form, February 14, 2011 and March 4, 2011.

1991 *Mathematics Subject Classification.* Primary 34-XX, 35-XX.

Key words and phrases. Liouville type results, supersolution, subsolution, comparison principle.

Du and Ma in [10] considered problem (1.1) where $p = 2$ under the following assumptions: f is a real-valued C^1 function satisfying

$$(H_1)\begin{cases} \bullet \quad f(1) = f(0) = 0, & \\ \bullet \quad f(u) > 0, & \forall u \in (0,1), \\ \bullet \quad f(u) < 0, & \forall u > 1, \end{cases}$$

and the Keller-Osserman condition:
For some large constant $M > 1$, we have

$$(H_2)\begin{cases} \bullet \quad \lim\limits_{u\to 0^+} \dfrac{f(u)}{u^{1+\frac{2}{n}}} \in (0,+\infty], & \\ \bullet \quad f(u) \le g(u) < 0, & \text{on } [M,+\infty), \\ \bullet \quad g \text{ is decreasing in } [M,+\infty) & \text{and} \quad \int_M^\infty (\int_M^u |g(s)|ds)^{\frac{-1}{2}} du < \infty. \end{cases}$$

Under the assumptions (H_1) and (H_2), Du and Ma proved that every positive solution in $C^2(\mathbb{R}^N)$ of $\Delta u + f(u) = 0$ is a constant $(u = 1)$.

Several results have been obtained starting with the famous paper by Gidas and Nirenberg [11] where, among other things, it is proved that if Ω is a ball, the solutions of the problem $\Delta u = \lambda u - u^p$ are radially symmetric and strictly radially decreasing and (see in [3]) when Ω is unbounded.

Serrin and Zou [17] considered the case when f is subcritical, i.e. when $1 < p < N$ and there exists a number $1 < \alpha < \frac{pN}{N-p}$ such that

$$f(u) \ge 0, \quad (\alpha - 1)f(u) - uf^{'}(u) \ge 0, \quad \text{for} \quad u > 0.$$

They proved the following:

i) If $f(u) > 0$ for all $u > 0$, then every bounded solution of (1.1) is trivial.

ii) If

$$\text{either} \;\; N = 2 \;\; \text{and} \;\; p > \frac{1}{4}(1+\sqrt{17}) \;\; \text{or } N \in [3, 2p), \;\; p > \frac{3}{2},$$

then every solution of (1.1) is constant.

Du and Guo [9] considered problem (1.1) when $f(u) = a|u|^{p-2}u - b|u|^{q-1}u$, with $q > p - 1 > 0$, a is a real parameter, $b > 0$ is a constant $(0 < \alpha < 1,\ N \ge 2)$. They showed that if $u \in C^1(\mathbb{R}^N)$ is a nonnegative solution of the equation (1.1), then u must be a constant.

Du and Guo [8] considered problem (1.1) under the following assumptions on f:
(F1) For some $a > 0$,

$$f(0) = f(a) = 0, \;\; f(s) > 0 \;\; \text{in} \;\; (0,a), \;\; f(s) < 0 \;\; \text{in} \;\; (a,\infty).$$

(F2) For some small $\delta > 0$, there exists a constant $\sigma > 0$ such that

$$f(s) \ge \sigma s^{p-1}, \quad \forall s \in (0,\delta).$$

(F3) For some large $M > 0$, there exists a continuous function $g(s)$ such that

$$f(s) \le g(s) < 0, \;\; \forall s \in [M,\infty),$$

$g(s)$ is nonincreasing in $[M,\infty)$ and $\int_M^\infty (\int_M^u |g(s)|ds)^{\frac{-1}{p}} du < \infty$. Then any nonnegative solution of (1.1) is a constant.

Zhao [19] considered the problem

$$\Delta_p u + u^{p-1} - u^m = 0, \quad \text{on} \;\; \mathbb{R}^N, \quad \text{when} \quad m > p - 1 > 0.$$

We notice that if we take $f_1(u) = \alpha u^{q-1}$ and $f_2(u) = \beta u^m$, (1.1) is equivalent to

$$\Delta_p u + \alpha u^{q-1} - \beta u^m = 0 \quad \text{on} \quad \mathbb{R}^N,\ N > 1. \tag{1.4}$$

We remark that if $q > p$, assumption (F2) is not fulfilled.

Remark 1.1. We notice that the case $q \neq p$ was not treated in the literature even when $p = 2$.

In this paper, we will study the qualitative properties of the solutions of problem (1.4) and we will deduce those of (1.1). In particular we will prove the following Liouville type theorems.

Theorem 1.1. *Assume that $m > q - 1$. If u is a solution of the problem* (1.1) *on $\mathbb{R}^N$, then $u \leq (\frac{\alpha}{\beta})^{\frac{1}{m-q+1}}$. Moreover, if f is odd, we have $|u| \leq (\frac{\alpha}{\beta})^{\frac{1}{m-q+1}}$.*

Theorem 1.2 (Liouville type property)**.** *Assume that $m > q-1$. Suppose that u is a positive solution of* (1.4) *such that $\inf_{\mathbb{R}^N} u > 0$. Then u is a constant, $u = (\frac{\alpha}{\beta})^{\frac{1}{m-q+1}}$.*

Theorem 1.3. *If $m = q - 1$, $0 < \alpha < \beta$, $m > p$, then we get $u = 0$.*

A similar result for Theorem 1.2 is obtained by Zhao [19] for the case $q = p$ and $\alpha = \beta = 1$ when $m > p - 1 > 0$.

The results of this paper are based on a comparison principle. Throughout this paper, $\mathbb{R}^N$ is the real Euclidean N-space, $N \geq 2$. For $x_0 \in \mathbb{R}^N$, we denote by $B(x_0, R)$ the ball with center x_0 and radius R and $S(x_0, R) = \{x \in \mathbb{R}^N,\ |x - x_0| = R\}$.

$u \vee v$ and $u \wedge v$ designate respectively the supremum and the infinimum of u and v.

If $1 < p < N$ is a real number, we denote by $p^\star = \frac{NP}{N-p}$, $p' = \frac{p}{p-1}$.

2. Preliminaries

We start this section with some preliminary results. For the reader's convenience, we recall the definition of the supersolution and the subsolution of (1.1). To this end, let Ω denote a bounded domain of $\mathbb{R}^N$ and f a function satisfying condition (1.2).

Definition 2.1. Let $p > 1$. Consider the problem

$$\Delta_p u + f(u) = 0, \quad \text{in } \Omega. \tag{2.1}$$

We say that a function $u \in W^{1,p}_{loc}(\Omega)$ is a weak supersolution (resp. subsolution) of (2.1) if

$$\begin{cases} f(u) \in (L^{p^\star})', \\ \int_\Omega |\nabla u|^{p-2} \nabla u \nabla \varphi - \int_\Omega f(u)\varphi \geq 0 \quad (\text{resp.} \quad \leq 0) \end{cases}$$

for every nonnegative function $\varphi \in W^{1,p}_0(\Omega)$.

A function u is a solution of (2.1) if and only if u is a supersolution and a subsolution of (2.1).

Next, we recall the following result proved in [2] (see Corollary 6.1).

Proposition 2.1. *Let u and v be two subsolutions of* (2.1) *in Ω. Then $u \vee v$ is also a subsolution. A similar statement holds for the minimum of two supersolutions.*

Proposition 2.2. *Let u be a subsolution of* (2.1) *and v be a supersolution of* (2.1) *such that*

$$\liminf_{x\to z} v(x) \geq \limsup_{x\to z} u(x) \tag{2.2}$$

for all $z \in \partial\Omega$ and both sides of the inequality are not simultaneously $+\infty$ or $-\infty$. Moreover assume that $f(v) \geq f(u)$ on Ω. Then $v \geq u$ on Ω.

Proof. Let $\epsilon > 0$ and K be a compact subset of Ω such that $u-v \leq \epsilon$ on $\Omega\setminus K$. Then the function $\varphi = (u-v-\epsilon)^+ \in W_0^{1,p}(\Omega)$. Using the fact that u is a subsolution and v is a supersolution of (2.1) and testing in (2.1) by $(\varphi = u-v-\epsilon)^+$, we obtain

$$\int_{u>v+\epsilon} |\nabla u|^{p-2}\nabla u \nabla\varphi dx - \int_{u>v+\epsilon} f(u)\varphi dx \leq 0 \tag{2.3}$$

and

$$\int_{u>v+\epsilon} |\nabla v|^{p-2}\nabla v \nabla\varphi dx - \int_{u>v+\epsilon} f(v)\varphi dx \geq 0. \tag{2.4}$$

Denote by

$$I_1 = \int_{u>v+\epsilon} (|\nabla v|^{p-2}\nabla v - |\nabla u|^{p-2}\nabla u)\nabla(u-v)dx \tag{2.5}$$

and

$$I_2 = \int_{u>v+\epsilon} (f(u)-f(v))(u-v-\epsilon)dx. \tag{2.6}$$

Combining equations (2.3) and (2.4), we get $0 \leq I_1 + I_2$.

On the other hand, we have

$$I_1 \leq 0 \quad \text{and} \quad I_2 \leq 0.$$

Hence, we get $(u-v-\epsilon)^+ = 0$ a.e. in Ω. Letting $\epsilon \to 0$, we have $u \leq v$ a.e. in Ω. □

Corollary 2.1. *Let $\delta > 0$. Assume that f is nonincreasing on $[\delta, \infty)$ and let u be a subsolution of* (2.1) *and v be a supersolution of* (2.1) *and such that*

$$\liminf_{x\to z}(v(x)-u(x)) \geq 0 \quad \forall z \in \partial\Omega$$

and

$$v \geq \delta \quad \text{on} \quad \Omega.$$

Then,

$$v \geq u \quad \text{on} \quad \Omega.$$

Proof. Assume that the set $U = \{x \in \Omega \ : \ v(x) < u(x)\}$ is nonempty. We have $\liminf_{x\to z}(v-u)(x) \geq 0 \quad \forall z \in \partial U$. Since $f(u) \leq f(v)$ on U, we get from Proposition 2.2 that $v \geq u$ on U, which is impossible. Hence $u \leq v$ on Ω. □

In the sequel, we consider a real $R \geq 1$ and $x_0 \in \mathbb{R}^N$.

Proposition 2.3. *Let $R \geq 1$ and let*

$$v_\lambda(x) = \frac{\lambda}{(R^2 - |x-x_0|^2)^{\frac{p}{m-p+1}}} \quad \forall x \in B(x_0, R), \tag{2.7}$$

where λ is a positive constant. We have

$$\lim_{x\to z} v_\lambda(x) = +\infty \quad \forall z \in \partial B(x_0, R) \tag{2.8}$$

and

$$\Delta_p v_\lambda + \alpha v_\lambda^{q-1} - \beta v_\lambda^m \le 0 \quad in \quad B(x_0, R), \tag{2.9}$$

for some $\lambda > 1$.

Proof. For each $x \in B(x_0, R)$, we denote $r = |x - x_0|$ and we consider the v_λ defined by $v_\lambda(x) = \frac{\lambda}{(R^2-r^2)^{\frac{p}{m-p+1}}}$, for some $\lambda > 1$. So it is an easy task to show that v_λ fulfills the first part of the proposition. By means of a straightforward calculation we verify that (2.9) is equivalent to

$$\beta\lambda^{m-p+1} - \alpha\lambda^{q-p}(R^2 - r^2)^{p\frac{m-q+1}{m-p+1}} \ge \delta_1 r^p + \delta_2(R^2 - r^2)r^{p-2}, \tag{2.10}$$

where $\delta_1 = (\frac{2}{m-p+1})^p(m+1)(p-1)p^{p-1}$ and $\delta_2 = (N+p-2)(\frac{2p}{m-p+1})^{p-1}$.

Clearly (2.10) holds if

$$\beta\lambda^{m-p+1} - \alpha\lambda^{q-p}R^{2p\frac{m-q+1}{m-p+1}} \ge (\delta_1 + \delta_2)R^p. \tag{2.11}$$

Case $m > q - 1$.

Set

$$h(\lambda) = \beta\lambda^{m-p+1} - \alpha\lambda^{q-p}R^{2p\frac{m-q+1}{m-p+1}}, \tag{2.12}$$

and for some $\gamma > 0$, let

$$\lambda_0 = (\frac{\alpha}{\beta})^{\frac{1}{m-q+1}}[R^{2p} + \gamma R^p]^{\frac{1}{m-p+1}}. \tag{2.13}$$

We obtain

$$h(\lambda_0) = \beta(\frac{\alpha}{\beta})^{\frac{m-p+1}{m-q+1}}(R^{2p} + \gamma R^p) - \alpha(\frac{\alpha}{\beta})^{\frac{q-p}{m-q+1}}R^{2p}(1 + \frac{\gamma}{R^p})^{\frac{q-p}{m-p+1}}.$$

Using the fact that $m \ge q - 1$ and $(1 + \frac{\gamma}{R^p})^{\frac{q-p}{m-p+1}} \le 1 + \frac{q-p}{m-p+1}\frac{\gamma}{R^p}$, we get

$$h(\lambda_0) \ge \gamma R^p\alpha(\frac{\alpha}{\beta})^{\frac{q-p}{m-q+1}}(\frac{m-q+1}{m-p+1}).$$

Then, it is easily checked that (2.9) holds if

$$\gamma \ge \frac{1}{\alpha}(\frac{\beta}{\alpha})^{\frac{q-p}{m-q+1}}(\frac{m-p+1}{m-q+1})(\delta_1 + \delta_2). \tag{2.14}$$

Case $m = q - 1$.

We should have $(\beta-\alpha)\lambda^{m-p+1} \ge (\delta_1+\delta_2)R^p$. So (2.9) holds if $\lambda \ge [\frac{\delta_1+\delta_2}{\beta-\alpha}R^p]^{\frac{1}{m-p+1}}$. □

Remark 2.1. Let us observe that for every $\lambda \ge \lambda_0$ given in (2.13), the function $v_\lambda(x) = \frac{\lambda}{(R^2-r^2)^{\frac{p}{m-p+1}}}$ is a supersolution of (2.9). Indeed, since the function h defined in (2.12) is nondecreasing on $[(\frac{\alpha}{\beta})^{\frac{1}{m-q+1}}(\frac{q-p}{m-p+1})^{\frac{m-q+1}{m-p+1}}R^{\frac{2p}{m-q+1}}, \infty)$, then for every $\lambda \ge \lambda_0$, (2.11) holds provided γ satisfies (2.14). Moreover, we have for $\lambda \ge \lambda_0$,

$$v_\lambda \ge v_{\lambda_0} = \frac{(\frac{\alpha}{\beta})^{\frac{1}{m-q+1}}[R^{2p} + \gamma R^p]^{\frac{1}{m-p+1}}}{(R^2 - r^2)^{\frac{p}{m-p+1}}} \ge (\frac{\alpha}{\beta})^{\frac{1}{m-q+1}}, \text{ for } m > q - 1. \tag{2.15}$$

Proposition 2.4. *Let $x_0 \in \mathbb{R}^N$ and let u be a subsolution of* (1.4) *on $B(x_0, R)$ for some $R > 1$. Then*

$$u(x_0) \leq (\frac{\alpha}{\beta})^{\frac{1}{m-q+1}}(1 + \frac{\gamma}{R^p})^{\frac{1}{m-p+1}},$$

where γ is the constant satisfying (2.14)*, if $m > q-1$ and*

$$u(x_0) \leq (\frac{\delta_1 + \delta_2}{\alpha - \beta})^{\frac{1}{m-p+1}} R^{\frac{-p}{m-p+1}}, \quad \textit{if} \;\; m = q-1.$$

Proof. Case $m > q-1$. Let $\lambda_0 = (\frac{\alpha}{\beta})^{\frac{1}{m-q+1}}[R^{2p} + \gamma R^p]^{\frac{1}{m-p+1}}$ satisfy (2.11). Then the function v_{λ_0} is a supersolution of (1.4) on $B(x_0, R)$ satisfying $\lim\limits_{x \to z} v_{\lambda_0}(x) = +\infty \geq \limsup\limits_{x \to z} u(x), \quad \forall z \in S(x_0, R)$. On the other hand, due to equation (2.15), we have $v_{\lambda_0} \geq (\frac{\alpha}{\beta})^{\frac{1}{m-q+1}}$. Since the function f defined by $f(y) = \alpha y^{q-1} - \beta y^m$ is decreasing on $[(\frac{\alpha}{\beta})^{\frac{1}{m-q+1}}, \infty)$, it follows from Proposition 2.2 that

$$u(x_0) \leq v_{\lambda_0}(x_0) = (\frac{\alpha}{\beta})^{\frac{1}{m-q+1}}(1 + \frac{\gamma}{R^p})^{\frac{1}{m-p+1}}.$$

If $m = q-1$, the function $f(y) = (\alpha - \beta)y^m$ is decreasing. Then by Proposition 2.2 we get that $u(x_0) \leq v_{\lambda_0}(x) = (\frac{\delta_1+\delta_2}{\alpha-\beta})^{\frac{1}{m-p+1}} R^{\frac{-p}{m-p+1}}$. □

An easy consequence of Proposition 2.4 is the following corollary.

Corollary 2.2. *If u is a subsolution of the problem* (1.4) *on $\mathbb{R}^N$, then*

$$u \leq (\frac{\alpha}{\beta})^{\frac{1}{m-q+1}} \quad \textit{when} \;\; m > q-1$$

and

$$u \leq 0 \quad \textit{when} \;\; m = q-1.$$

Now, we are able to prove our first two theorems.

Proof of Theorem 1.1. First, we remark from condition (1.2) that 0 is a solution of (1.1). Let u be a solution of the problem (1.1). By means of Proposition 2.1, we get that $\tilde{u} = u \vee 0$ is also a subsolution of problem (1.1). Using (1.2), we get that $\tilde{u}$ is a subsolution of (1.4). Thus, we obtain from Proposition 2.4 that $\tilde{u} \leq \left(\frac{\alpha}{\beta}\right)^{\frac{1}{m-q+1}}$, which leads to $u \leq \left(\frac{\alpha}{\beta}\right)^{\frac{1}{m-q+1}}$.

The assumption that f is odd implies that if f is a solution of (1.1), then $-u$ is a solution of (1.1), whence $-u \leq \left(\frac{\alpha}{\beta}\right)^{\frac{1}{m-q+1}}$. This proves Theorem 1.1. □

Proof of Theorem 1.3. This is trivial, using the same argument as in the proof of Theorem 1.1. □

Proof of Theorem 1.2. The basic ingredients in the proof consist of the following lemma. This lemma is given in much more general form than is required in the proof of Theorem 1.2.

The proof of Theorem 1.2 relies essentially on a comparison principle for concave sublinear problems and involves boundary blow-up solutions. □

Lemma 2.1 (Comparison principle)**.** *Suppose that Ω is a bounded domain in $\mathbb{R}^N$. Let $u_1, u_2 \in C^1(\Omega)$ be positive in Ω and satisfy in the sense of distributions*

$$-\Delta_p u_1 - \alpha_1 u_1^{p-1} + \beta_1 u_1^r \geq 0 \geq -\Delta_p u_2 - \alpha_1 u_2^{p-1} + \beta_1 u_2^r, \quad x \in \Omega \tag{2.16}$$

and $\limsup_{x\to\partial\Omega} u_2^p - u_1^p \leq 0$, *where $r > p-1 > 0$ and α_1, β_1 are positive constants. Then $u_2 \leq u_1$ in Ω.*

Proof. This follows from a simple modification of the arguments in the proof of Proposition 2.2 in [9]. Using the notation there, we see that it suffices to show that the function $\frac{g(s)}{s^{p-1}}$ is increasing, where $g(s) = s^r$. □

Proof of Theorem 1.2. Suppose that $\lambda > 0$ and let u be an arbitrary positive solution of (1.4) such that $\tau = \inf_{\mathbb{R}^N} u > 0$. We will show that $u(x_0) = (\frac{\alpha}{\beta})^{\frac{1}{m-q+1}}$, for x_0 an arbitrary point in $\mathbb{R}^N$. Let us define

$$u_\lambda(x) = u((x_0 + \lambda(x - x_0)).$$

It is easily checked that u_λ satisfies

$$-\Delta_p u_\lambda = \lambda^{p-1}(\alpha u_\lambda^{q-1} - \beta u_\lambda^m) = \lambda^{p-1} u_\lambda^{q-p}(\alpha u_\lambda^{p-1} - \beta u_\lambda^{m-q+p}). \tag{2.17}$$

Taking into account that $\tau = \inf_{\mathbb{R}^N} u > 0$, we obtain

$$-\Delta_p u_\lambda \geq \lambda^{p-1}\tau^{q-p}(\alpha u_\lambda^{p-1} - \beta u_\lambda^{m-q+p}). \tag{2.18}$$

That is,

$$\Delta_p u_\lambda + \lambda^{p-1}\tau^{q-p}(\alpha u_\lambda^{p-1} - \beta u_\lambda^{m-q+p}) \leq 0.$$

Next, we apply Theorem 2.5 in Du and Guo [8] for $f(s) = \tau^{q-p}(\alpha s^{p-1} - \beta s^{m-q+p})$. We get that for λ large the problem

$$(\star)\begin{cases} \Delta_p v + \lambda^{p-1}\tau^{q-p}(\alpha v^{p-1} - \beta v^{m-q+p}) = 0 & \text{in } B(x_0, R), \\ v = 0 & \text{on } \partial B(x_0, R) \end{cases}$$

has a unique positive solution v_λ such that $v_\lambda(x) \to (\frac{\alpha}{\beta})^{\frac{1}{m-q+1}}$ uniformly on any compact subset of Ω as $\lambda \to \infty$.

From Lemma 2.1 we have $u_\lambda \geq v_\lambda$ on Ω. Then

$$u(x_0) = u_\lambda(x_0) \geq v_\lambda(x_0) \to (\frac{\alpha}{\beta})^{\frac{1}{m-q+1}}, \quad \text{as } \lambda \to \infty.$$

On the other hand, we have by Theorem 1.1 that $u \leq (\frac{\alpha}{\beta})^{\frac{1}{m-q+1}}$. Therefore we must have $u = (\frac{\alpha}{\beta})^{\frac{1}{m-q+1}}$, as required. □

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