PROCEEDINGS OF THE
AMERICAN MATHEMATICAL SOCIETY
Volume 140, Number 8, August 2012, Pages 2817–2824
S 0002-9939(2011)11146-8
Article electronically published on December 28, 2011

ISOMETRIES OF THE ZYGMUND F-ALGEBRA

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(Communicated by Richard Rochberg)

ABSTRACT. In his monograph A. Zygmund introduced the space $N\log^\alpha N$ $(\alpha > 0)$ of holomorphic functions on the unit ball that satisfy

$$\sup_{0\le r<1}\int_{\mathbb{S}}\varphi_\alpha(\log(1+|f(r\zeta)|))d\sigma(\zeta) < \infty,$$

where $\varphi_\alpha(t) = t\{\log(\gamma_\alpha + t)\}^\alpha$ for $t \in [0,\infty)$ and $\gamma_\alpha = \max\{e, e^\alpha\}$. In 2002, O.M. Eminyan provided some basic properties of $N\log^\alpha N$. In this paper we will characterize injective and surjective linear isometries of $N\log^\alpha N$. As an application, we will consider isometrically equivalent composition operators or multiplication operators on $N\log^\alpha N$.

1. Introduction and main result

Throughout this paper, let $\mathbb{B}$ and $\mathbb{S}$ denote the open unit ball and the unit sphere in the N-dimensional complex Euclidean space $\mathbb{C}^N$, respectively, and $d\sigma$ the normalized Lebesgue measure on $\mathbb{S}$.

Let X be a space of all holomorphic functions on some domain. The studies on linear isometries of X have been studied since the 1960s. When X is the Hardy space H^p $(0 < p \le \infty, p \ne 2)$ on the unit disc, D. deLeeuw, W. Rudin, and J. Wermer [16] $(p = 1, \infty)$ and F. Forelli [6] $(1 \le p < \infty)$ characterized the linear isometries. J. Cima and W.R. Wogen [2] obtained the form of the isometries of the Bloch space. Also W. Hornor and J.E. Jamison [11] considered the isometries of the Dirichlet space and the $\mathcal{S}^p$-space. For the details on these studies, we can also refer to the monograph [5]. For the several variables case, Forelli [7] and Rudin [17] have determined the injective and/or surjective isometries of H^p. For the case when X is the weighted Bergman spaces A^p_α $(0 < p < \infty, p \ne 2)$, the isometries were completely characterized in a sequence of papers by C.J. Kolaski [13, 14, 15]. By these works we see that the isometries on these holomorphic function spaces are described as weighted composition operators defined by $\Psi C_\Phi(f) = \Psi \cdot (f \circ \Phi)$ for some holomorphic function Ψ and holomorphic self-map Φ of the unit ball, which is one of the reasons why these operators have been investigated so much recently in the settings of the unit ball. The case when X is not a Banach space has also been studied by many authors. The *Smirnov class* N^* and the *Privalov space* N^p $(1 < p < \infty)$ which are contained in the *Nevanlinna class* N are examples of such spaces. These types of spaces are F-spaces in the sense of Banach with respect to a suitable metric on them. K. Stephenson [19], Y. Iida and N. Mochizuki [12], and A.V. Subbotin [21] have studied linear isometries of these spaces. Their works

Received by the editors October 1, 2010 and, in revised form, March 16, 2011.
2010 *Mathematics Subject Classification.* Primary 32A37; Secondary 47B33.
Key words and phrases. Isometries, Zygmund F-algebra, composition operators.

showed that the injective isometries are weighted composition operators induced by some inner functions and inner maps of $\mathbb{B}$ whose radial limit maps satisfy a measure-preserving property.

Motivated by these works, in this paper we will investigate injective and surjective linear isometries of the Zygmund F-algebra.

Take an arbitrary $\alpha > 0$ and consider the function $\varphi_\alpha(t) = t\{\log(\gamma_\alpha + t)\}^\alpha$ for $t \in [0, \infty)$ where $\gamma_\alpha = e$ if $0 < \alpha \leq 1$, $\gamma_\alpha = e^\alpha$ if $\alpha > 1$. The *Zygmund F-algebra* $N\log^\alpha N$ consists of holomorphic functions f on $\mathbb{B}$ for which

$$\sup_{0 \leq r < 1} \int_{\mathbb{S}} \varphi_\alpha(\log^+ |f(r\zeta)|) d\sigma(\zeta) < \infty,$$

where $\log^+ x = \max\{0, \log x\}$ for $x \geq 0$. It is easily verified that the above condition is equivalent to the condition

$$\sup_{0 \leq r < 1} \int_{\mathbb{S}} \varphi_\alpha(\log(1 + |f(r\zeta)|)) d\sigma(\zeta) < \infty.$$

This class was considered by A. Zygmund [22] first. Recently O.M. Eminyan [4] studied linear space properties of this class. Since the function $\varphi_\alpha(\log(1 + x))$ satisfies

$$\varphi_\alpha(\log(1 + x)) \leq (\log \gamma_\alpha)^\alpha x \quad \text{for } x \geq 0, \tag{1}$$

we see that the inclusion $H^1 \subset N\log^\alpha N$ holds. More precisely it is known that it holds the following relation:

$$\bigcup_{p>0} H^p \subset N\log^\alpha N \subset N^* \subset N.$$

This implies that the boundary function f^* exists for any $f \in N\log^\alpha N$. By using this boundary value of f, we can define the quasi-norm $\|f\|_\alpha$ on $N\log^\alpha N$ by

$$\|f\|_\alpha = \int_{\mathbb{S}} \varphi_\alpha(\log(1 + |f^*(\zeta)|)) d\sigma(\zeta).$$

Since this quasi-norm satisfies the triangle inequality, $d_\alpha(f, g) := \|f - g\|_\alpha$ defines a translation invariant metric on $N\log^\alpha N$. So $N\log^\alpha N$ is an F-space in the sense of Banach with respect to this metric d_α. Moreover Eminyan [4] proved that $N\log^\alpha N$ forms an F-algebra with respect to d_α. We will consider a linear isometry of $N\log^\alpha N$ in this metric. The following is the main result in this paper.

Theorem 1. *If T is a linear isometry of $N\log^\alpha N$ into itself, then there exist an inner function Ψ and an inner map Φ on $\mathbb{B}$ which Φ^* satisfies the measure-preserving property on $\mathbb{S}$ such that $T = \Psi C_\Phi$ on $N\log^\alpha N$.*

Conversely, for given such Ψ and Φ, the weighted composition operator ΨC_Φ is an injective linear isometry of $N\log^\alpha N$.

2. Proofs of main result

In this section we will prove Theorem 1. As a corollary, we also give the complete characterization of the surjective isometry of $N\log^\alpha N$. To prove Theorem 1 we need some lemmas.

Lemma 1. *If T is a linear isometry of $N\log^\alpha N$, then the restriction of T to H^1 is also a linear isometry of H^1 into H^1.*

Proof. Take an $f \in H^1$ and put $g = Tf$. For each positive integer m we have $g/m = T(f/m)$, and so we obtain

$$\int_{\mathbb{S}} \varphi_\alpha\left(\log\left(1+\frac{|f^*(\zeta)|}{m}\right)\right) d\sigma(\zeta) = \int_{\mathbb{S}} \varphi_\alpha\left(\log\left(1+\frac{|g^*(\zeta)|}{m}\right)\right) d\sigma(\zeta). \tag{2}$$

By inequality (1), we have

$$m\varphi_\alpha\left(\log\left(1+\frac{|f^*(\zeta)|}{m}\right)\right) \le (\log\gamma_\alpha)^\alpha |f^*(\zeta)|,$$

for each positive integer m and almost all $\zeta \in \mathbb{S}$. By using the definition of φ_α, we see that

$$\lim_{m\to\infty} m\varphi_\alpha\left(\log\left(1+\frac{|f^*(\zeta)|}{m}\right)\right) = (\log\gamma_\alpha)^\alpha |f^*(\zeta)|,$$

for almost all $\zeta \in \mathbb{S}$. The Lebesgue dominated convergence theorem gives

$$\lim_{m\to\infty} \int_{\mathbb{S}} m\varphi_\alpha\left(\log\left(1+\frac{|f^*(\zeta)|}{m}\right)\right) d\sigma(\zeta) = (\log\gamma_\alpha)^\alpha \int_{\mathbb{S}} |f^*(\zeta)| d\sigma(\zeta). \tag{3}$$

On the other hand, Fatou's lemma, (2) and (3) give

$$\begin{aligned}(\log\gamma_\alpha)^\alpha \int_{\mathbb{S}} |g^*(\zeta)| d\sigma(\zeta) &\le \liminf_{m\to\infty} \int_{\mathbb{S}} m\varphi_\alpha\left(\log\left(1+\frac{|g^*(\zeta)|}{m}\right)\right) d\sigma(\zeta) \\ &= (\log\gamma_\alpha)^\alpha \int_{\mathbb{S}} |f^*(\zeta)| d\sigma(\zeta),\end{aligned}$$

and so $g \in H^1$. By applying the Lebesgue dominated convergence theorem once again, we have

$$\lim_{m\to\infty} \int_{\mathbb{S}} m\varphi_\alpha\left(\log\left(1+\frac{|g^*(\zeta)|}{m}\right)\right) d\sigma(\zeta) = (\log\gamma_\alpha)^\alpha \int_{\mathbb{S}} |g^*(\zeta)| d\sigma(\zeta). \tag{4}$$

By (2), (3) and (4), we see that T is a linear isometry of H^1 into H^1. □

Lemma 2. *There exist a bounded continuous function θ_α on $[0,\infty)$ and a positive constant K_α such that*

$$\varphi_\alpha(\log(1+x)) = (\log\gamma_\alpha)^\alpha x - K_\alpha x^2 + x^3\theta_\alpha(x) \quad \textit{for } x \in [0,\infty).$$

Proof. By the application of Taylor's theorem of $\varphi_\alpha(\log(1+x))$, we have

$$\varphi_\alpha(\log(1+x)) = \omega'_\alpha(0)x + \frac{\omega''_\alpha(0)}{2!}x^2 + \frac{\omega'''_\alpha(0)}{3!}x^3 + R_4(x),$$

where $\omega_\alpha(x) = \varphi_\alpha(\log(1+x))$ and $R_4(x)$ denotes the remainder term of order 4. Since we see that $\omega'_\alpha(0) = (\log\gamma_\alpha)^\alpha$ and $\omega''_\alpha(0)/2!$ is equal to $\frac{2\alpha-e}{2e}(<0)$ if $0<\alpha\le 1$, $\alpha^\alpha\frac{2-e^\alpha}{2e^\alpha}(<0)$ if $\alpha > 1$, we put

$$\theta_\alpha(x) = \frac{\varphi_\alpha(\log(1+x)) - (\log\gamma_\alpha)^\alpha x + K_\alpha x^2}{x^3},$$

where $K_\alpha = -\omega''_\alpha(0)/2!$. Since $R_4(x)/x^3 \to 0$ as $x \to 0^+$, $\theta_\alpha(x)$ has a finite limit $\omega'''_\alpha(0)/3!$ as $x \to 0^+$, and so this θ_α is a desired function. □

Proof of Theorem 1. Assume that T is a linear isometry of $N\log^{\alpha}N$. Since T is also an isometry of H^1 by Lemma 1, Rudin's theorem [17] implies that $T = \Psi C_{\Phi}$ on H^1 where $\Psi = T(1)$ and Φ is an inner map of $\mathbb{B}$ which satisfies

$$\int_{\mathbb{S}} h\, d\sigma = \int_{\mathbb{S}} (h \circ \Phi^*)|\Psi^*| d\sigma, \tag{5}$$

for every bounded Borel function h on $\mathbb{S}$.

First we will prove that $T = \Psi C_{\Phi}$ on $N\log^{\alpha}N$. Fix an $f \in N\log^{\alpha}N$ and consider dilated functions $\{f_r\}_{0<r<1}$ of f ($f_r(z) = f(rz)$). Since each f_r are in the ball algebra, we have $T(f_r)(z) = \Psi(z)\cdot f(r\Phi(z))$ for all $r \in (0,1)$ and $z \in \mathbb{B}$. Since the convergence in $N\log^{\alpha}N$ implies the uniform convergence on compact subsets of $\mathbb{B}$ and $\|f_r - f\|_{\alpha} \to 0$ as $r \to 1^-$ (see [4]),

$$\lim_{r\to 1} T(f_r)(z) = \lim_{r\to 1} \Psi(z)\cdot f(r\Phi(z)) = \Psi(z)\cdot f(\Phi(z)). \tag{6}$$

On the other hand, the assumption on T which is an isometry of $N\log^{\alpha}N$ gives

$$\lim_{r\to 1} \|T(f_r) - T(f)\|_{\alpha} = \lim_{r\to 1} \|f_r - f\|_{\alpha} = 0.$$

Combining this with (6), we see that $T = \Psi C_{\Phi}$ on $N\log^{\alpha}N$.

Next we will prove that Ψ is an inner function on $\mathbb{B}$. Since $\Psi \in H^1$, it is enough to prove that $|\Psi^*| = 1$ a.e. on $\mathbb{S}$. Note that $1 = \|\Psi\|_{H^1} \le \|\Psi\|_{H^2}$, $\|t\Psi\|_{H^1} = t$, and $\|t\Psi\|_{\alpha} = \varphi_{\alpha}(\log(1+t))$ for any $t > 0$. By Lemma 2 we obtain

$$\begin{aligned}\int_{\mathbb{S}} \{K_{\alpha}|t\Psi^*|^2 - |t\Psi^*|^3\theta_{\alpha}(|t\Psi^*|)\} d\sigma &= (\log\gamma_{\alpha})^{\alpha} t - \varphi_{\alpha}(\log(1+t)) \\ &= K_{\alpha}t^2 - t^3\theta_{\alpha}(t),\end{aligned}$$

and so we have

$$\int_{\mathbb{S}} \{K_{\alpha}|\Psi^*|^2 - t|\Psi^*|^3\theta_{\alpha}(|t\Psi^*|)\} d\sigma = K_{\alpha} - t\theta_{\alpha}(t).$$

Since $K_{\alpha}|\Psi^*|^2 - t|\Psi^*|^3\theta_{\alpha}(|t\Psi^*|) = \{(\log\gamma_{\alpha})^{\alpha}|t\Psi^*| - \varphi_{\alpha}(\log(1+|t\Psi^*|))\}/t^2 \ge 0$ a.e. on $\mathbb{S}$, Fatou's lemma gives

$$\int_{\mathbb{S}} K_{\alpha}|\Psi^*|^2 d\sigma \le \liminf_{t\to 0}\{K_{\alpha} - t\theta_{\alpha}(t)\} = K_{\alpha}.$$

Thus we have $\|\Psi\|_{H^2} \le 1$ and $\|\Psi\|_{H^2} = \|\Psi\|_{H^1} = 1$. This implies that $|\Psi^*| = 1$ a.e. on $\mathbb{S}$. Furthermore by combining $|\Psi^*| = 1$ a.e. on $\mathbb{S}$ and (5), we see that Φ^* is a measure-preserving map on $\mathbb{S}$.

For the converse, we assume that Ψ is an inner function and Φ is an inner map with Φ^* satisfy the measure-preserving property. If f is in the ball algebra, then it holds that $\|\Psi C_{\Phi}(f)\|_{\alpha} = \|f\|_{\alpha}$ since $f \circ \Phi^* = (f\circ\Phi)^*$ a.e. on $\mathbb{S}$. Fix $f \in N\log^{\alpha}N$ and take a sequence $\{r_j\}$ with $r_j \to 1$ as $j \to \infty$. Since $f_{r_j} \to f$ in $N\log^{\alpha}N$ as $j \to \infty$, $\{\Psi C_{\Phi}(f_{r_j})\}$ is a Cauchy sequence in $N\log^{\alpha}N$ and $g := \lim_{j\to\infty} \Psi C_{\Phi}(f_{r_j}) \in N\log^{\alpha}N$. The uniform convergence on compact subsets of $\mathbb{B}$ shows that $g(z) = \Psi C_{\Phi}(f)(z)$ for each $z \in \mathbb{B}$ and $\|\Psi C_{\Phi}(f)\|_{\alpha} = \|f\|_{\alpha}$; namely ΨC_{Φ} is an isometry of $N\log^{\alpha}N$. $\square$

To obtain the form of the surjective isometry of $N\log^{\alpha}N$, we need some characterization for which a holomorphic function f belongs to $N\log^{\alpha}N$.

Lemma 3. *Let N denote the Nevanlinna class. For any function $f \in N$, $f \in N\log^{\alpha}N$ if and only if $\varphi_\alpha(\log^+|f^*|) \in L^1(d\sigma)$ and*

$$\varphi_\alpha(\log^+|f(z)|) \leq \int_{\mathbb{S}} P(z,\zeta)\varphi_\alpha(\log^+|f^*(\zeta)|)d\sigma(\zeta) \quad \text{for } z \in \mathbb{B}, \tag{7}$$

where $P(z,\zeta)$ denotes the invariant Poisson kernel of $\mathbb{B}$.

Proof. The result which replaced $\varphi_\alpha(\log^+ x)$ with $\{\log^+ x\}^p$ $(p > 1)$ can be found in [20]. For the reader's benefit, however, we will give the proof.

Now we assume that $f \in N\log^{\alpha}N$. Fatou's lemma shows that $\varphi_\alpha(\log^+|f^*|) \in L^1(d\sigma)$. The inclusion $N\log^{\alpha}N \subset N^*$ implies that $\log^+|f|$ has the least $\mathcal{M}$-harmonic majorant. Since the least $\mathcal{M}$-harmonic majorant of $\log^+|f|$ is the Poisson integral $P[\log^+|f^*|]$, we have the following inequality:

$$\log^+|f(z)| \leq \int_{\mathbb{S}} P(z,\zeta)\log^+|f^*(\zeta)|d\sigma(\zeta) \quad \text{for } z \in \mathbb{B}. \tag{8}$$

Note that $\varphi_\alpha(t)$ is strictly increasing and convex on $[0,\infty)$ and the Poisson kernel has the normalization

$$\int_{\mathbb{S}} P(z,\zeta)d\sigma(\zeta) = 1. \tag{9}$$

Applying Jensen's inequality to (8), we obtain the inequality (7).

Conversely, we put $z = r\eta$ $(0 \leq r < 1, \eta \in \mathbb{S})$ in (7). By integrating with respect to η and applying Fubini's theorem, we have that

$$\int_{\mathbb{S}} \varphi_\alpha(\log^+|f(r\eta)|)d\sigma(\eta) \leq \int_{\mathbb{S}} \varphi_\alpha(\log^+|f^*(\zeta)|)d\sigma(\zeta) \int_{\mathbb{S}} P(r\eta,\zeta)d\sigma(\eta).$$

By the symmetric property $P(r\eta,\zeta) = P(r\zeta,\eta)$ and (9), we obtain that

$$\sup_{0\leq r<1} \int_{\mathbb{S}} \varphi_\alpha(\log^+|f(r\eta)|)d\sigma(\eta) \leq \int_{\mathbb{S}} \varphi_\alpha(\log^+|f^*(\zeta)|)d\sigma(\zeta).$$

Hence the condition $\varphi_\alpha(\log^+|f^*|) \in L^1(d\sigma)$ implies that $f \in N\log^{\alpha}N$. □

Corollary 1. *An isometry T of $N\log^{\alpha}N$ is surjective if and only if $T = aC_{\mathcal{U}}$, where $a \in \mathbb{C}$ with $|a| = 1$ and $\mathcal{U}$ is a unitary transformation.*

Proof. Since the surface measure $d\sigma$ is unitary invariant, it is clear that the operator $aC_{\mathcal{U}}$ is a surjective isometry of $N\log^{\alpha}N$.

Suppose that T is surjective. Then Theorem 1 gives that $T = \Psi C_\Phi$. This assumption shows that Φ is an automorphism of $\mathbb{B}$. So it is enough to prove that Φ fixes the origin because the automorphism which fixes the origin is a unitary transformation. Let Φ_j $(1 \leq j \leq N)$ be the component of Φ. For each $j \in \{1,\dots,N\}$ and $r \in (0,1)$ we have

$$\int_{\mathbb{S}} \Phi_j(r\zeta)d\sigma(\zeta) = \int_{\mathbb{S}} d\sigma(\zeta)\frac{1}{2\pi}\int_{-\pi}^{\pi} \Phi_j(re^{i\theta}\zeta)d\theta.$$

Since the mean value theorem shows that the right-hand side term is equal to $\Phi_j(0)$, the Lebesgue dominated convergence theorem gives $\int_{\mathbb{S}} \Phi_j^*(\zeta)d\sigma(\zeta) = \Phi_j(0)$.

On the other hand, by applying the measure-preserving property of Φ^* to a bounded Borel function $h(w) = \langle w, e_j\rangle$ where e_j is the standard orthonormal base vector in $\mathbb{C}^N$, we have

$$\Phi_j(0) = \int_{\mathbb{S}} \Phi_j^*(\zeta) d\sigma(\zeta) = \int_{\mathbb{S}} \langle \Phi^*(\zeta), e_j \rangle d\sigma(\zeta) = \int_{\mathbb{S}} \langle \zeta, e_j \rangle d\sigma(\zeta).$$

From [18, p. 15, §1.4.5 (2)] we have that

$$\int_{\mathbb{S}} \langle \zeta, e_j \rangle d\sigma(\zeta) = \frac{N-1}{\pi} \int_{-\pi}^{\pi} \int_0^1 (1-t^2)^{N-2} t^2 e^{i\theta} dt\, d\theta = 0,$$

and so Φ fixes the origin.

Next we will prove that Ψ is a unimodular constant. If $f \in N\log^\alpha N$ such that $1 = T(f) = \Psi C_\Phi(f)$, then $1/\Psi = f \circ \Phi \in N\log^\alpha N$. Inequality (7) in Lemma 3 gives that

$$\varphi_\alpha \left(\log^+ \frac{1}{|\Psi(z)|} \right) \le \int_{\mathbb{S}} P(z,\zeta) \varphi_\alpha \left(\log^+ \frac{1}{|\Psi^*(\zeta)|} \right) d\sigma(\zeta) = 0,$$

and so we have $1/|\Psi| \le 1$ on $\mathbb{B}$. Since Ψ is inner, Ψ is a unimodular constant. $\square$

3. Isometrically equivalent composition operators and multiplication operators

For two continuous operators S, T on a Banach space X, the isometric equivalence problem is defined as follows: What are the necessary and sufficient conditions such that $SU_1 = U_2T$ on X for some surjective isometries U_1, U_2 of X? This problem for composition operators on various holomorphic function spaces has been considered by several authors [1, 8, 10]. Recently N.J. Gal, J.E. Jamison, and A.G. Siskakis [9] considered this problem for an integral operator on the Hardy space and the Bergman space. By taking the form of the surjective isometry of $N\log^\alpha N$ into consideration, we can also consider this problem for operators on $N\log^\alpha N$. Hence we will define the $N\log^\alpha N$-isometric equivalence for operators S, T as follows.

Definition. For two continuous operators S, T on $N\log^\alpha N$, we say that S and T are $N\log^\alpha N$-isometrically equivalent if and only if there are two surjective isometries U_1, U_2 of $N\log^\alpha N$ such that $SU_1 = U_2T$ on $N\log^\alpha N$.

In this section, we will consider the isometric equivalence problem for composition operators C_Φ and multiplication operators M_g on $N\log^\alpha N$.

Composition operators are very popular studies in the field of analytic functions and operator theory. For the case $N\log^\alpha N$ on the unit disc $\mathbb{D}$, since the function $\varphi_\alpha(\log(1+|f|))$ is subharmonic on $\mathbb{D}$, Littlewood's subordination principle shows that every analytic self-map of $\mathbb{D}$ induce a continuous composition operator on $N\log^\alpha N$. For the unit ball case, however, every holomorphic self-map of $\mathbb{B}$ does not always induce a continuous composition operator on $N\log^\alpha N$. This situation is the same as the Hardy space case (see [3]).

In contrast to the case of composition operators, multiplication operators M_g defined by $M_g(f) = g \cdot f$ are always continuous on $N\log^\alpha N$ for each $g \in N\log^\alpha N$. Because $N\log^\alpha N$ forms an algebra, the closed graph theorem implies that M_g is continuous on $N\log^\alpha N$.

Hence we will assume that a holomorphic self-map of $\mathbb{B}$ induces a continuous composition operator on $N\log^\alpha N$ and a multiplier function g belongs to $N\log^\alpha N$ when we consider the isometric equivalence problem for these operators.

Theorem 2. *Suppose that Φ and Ψ are holomorphic self-maps of $\mathbb{B}$ such that C_Φ and C_Ψ are continuous on $N\log^\alpha N$. Then C_Φ and C_Ψ are $N\log^\alpha N$-isometrically equivalent if and only if*

$$\Psi(z) = (\mathcal{U}_1 \circ \Phi \circ \mathcal{U}_2^*)(z),$$

for some unitary transformations $\mathcal{U}_1, \mathcal{U}_2$.

Proof. Now suppose that $C_\Phi T_1 = T_2 C_\Psi$ for some surjective isometries T_1, T_2 of $N\log^\alpha N$. By Corollary 1, T_j $(j = 1, 2)$ has the form $T_j = a_j C_{\mathcal{U}_j}$ where a_j are unimodular constants and $\mathcal{U}_j$ are unitary transformations. Hence we have that $a_1 f(\mathcal{U}_1 \circ \Phi) = a_2 f(\Psi \circ \mathcal{U}_2)$ for any $f \in N\log^\alpha N$. By taking $f \equiv 1$ in this relation, we see that $a_1 = a_2$. By taking $f(z) = z_j$ $(1 \le j \le N)$ in this relation, we also obtain that $\Psi_j(\mathcal{U}_2(z)) = \sum_{k=1}^N a_{jk}\Phi_k(z)$, where a_{jk} denotes components of the unitary matrix $\mathcal{U}_1$. This implies that $\Psi = \mathcal{U}_1 \circ \Phi \circ \mathcal{U}_2^*$ on $\mathbb{B}$.

To prove the other direction, take a unimodular constant λ and unitary transformations $\mathcal{U}_1$ and $\mathcal{U}_2$ and consider the surjective isometries of $N\log^\alpha N$ given by

$$T_1 = \lambda C_{\mathcal{U}_1} \quad \text{and} \quad T_2 = \lambda C_{\mathcal{U}_2}.$$

Then we see that these isometries satisfy the relation $C_\Phi T_1 = T_2 C_\Psi$ on $N\log^\alpha N$. $\square$

Theorem 3. *Let $g, h \in N\log^\alpha N$. Then M_g and M_h are $N\log^\alpha N$-isometrically equivalent if and only if*

$$g(z) = \lambda h(\mathcal{U}z),$$

for some unimodular constant λ and unitary transformation $\mathcal{U}$.

Proof. To prove the sufficiency, we may consider the surjective isometries

$$T_1 = \overline{\lambda} C_{\mathcal{U}} \quad \text{and} \quad T_2 = C_{\mathcal{U}}.$$

To prove the necessity, we may apply the relation $M_g T_1 = T_2 M_h$ to the constant function $f \equiv 1$. $\square$

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