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# EQUIDISTRIBUTION OF HECKE POINTS ON THE SUPERSINGULAR MODULE

RICARDO MENARES

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Abstract. For a fixed prime $p$, we consider the (finite) set of supersingular elliptic curves over $\overline{\mathbb{F}}_p$. Hecke operators act on this set. We compute the asymptotic frequence with which a given supersingular elliptic curve visits another under this action.

## 1. Introduction

Let $p$ be a prime number. We denote by $E = \{E_1, \dots, E_n\}$ the set of isomorphism classes of supersingular elliptic curves over $\overline{\mathbb{F}}_p$. We denote by $S := \bigoplus_{i=1}^n \mathbb{Z}E_i$ the supersingular module in characteristic $p$ (i.e. $S$ is the free abelian group spanned by the elements of $E$). Hecke operators act on $S$ by

$$T_1 := id, \quad T_m(E_i) = \sum_C E_i/C, \quad m \geq 2,$$

where $C$ runs through the subgroup schemes of $E_i$ of rank $m$. This definition is extended by linearity to $S$ and to $S_{\mathbb{R}} := S \otimes \mathbb{R}$. For an integer $m \geq 1$ we put

$$B_{i,j}(m) = |\{C \subset E_i, \quad |C| = m \text{ and } E_i/C \cong E_j\}|.$$

We have that $T_m E_i = \sum_{j=1}^n B_{i,j}(m) E_j$. The the matrix $\big(B_{i,j}(m)\big)_{i,j=1}^n$ is known as the Brandt matrix of order $m$.

For a given $D = \sum_{i=1}^n a_i E_i \in S_{\mathbb{R}}$, we put $\deg D = \sum_{i=1}^n a_i$. We have that ([4], Proposition 2.7)

$$\deg T_m E_i = \sum_{\substack{d \mid m \\ p \nmid d}} d =: \sigma(m)_p,$$

leading to define $\deg T_m := \sigma(m)_p$.

Let $M$ be the set of probability measures on $E$. For every $i = 1, \dots, n$, we denote by $\delta_{E_i} \in M$ the Dirac measure supported on $E_i$. Let

$$S^+ := \Big\{ \sum_{i=1}^n a_i E_i \in S_{\mathbb{R}} \text{ such that } a_i \geq 0 \Big\} - \{0\}.$$

For any $D = \sum_{i=1}^n a_i E_i \in S^+$, we put

$$\Theta_D := \frac{1}{\deg D} \sum_{i=1}^n a_i \delta_{E_i}.$$

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We have that $\Theta_D$ is a probability measure on $E$, and every element of $M$ has this form. Hence, there is a natural action of the Hecke operators on $M$, given by $T_m\Theta_D := \Theta_{T_m D}$.

Each $E_i$ has a finite number of automorphisms. We define

$$w_i := |\mathrm{Aut}(E_i)/\{\pm 1\}|, \quad W := \sum_{i=1}^{n} \frac{1}{w_i}.$$

The element $e := \sum_{i=1}^{n} \frac{1}{w_i} E_i \in S \otimes \mathbb{Q}$ is Eisenstein ([4], p. 139), i.e.

$$T_m(e) = \deg T_m e. \tag{1.1}$$

We denote $\Theta := \Theta_e$. Equation (1.1) implies that $T_m\Theta = \Theta$ for all $m \geq 1$.

Let $C(E) \cong \mathbb{C}^n$ be the space of complex valued functions on $E$. For $f \in C(E)$, we denote $\|f\| = \max_i |f(E_i)|$ and

$$\Theta_D(f) := \int_E f\Theta_D = \frac{1}{\deg D} \sum_{i=1}^{n} a_i f(E_i).$$

For a positive integer $m$, we write $m = p^k m_p$ with $p \nmid m_p$. In this paper, we will prove the following result:

**Theorem 1.1.** *For all $i = 1, \ldots, n$, the sequence of measures $\{\Theta_{T_m E_i}\}$, where $m$ runs through a set of positive integers such that $m_p \to \infty$, is equidistributed with respect to $\Theta$. More precisely, for all $\varepsilon > 0$ there exists $C_\varepsilon > 0$ such that for every $f \in C(E)$, and for every sequence of integers $m$ such that $m_p \to \infty$, we have that*

$$|\Theta_{T_m E_i}(f) - \Theta(f)| \leq C_\varepsilon \|f\| n m^{-\frac{1}{2}+\varepsilon}.$$

We study the asymptotic frequence of the multiplicity of $E_j$ inside $T_m E_i$. That is, we investigate the behavior of the ratio $B_{i,j}(m)/\deg(T_m)$ when $m$ varies. We will prove Theorem 1.1 in the equivalent formulation:

**Theorem 1.2.** *For all $\varepsilon > 0$, there exists $C_\varepsilon > 0$ such that for every sequence of integers $m$ such that $m_p \to \infty$, we have that*

$$\Big| \frac{B_{i,j}(m)}{\deg T_m} - \frac{12}{w_j(p-1)} \Big| \leq C_\varepsilon m^{-\frac{1}{2}+\varepsilon}. \tag{1.2}$$

*In particular,*

$$\lim_{m_p \to \infty} \frac{B_{i,j}(m)}{\deg T_m} = \frac{12}{w_j(p-1)}. \tag{1.3}$$

The proof of this assertion is found in section 1.2.

*Remark* 1.3. The equality $\sum_{j=1}^{n} \frac{B_{i,j}(m)}{\deg T_m} = 1$ combined with equation (1.3) implies the mass formula of Deuring and Eichler:

$$W = \sum_{j=1}^{n} \frac{1}{w_j} = \frac{p-1}{12}.$$

Theorem 1.1 can be deduced from Theorem 1.2 as follows: Remark 1.3 implies that $\Theta = \sum_{j=1}^{n} \frac{12}{w_j(p-1)} \delta_{E_j}$. Take $f \in C^0(E)$. We have that

$$|\Theta_{T_m E_i}(f) - \Theta(f)| \leq \|f\| \sum_{j=1}^{n} \Big| \frac{B_{i,j}(m)}{\deg T_m} - \frac{12}{w_j(p-1)} \Big|.$$

Hence, inequality (1.2) implies Theorem 1.1.

Let $h : E \to E$ be a function. Then $h$ defines an endomorphism of $S$ and of $S_{\mathbb{R}}$ by the rule

$$h\big(\sum a_i E_i\big) := \sum a_i h(E_i).$$

We will also consider the action induced on $M$ by $h^*\Theta_D := \Theta_{h(D)}$.

**Corollary 1.4.** *Let $q \neq p$ be a prime number. Let $h : E \to E$ be a function such that $h \circ T_q = T_q \circ h$. Then $h^*\Theta = \Theta$. In other words, $h$ can be identified with a permutation $\tau \in S_n$ by $h(E_i) = E_{\tau(i)}$, and we have that $w_i = w_{\tau(i)}$ for all $i = 1, \ldots, n$.*

*Proof.* Since $T_{q^k}$ is a polynomial in $T_q$, we also have that $h \circ T_{q^k} = T_{q^k} \circ h$. Let $f \in C(E)$. We have that

$$\begin{aligned} h^*\Theta(f) &= \lim_{k\to\infty} h^*\Theta_{T_{q^k}E_1}(f) && (1.4)\\ &= \lim_{k\to\infty} \Theta_{h\circ T_{q^k}E_1}(f)\\ &= \lim_{k\to\infty} \Theta_{T_{q^k}\left(h(E_1)\right)}(f)\\ &= \Theta(f), && (1.5) \end{aligned}$$

where we have used Theorem 1.1 in (1.4) and (1.5). □

Statement of Theorem 1.1, using the Hecke invariant measure $\Theta$, has been included to emphasize the analogy with the fact that Hecke orbits are equidistributed on the modular curve $SL_2(\mathbb{Z})\backslash\mathbb{H}$ with respect to the hyperbolic measure, which is Hecke invariant (e.g. see [1], Section 2).

1.1. **Weight 2 Eisenstein series for $\Gamma_0(p)$.** The modular curve $X_0(p)$ has two cusps, represented by 0 and $\infty$. We denote by $\Gamma_\infty$ (resp. $\Gamma_0$) the stabilizer of $\infty$ (resp. 0). The associated weight 2 Eisenstein series are given by

$$\begin{aligned} \mathbf{E}_\infty(z) &= \frac{1}{2}\lim_{\varepsilon\to 0^+} \sum_{\gamma\in\Gamma_\infty\backslash\Gamma_0(p)} j_\gamma(z)^{-2}|j_\gamma(z)|^{-2\varepsilon},\\ \mathbf{E}_0(z) &= \frac{1}{2}\lim_{\varepsilon\to 0^+} \sum_{\gamma\in\Gamma_0\backslash\Gamma_0(p)} j_{\sigma_0^{-1}\gamma}(z)^{-2}|j_{\sigma_0^{-1}\gamma}(z)|^{-2\varepsilon}, \end{aligned}$$

where $\sigma_0 = \begin{pmatrix} 0 & -1/\sqrt{p} \\ \sqrt{p} & 0 \end{pmatrix}$ and $j_\eta(z) = cz + d$ for $\eta = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$.

The functions $\mathbf{E}_\infty$ and $\mathbf{E}_0$ are weight 2 modular forms for $\Gamma_0(p)$, and they are Hecke eigenforms. The Fourier expansions at $i\infty$ are ([5], Theorem 7.2.12, p. 288)

$$\begin{aligned} \mathbf{E}_\infty(z) &= 1 - \frac{3}{\pi y(p+1)} + \frac{24}{p^2-1}\sum_{n=1}^{\infty} b_n q^n,\\ \mathbf{E}_0(z) &= -\frac{3}{\pi y(p+1)} - \frac{24p}{p^2-1}\sum_{n=1}^{\infty} a_n q^n, \end{aligned}$$

with the sequences $a_n$ and $b_n$ given by:

- if $p \nmid n$, then $a_n = b_n = \sigma_1(n) = \sum_{d|n} d$;
- if $k \geq 1$, then $b_{p^k} = p + 1 - p^{k+1}$ and $a_{p^k} = p^k$;

- if $p \nmid m$ and $k \geq 1$, then $b_{p^k m} = -b_{p^k} b_m$ and $a_{p^k m} = a_{p^k} a_m$.

By taking an appropriate linear combination, we obtain a noncuspidal, holomorphic at $i\infty$ modular form

$$\begin{aligned} f_0(z) &:= \mathbf{E}_\infty(z) - \mathbf{E}_0(z) \\ &= 1 + \frac{24}{p^2-1} \sum_{n=1}^{\infty} (pa_n + b_n) q^n. \end{aligned}$$

Since we have that

$$\begin{aligned} \mathbf{E}_\infty|_{\sigma_0}(z) &= \mathbf{E}_0(z), \\ \mathbf{E}_0|_{\sigma_0}(z) &= \mathbf{E}_\infty(z), \end{aligned}$$

this shows that $f_0$ is holomorphic at $\Gamma_0(p)0$ as well. Since

$$\dim_{\mathbb{C}} M_2(\Gamma_0(p)) = 1 + \dim_{\mathbb{C}} S_2\big(\Gamma_0(p)\big)$$

and since $f_0$ is holomorphic, nonzero and noncuspidal, we have the decomposition

$$M_2(\Gamma_0(p)) = S_2(\Gamma_0(p)) \oplus \mathbb{C} f_0. \tag{1.6}$$

1.2. **Proof of Theorem 1.2.** Recall that we write $m = p^k m_p$ with $p \nmid m_p$. We have that $B(p^k)$ is a permutation matrix of order dividing 2 and that $B(m) = B(p^k)B(m_p)$ ([4], Proposition 2.7). It follows that $\deg(T_m) = \deg(T_{m_p})$ and that we can define, for each $i = 1, \ldots, n$, an index $i(k) \in \{1, \ldots, n\}$ such that $B_{i,l}(p^k) = \delta_{i(k),l}$. Furthermore, $i(k) = i$ if $k$ is even. We have that

$$\begin{aligned} \frac{B_{i,j}(m)}{\deg T_m} &= \sum_{l=1}^{n} \frac{B_{i,l}(p^k) B_{l,j}(m_p)}{\deg T_{m_p}} \\ &= \frac{B_{i(k),j}(m_p)}{\deg T_{m_p}}. \end{aligned}$$

Hence, to prove Theorem 1.2 we may assume $p \nmid m$, which is what we will do in what follows.

Our method is based on the interpretation of the multiplicities $B_{i,j}(m)$ as Fourier coefficients of a modular form.

**Theorem 1.5.** *For every $0 \leq i, j \leq n$, there exists a weight 2 modular form $f_{i,j}$ for $\Gamma_0(p)$ such that its $q$-expansion at $\infty$ is*

$$f_{i,j}(z) := \frac{1}{2w_j} + \sum_{m=1}^{\infty} B_{i,j}(m) q^m, \quad q = e^{2\pi i z}.$$

*Proof.* This fact is stated in [4], p. 118. It is a particular case of [3], Chapter II, Theorem 1 ($D = p, H = 1, l = 0$ in Eichler's notation). We remark that the theorem in [3], Chapter II, states the modularity of a theta series constructed from an order in a quaternion algebra. The fact that this theta series is the same as our $f_{i,j}$ is a consequence of [4], Proposition 2.3. □

Using (1.6), we can decompose

$$f_{i,j} = g_{i,j} + c_{i,j} f_0, \quad g_{i,j} \in S_2(\Gamma_0(p)), \quad c_{i,j} \in \mathbb{C}.$$

Comparing the $q$-expansions, we get $c_{i,j} = \frac{1}{2w_j}$. We have that

$$g_{i,j} = f_{i,j} - c_{i,j} f_0 = \sum_{m=1}^{\infty} c_m q^m,$$

where

$$c_m = B_{i,j}(m) - \frac{12}{w_j(p^2-1)}(pa_m + b_m).$$

The coefficient $c_m$ depends on $(i,j)$, but we do not include this dependence in the notation in order to simplify it. Since $p \nmid m$, we have that $\deg(T_m) = \sigma_1(m)$ and

$$c_m = B_{i,j}(m) - \frac{12}{w_j(p-1)}\sigma_1(m).$$

Hence,

$$\begin{aligned} \Big| \frac{B_{i,j}(m)}{\deg T_m} - \frac{12}{w_j(p-1)} \Big| &= \frac{|c_m|}{\sigma_1(m)} \\ &\leq \frac{|c_m|}{m}. \end{aligned}$$

Using Deligne's theorem ([2], théorème 8.2, previously Ramanujan's conjecture), we have that

$$c_m = O_\varepsilon(m^{1/2+\varepsilon}),$$

concluding the proof. □

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Facultad de Matematicas, Pontificia Universidad Católica de Chile, Avda. Vicuña Mackenna 4860, Santiago, Chile

*E-mail address*: `remenares@mat.puc.cl`