

3-MANIFOLDS WITH POSITIVE FLAT CONFORMAL STRUCTURE

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ABSTRACT. In this paper, we consider a closed 3-manifold M with flat conformal structure C . We will prove that if the Yamabe constant of (M, C) is positive, then (M, C) is Kleinian.

1. INTRODUCTION AND MAIN THEOREM

In 1988, Schoen and Yau [19] gave a final resolution for the *Yamabe Problem* (cf. [3, 15, 18]). In [19, Proposition 3.3], they also proved that *any closed n -manifold with flat conformal structure of positive Yamabe constant is Kleinian, provided that $n \geq 4$* . Moreover, under the assumption that an extended Positive Mass Theorem holds (but a proof has not yet appeared), they showed that the above assertion still holds even when $n = 3$ (see [19, Proposition 4.4'] and the paragraph just before it). On the other hand, there are enormous examples of closed 3-manifolds with flat conformal structures which are not Kleinian (see [8, Remark 7.4]).

The purpose of this brief note is to prove the above assertion for the remaining case $n = 3$.

Theorem 1.1. *Let M be a closed 3-manifold with flat conformal structure C . If its Yamabe constant is positive, then (M, C) is Kleinian.*

This assertion can be obtained by an argument in the proof of [1, the second assertion of Theorem 1.4], which is a combination of a result [19, Proposition 4.2], a positive mass theorem [1, the first assertion of Theorem 1.4] (different from the one Schoen and Yau mentioned in [19]) and a classification of 3-manifolds with positive scalar curvature [7, 10, 11]. Here, we will explicitly give a proof of it (see also Remark 2.2 below).

The remaining sections are organized as follows. Section 2 contains some necessary definitions and preliminary geometric results. Section 3 is devoted to the proof of Theorem 1.1.

2. PRELIMINARIES

Let M be a closed 3-manifold, that is, a compact 3-manifold without boundary. To simplify the presentation and the argument, we always assume that $\dim M = 3$

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throughout this paper. For each conformal class C on M , the *Yamabe constant* $Y(M, C)$ of (M, C) is defined by

$$Y(M, C) := \inf_{g \in C} E(g), \quad E(g) := \frac{\int_M R_g d\mu_g}{\text{Vol}_g(M)^{1/3}},$$

where R_g, μ_g and $\text{Vol}_g(M)$ denote respectively the scalar curvature, the volume element of g and the volume of (M, g) . It is a finite-valued conformal invariant of C . The Yamabe constant $Y(M, C)$ is positive if and only if there exists a positive scalar curvature metric $g \in C$ (cf. [3]). A remarkable theorem [22, 20, 2, 17, 19] of Yamabe, Trudinger, Aubin and Schoen asserts that each conformal class C contains a minimizer \check{g} of $E|_C$, called a *Yamabe metric* (or a *solution of the Yamabe Problem*), which is of constant scalar curvature

$$R_{\check{g}} = Y(M, C) \cdot \text{Vol}_{\check{g}}(M)^{-2/3}.$$

Let M_∞ be an infinite covering of M . We shall say that the fundamental group $\pi_1(M)$ of M has a *descending chain of finite index subgroups tending to* $\pi_1(M_\infty)$ if it satisfies the following: There exists a family of subgroups $\{\Gamma_i\}_{i \geq 1}$ of $\pi_1(M)$ such that

- (i) each Γ_i is of finite index in $\pi_1(M)$ with $\Gamma_i \supset \pi_1(M_\infty)$,
- (ii) $\pi_1(M) = \Gamma_1 \supsetneq \Gamma_2 \supsetneq \cdots \supsetneq \Gamma_i \supsetneq \Gamma_{i+1} \supsetneq \cdots$,
- (iii) $\bigcap_{i=1}^\infty \Gamma_i = \pi_1(M_\infty)$.

Assume that $Y(M, C) > 0$. Take a positive scalar curvature metric $g \in C$ and any point $p \in M$. Then, there exists the *normalized Green's function* G_p for L_g with a pole at p , that is,

$$L_g G_p = c_0 \cdot \delta_p \quad \text{on } M \quad \text{and} \quad \lim_{q \rightarrow p} \text{dist}(q, p) G_p(q) = 1.$$

Here, $L_g := -8\Delta_g + R_g, c_0 > 0$ and δ_p stand respectively for the *conformal Laplacian*, a specific universal positive constant and the Dirac δ -function at p . Assume also that the covering $P_\infty : M_\infty \rightarrow M$ is normal. Let g_∞ denote the lift of g to M_∞ , and p_∞ a point in M_∞ with $P_\infty(p_\infty) = p$. Then, there exists uniquely also a *normalized minimal positive Green's function* G_∞ on M_∞ for $L_{g_\infty} := -8\Delta_{g_\infty} + R_{g_\infty}$ with pole at p_∞ (cf. [19]), which satisfies the following:

$$(P_\infty)^* G_p = \sum_{\gamma \in \mathcal{G}} G_\infty \circ \gamma \quad \text{on } M_\infty.$$

Here, \mathcal{G} stands for the group of deck transformations for the normal covering $M_\infty \rightarrow M$. Set

$$g_{\infty, AF} := G_\infty^4 \cdot g_\infty \quad \text{on } M_\infty^* := M_\infty - \{p_\infty\}.$$

Then, $g_{\infty, AF}$ defines a scalar-flat, asymptotically flat metric on M_∞^* (cf. [15]). Note that this asymptotically flat 3-manifold $(M_\infty^*, g_{\infty, AF})$ has *infinitely many* singularities created by the ends of M_∞^* . However, the mass $\mathfrak{m}_{ADM}(g_{\infty, AF})$ of $(M_\infty^*, g_{\infty, AF})$ can be defined in the usual way (cf. [4]). Note also that the positive mass theorem for asymptotically flat 3-manifolds with singularities does not always hold (see [1, Remark 1.5-(2)] for instance).

Once this is understood, the following positive mass theorem holds as a special case of [1, the first assertion of Theorem 1.4]:

Proposition 2.1. *Let (M, C) be a closed 3-manifold with $Y(M, C) > 0$. Let (M_∞, g_∞) be a normal infinite Riemannian covering of (M, g) such that $\pi_1(M)$ has a descending chain of finite index subgroups tending to $\pi_1(M_\infty)$, where $g \in C$ is a positive scalar curvature metric and g_∞ is its lift to M_∞ . For any point $p_\infty \in M_\infty$, let G_∞ denote the normalized minimal positive Green's function on M_∞^* with pole at p_∞ . Then, the asymptotically flat 3-manifold $(M_\infty^*, g_{\infty, AF})$ has nonnegative mass*

$$m_{ADM}(g_{\infty, AF}) \geq 0.$$

Remark 2.2. Assume that $M = \#\ell(S^1 \times S^2)$ for $\ell \geq 2$ and M_∞ is its universal covering. Note that M_∞ is spin. For each small $\sigma > 0$, consider the complete metric $g_{\sigma, AF} := (G_\infty + \sigma)^4 \cdot g_\infty$ with $R_{g_{\sigma, AF}} \geq 0$ on M_∞^* (cf. [19, Proposition 4.4']). Then, only one end of $(M_\infty^*, g_{\sigma, AF})$ is asymptotically flat and the other infinitely many ends are merely complete. Gilles Carron and the referee kindly pointed out that Witten's approach [21] (cf. [16]) to the Positive Mass Theorem is still valid for the family $\{(M_\infty^*, g_{\sigma, AF})\}_{0 < \sigma < 1}$. It implies that a more general positive mass theorem than Proposition 2.1 is a *folk theorem* for experts in this field, and Theorem 1.1 is too. But Proposition 2.1 itself is a complete form, and hence, by using it, we will give here an explicit and self-contained proof of Theorem 1.1.

A conformal 3-manifold (M, C) is said to be *locally conformally flat* if, for any point $p \in M$, there exists a metric $\bar{g} \in C$ such that \bar{g} is flat on some neighborhood of p . A conformal class C on M is called a *flat conformal structure* if (M, C) is locally conformally flat. In [14], Kuiper proved that, for a simply connected locally conformally flat 3-manifold (X, C') , there is a conformal immersion into (S^3, C_0) called a *developing map*, which is unique up to composition with a Möbius transformation of (S^3, C_0) . Therefore, the universal covering of a locally conformally flat manifold (M, C) admits a developing map. Here, (S^3, C_0) denotes the 3-sphere S^3 with the conformal class $C_0 := [g_0]$ of the standard metric g_0 of constant curvature one. (M, C) is called *Kleinian* if (M, C) is conformal to Ω/Γ for some open set Ω of S^3 and some discrete subgroup Γ of the conformal transformation group $\text{Conf}(S^3, C_0)$, which leaves Ω invariant and acts freely and properly discontinuously on Ω . Note that, if the developing map of the universal covering of a locally conformally flat manifold (M, C) is injective, then (M, C) is Kleinian.

With this understanding, the following criterion also holds as a special case of [19, Proposition 4.2]:

Proposition 2.3. *Let (M, C) be a closed 3-manifold with $Y(M, C) > 0$, and (\tilde{M}, \tilde{g}) be the universal Riemannian covering of (M, g) , where $g \in C$ is a positive scalar curvature metric. For any point $\tilde{p} \in \tilde{M}$, let \tilde{G} denote the normalized minimal positive Green's function on \tilde{M} for $L_{\tilde{g}}$ with pole at \tilde{p} , and $(\tilde{M} - \{\tilde{p}\}, \tilde{g}_{AF} = \tilde{G}^4 \cdot \tilde{g})$ the asymptotically flat 3-manifold as above. If the mass $m_{ADM}(\tilde{g}_{AF})$ is nonnegative, then the developing map of $(\tilde{M}, [\tilde{g}])$ is injective. In particular, (M, C) is Kleinian.*

Remark 2.4. We remark that the mass $m_{ADM}(\tilde{g}_{AF})$ is equal to the ADM energy E of $(\tilde{M} - \{\tilde{p}\}, \tilde{g}_{AF})$ appearing in [19, page 64] up to a positive constant.

3. PROOF OF MAIN THEOREM

Proof of Theorem 1.1. Consider the universal covering \tilde{M} of M and denote the lift of the flat conformal structure C by \tilde{C} . If $|\pi_1(M)| < \infty$, then (\tilde{M}, \tilde{C}) is conformal

to (S^3, C_0) by Kuiper’s Theorem [14]. Hence, (M, C) is Kleinian. From now on, we assume that $|\pi_1(M)| = \infty$, that is, the degree of the covering map $P : \widetilde{M} \rightarrow M$ is infinite.

Take a unit-volume Yamabe metric $g \in C$, and consider its lift $\tilde{g} \in \widetilde{C}$ to \widetilde{M} . Note that $R_{\tilde{g}} = R_g = Y(M, C) > 0$. Take any base points $p \in M, \tilde{p} \in \widetilde{M}$ satisfying $P(\tilde{p}) = p$, and fix them. Then, let \tilde{G} denote the normalized minimal positive Green function on \widetilde{M} for $L_{\tilde{g}}$ with pole at \tilde{p} , and the mass $\mathfrak{m}_{\text{ADM}}(\tilde{g}_{AF})$ of the asymptotically flat 3-manifold $(\widetilde{M} - \{\tilde{p}\}, \tilde{g}_{AF} := \tilde{G}^4 \cdot \tilde{g})$.

Suppose that

$$\mathfrak{m}_{\text{ADM}}(\tilde{g}_{AF}) \geq 0.$$

Recall that we can choose the base point $\tilde{p} \in \widetilde{M}$ arbitrarily. It then follows from Proposition 2.3 that the developing map of $(\widetilde{M}, \widetilde{C})$ is injective, and hence (M, C) is Kleinian. In this case, especially $\mathfrak{m}_{\text{ADM}}(\tilde{g}_{AF}) = 0$. Therefore, it is enough to show that $\mathfrak{m}_{\text{ADM}}(\tilde{g}_{AF}) \geq 0$.

By combining [7, Theorem 8.1] (cf. [9]) with $Y(M, C) > 0$ (replacing M by its orientable double covering if necessary), M can be decomposed uniquely into *prime* closed 3-manifolds

$$M = N_1 \# \cdots \# N_{\ell_1} \# \ell_2(S^1 \times S^2),$$

where $\pi_1(N_i)$ is finite for $i = 1, \dots, \ell_1$ and ℓ_1, ℓ_2 are nonnegative integers. By applying the C -prime decomposition theorem for closed 3-manifolds with flat conformal structures [10, 11] to (M, C) , there exists a flat conformal structure C_i on each N_i ($i = 1, \dots, \ell_1$). Then, Kuiper’s Theorem [14] again implies that each (N_i, C_i) is a nontrivial quotient of (S^3, C_0) . After taking an appropriate finite covering M' of M , we have

$$M' = \#\ell(S^1 \times S^2) \quad \text{for some } \ell \geq 1.$$

Recall that \widetilde{M} is the infinite universal covering of M . Then, there exists (uniquely) an infinite universal covering $\widetilde{M} \rightarrow M'$. Moreover, since $\pi_1(M')$ is a finitely generated free group, it has a descending chain of finite index subgroups tending to $\pi_1(\widetilde{M}) = \{e\}$. Let g' be the lifting of g to M' . Applying Proposition 2.1 to the normal infinite Riemannian covering $(\widetilde{M}, \tilde{g}) \rightarrow (M', g')$, we have that

$$\mathfrak{m}_{\text{ADM}}(\tilde{g}_{AF}) \geq 0.$$

This completes the proof of Theorem 1.1. □

Remark 3.1. Even if we replace the positivity $Y(M, C) > 0$ in Theorem 1.1 by the nonnegativity $Y(M, C) \geq 0$, it seems that the same conclusion still holds. More precisely, we propose the following (cf. [5, 13]).

Conjecture. *Let M be a closed 3-manifold with flat conformal structure C . If its Yamabe constant is zero, then either (1) or (2) holds:*

- (1) *There exists a flat metric $\bar{g} \in C$.*
- (2) *There exists a smooth family $\{g_t\}_{0 \leq t \leq 1}$ of locally conformally flat metrics on M such that $g_0 \in C$ and $Y(M, [g_1]) > 0$ (possibly $Y(M, [g_t]) < 0$ for some $t \in (0, 1)$).*

In the case (1), the universal covering $(\widetilde{M}, \widetilde{C})$ of (M, C) is conformal to $(S^3 - \{p_N\}, C_0)$ where $p_N := (1, 0, 0, 0) \in S^3$, and hence (M, C) is Kleinian. In the case (2), Theorem 1.1 implies that $(M, [g_1])$ is Kleinian. The argument in the proof of

Theorem 1.1 also implies that there exists a torsion free subgroup Γ of finite index in $\pi_1(M)$ such that Γ is either a trivial group or a nontrivial finitely generated free group. Then, the *virtual cohomological dimension* $\text{vcd } \pi_1(M)$ of $\pi_1(M)$ is either 0 or 1 (see [6]). Therefore, $(M, [g_1])$ is a closed Kleinian 3-manifold with $\text{vcd } \pi_1(M) < 3$. The quasiconformal stability of Kleinian groups [12, Theorem 2] implies that any flat conformal structure on M which is a smooth deformation of $[g_1]$ is also Kleinian; in particular C is too.

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