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# BEREZIN TRANSFORM AND WEYL-TYPE UNITARY OPERATORS ON THE BERGMAN SPACE

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ABSTRACT. For $\mathbf{D}$ the open complex unit disc with normalized area measure, we consider the Bergman space $L_a^2(\mathbf{D})$ of square-integrable holomorphic functions on $\mathbf{D}$. Induced by the group $Aut(\mathbf{D})$ of biholomorphic automorphisms of $\mathbf{D}$, there is a standard family of Weyl-type unitary operators on $L_a^2(\mathbf{D})$. For all bounded operators $X$ on $L_a^2(\mathbf{D})$, the Berezin transform $\widetilde{X}$ is a smooth, bounded function on $\mathbf{D}$. The range of the mapping Ber: $X \to \widetilde{X}$ is invariant under $Aut(\mathbf{D})$. The "mixing properties" of the elements of $Aut(\mathbf{D})$ are visible in the Berezin transforms of the induced unitary operators. Computations involving these operators show that there is no real number $M > 0$ with $M\|\widetilde{X}\|_\infty \geq \|X\|$ for all bounded operators $X$ and are used to check other possible properties of $\widetilde{X}$. Extensions to other domains are discussed.

## 1. Introduction

For $H$ a complex Hilbert space with inner product $\langle \cdot, \cdot \rangle$ and norm $\|v\| = \langle v, v\rangle^{\frac{1}{2}}$, we consider the algebra of all bounded linear operators $Op(H)$. For the unit sphere $S = \{v \in H : \langle v, v\rangle = 1\}$ and for $X$ in $Op(H)$, we have the usual operator norm $\|X\| = \sup\{\|Xv\| : v \in S\}$. We will also be concerned with the numerical range $W(X) = \{\langle Xv, v\rangle : v \in S\}$ and the numerical radius $w(X) = \sup\{|\langle Xv, v\rangle| : v \in S\}$. The set $W(X)$ is convex (Hausdorff's Theorem) and its closure contains the spectrum of X. There is a standard norm estimate

$$w(X) \leq \|X\| \leq 2\,w(X)$$

and a not-so-standard power estimate (Berger's Theorem [5])

$$w(X^n) \leq w(X)^n.$$

In the case that $H$ is the Bergman Hilbert space $L_a^2(\mathbf{D})$ or one of a large family of "reproducing kernel Hilbert spaces", we have, for each $c$ in $\mathbf{D}$, a reproducing kernel function $K(\cdot, c)$ so that, for any $f$ in $L_a^2(\mathbf{D})$,

$$f(c) = \langle f, K(\cdot, c)\rangle.$$

The normalized kernel functions $k_c(\cdot) \equiv K(\cdot, c)K(c, c)^{-\frac{1}{2}}$ play an important role in the analysis of operators on $L_a^2(\mathbf{D})$ as well as on other reproducing kernel spaces. In particular, for every bounded linear operator, we define the Berezin transform by $\widetilde{X}(c) = \langle Xk_c, k_c\rangle$. The map $Ber(X) = \widetilde{X}$ is one-to-one and $\widetilde{X}(\cdot)$ is known to be

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real-analytic [4] as well as Lipschitz with respect to the Bergman metric distance function on $\mathbf{D}$ [6].

It is not hard to check that the range of $Ber$ contains all bounded holomorphic functions on $\mathbf{D}$. Clearly, the range of $\widetilde{X}$ is contained in $W(X)$ and, for $\|f\|_\infty = \sup\{|f(z)| : z \in \mathbf{D}\}$, we have $\|\widetilde{X}\|_\infty \leq w(X)$ so that $\widetilde{X}$ is in the Banach space [8, p. 121] of bounded continuous functions $BC(\mathbf{D})$.

Convexity of range$(\widetilde{X})$ is easily seen to fail (take $X$ to be multiplication by a suitable holomorphic function). It is a natural problem to determine to what extent $\|\widetilde{X}\|_\infty$ imitates $w(X)$. Is there a real number $M > 0$ with $M\|\widetilde{X}\|_\infty \geq \|X\|$ for all $X$? Is $\|\widetilde{X^2}\|_\infty \leq \|\widetilde{X}\|_\infty^2$ for all $X$? The unitary operators $V_{[\lambda,c]}$ and related operators discussed in the next section provide examples to show that these estimates do not hold for $Op(L_a^2(\mathbf{D}))$. Similar constructions yield the same result for the Segal-Bargmann space of Gaussian square-integrable entire functions on complex $n$-space $\mathbf{C}^n$. The extension of our analysis to general bounded symmetric domains is plausible but presents significant difficulties.

## 2. Weyl-type unitary operators on $L_a^2(\mathbf{D})$

We consider the full group $Aut(\mathbf{D})$, given for $\lambda$ in $\mathbf{C}$ with $|\lambda| = 1$ and $c, z$ in $\mathbf{D}$, by

$$[\lambda, c](z) = \lambda \frac{z-c}{1-\overline{c}z}.$$

The group $Aut(\mathbf{D})$ acts on $L_a^2(\mathbf{D})$ by

$$([\lambda.c]f)(z) = f(\lambda\frac{z-c}{1-\overline{c}z}),$$

and it is standard that

$$(V_{[\lambda,c]}f)(z) = k_c(z) f(\lambda\frac{z-c}{1-\overline{c}z})$$

is a unitary transformation from $L_a^2(\mathbf{D})$ to itself, where

$$k_c(z) = \frac{1-|c|^2}{(1-\overline{c}z)^2}$$

is the normalized Bergman kernel for evaluation at $c$.

Multiplication on $Aut(\mathbf{D})$ is given by

$$[\lambda, c][\mu, d] = \Big[\mu\lambda\frac{1+d\overline{c}\overline{\lambda}}{1+c\overline{d}\lambda}, \frac{\overline{\lambda}d + c}{1+d\overline{c}\overline{\lambda}}\Big], \tag{$*$}$$

and it is easy to check that the map $[\lambda, c] \to V_{[\lambda,c]}$ is a projective unitary representation of $Aut(\mathbf{D})$ with

$$V_{[\lambda,c]}V_{[\mu,d]} = \frac{1+d\overline{c}\overline{\lambda}}{1+c\overline{d}\lambda} V_{[\lambda,c][\mu,d]}. \tag{$**$}$$

We will be interested in the Berezin transform of $V_{[\lambda,c]}$. We first check that

$$V_{[\lambda,c]}k_a = \Big(\frac{1+a\overline{c}\overline{\lambda}}{1+c\overline{a}\lambda}\Big) k_{[\overline{\lambda},-\lambda c](a)}. \tag{$***$}$$

It follows, using the defining property of the reproducing kernel, that

$$\langle V_{[\lambda,c]}k_a, k_a\rangle = \frac{(1-|a|^2)^2(1-|c|^2)}{[(1-\lambda|a|^2)-(a\overline{c}-\overline{a}\lambda c)]^2}. \tag{$****$}$$

*Remarks.* It follows from the above discussion that

$$V^*_{[\lambda,c]} = V_{[\overline{\lambda},-\lambda c]} = V^{-1}_{[\lambda,c]}.$$

The involutive unitary operators $V_{[-1,c]}$ are standard objects in the analysis of $\mathbf{D}$ as the prototypical bounded symmetric domain. We will use the elementary formula $V^2_{[1,c]} = V_{[1,\frac{2c}{1+|c|^2}]}$ in our analysis.

Using the above remark, we first have

**Theorem 1.** *The range of Ber: $Op[L^2_a(\mathbf{D})] \to BC(\mathbf{D})$ is invariant under $Aut(\mathbf{D})$.*

*Proof.* Using $(***)$, we can check that

$$\begin{aligned}\widetilde{X}\{[\lambda,c](a)\} &= \langle X k_{[\lambda,c](a)}, k_{[\lambda,c](a)}\rangle \\ &= \langle X V^*_{[\lambda,c]} k_a, V^*_{[\lambda,c]} k_a\rangle \\ &= \langle V_{[\lambda,c]} X V^*_{[\lambda,c]} k_a, k_a\rangle.\end{aligned}$$

□

**Corollary.** *The projective unitary representation $[\lambda,c] \to V_{[\lambda,c]}$ of $Aut(\mathbf{D})$ on $L^2_a(\mathbf{D})$ is irreducible.*

*Proof.* For $a, b$ arbitrary in $\mathbf{D}$, $([-1,a][-1,b])(a) = b$. For $X$ in $Op[L^2_a(\mathbf{D})]$ with $XV_{[\lambda,c]} = V_{[\lambda,c]}X$ for all $[\lambda,c]$ in $Aut(\mathbf{D})$, we have $\widetilde{X}\{[\lambda,c](a)\} = \widetilde{X}(a)$ for all $a$ in $\mathbf{D}$. Taking $[\lambda,c] = [-1,a][-1,b]$ gives $\widetilde{X}(a) = \widetilde{X}(b)$ for arbitrary $b$. Thus, $\widetilde{X}$ must be a constant function so that $X$ is a scalar multiple of the identity operator. □

*Remark.* The irreducibility is certainly "well known".

Next, we explicitly calculate $\|\widetilde{V}_{[\lambda,c]}\|_\infty$ for $\lambda = \pm 1$.

**Theorem 2.** *We have $\|\widetilde{V}_{[1,c]}\|_\infty = 1 - |c|^2$ and $\|\widetilde{V}_{[-1,c]}\|_\infty = 1$ for all $c$ in $\mathbf{D}$.*

*Proof.* First, for $\widetilde{V}_{[1,c]}$, we note that $(****)$ gives

$$\langle V_{[1,c]} k_a, k_a\rangle = \frac{(1-|a|^2)^2(1-|c|^2)}{[(1-|a|^2) + (\overline{a}c - a\overline{c})]^2}.$$

Since $i(\overline{a}c - a\overline{c})$ is real, we see that

$$|(1-|a|^2) + (\overline{a}c - a\overline{c})|^2 = (1-|a|^2)^2 + |\overline{a}c - a\overline{c}|^2,$$

so

$$|\langle V_{[1,c]} k_a, k_a\rangle| = \frac{(1-|a|^2)^2(1-|c|^2)}{(1-|a|^2)^2 + |\overline{a}c - a\overline{c}|^2}.$$

Since $\widetilde{V}_{[1,c]}(0) = 1 - |c|^2$ , it follows that $\|\widetilde{V}_{[1,c]}\|_\infty = 1 - |c|^2$.

For $\widetilde{V}_{[-1,c]}$, we have from $(***)$ that

$$V_{[-1,c]} k_a = \Big(\frac{1 - a\overline{c}}{1 - c\overline{a}}\Big) k_{[-1,c](a)}.$$

We note that, for each $c$ in $\mathbf{D}$, the equation $[-1,c](a) = a$ has a unique solution in $\mathbf{D}$, namely $a(0) = 0$ and

$$a(c) = \frac{1 - \sqrt{1-|c|^2}}{\overline{c}} \tag{†}$$

for $c \neq 0$. Thus, we have $\widetilde{V}_{[-1,c]}(a(c)) = 1$. Unitarity of $V_{[-1,c]}$ now implies that $\|\widetilde{V}_{[-1,c]}\|_\infty = 1$ for all $c$ in $\mathbf{D}$. □

**Corollary 1.** *There is no real number $M > 0$ so that $M\|\widetilde{X}\|_\infty \geq \|X\|$ for all $X$ in $Op(L_a^2(\mathbf{D}))$. Equivalently, $range(Ber)$ is a non-closed linear subspace of $BC(\mathbf{D})$.*

*Proof.* Since $V_{[1,c]}$ is unitary, we have $\|V_{[1,c]}\| = 1$ for all $c$ in $\mathbf{D}$. But $\|\widetilde{V}_{[1,c]}\|_\infty = 1 - |c|^2$ can be made arbitrarily small for $c$ in $\mathbf{D}$. Thus, there is no real number $M > 0$ so that $M\|\widetilde{X}\|_\infty \geq \|X\|$ for all $X$ in $Op(L_a^2(\mathbf{D}))$.

By the Schwarz inequality, $Ber$ is a bounded linear transformation from the Banach space $Op[L_a^2(\mathbf{D})]$ into the Banach space $BC(\mathbf{D})$. If $range(Ber)$ were closed in $BC(\mathbf{D})$, Ber would be a 1-1 bounded linear mapping onto the Banach space $\{range(Ber), \|\cdot\|_\infty\}$. The open mapping theorem would then give a norm estimate for $Ber^{-1}\widetilde{X} = X$ of the form $\|X\| \leq M\|\widetilde{X}\|_\infty$ for all $X$ in $Op[L_a^2(\mathbf{D})]$. Conversely, if $\|X\| \leq M\|\widetilde{X}\|_\infty$ for all $X$ in $Op[L_a^2(\mathbf{D})]$, then a standard argument shows that $range(Ber)$ is closed in $BC(\mathbf{D})$. $\square$

*Remark.* An unpublished proof of this result, using Toeplitz operators, is due to Fedor Nazarov.

**Corollary 2.** *The range of $\widetilde{V}_{[-1,c]}$ is exactly the interval $(0, 1]$.*

*Proof.* Note that $(****)$ gives

$$\widetilde{V}_{[-1,c]}(a) = \Big[\frac{1 - |a|^2}{(1 - |c|^2) + |c - a|^2}\Big]^2 (1 - |c|^2).$$

Hence, the range of $\widetilde{V}_{[-1,c]}$ is a connected subset of the positive real line which includes $\{1\}$, is bounded by 1, and, by taking $|a|$ near 1, has points arbitrarily close to 0. $\square$

*Remark.* The unitary operator $V_{[-1,c]}$ has spectrum $\{+1, -1\}$ and is certainly not a positive operator despite the positivity of $\widetilde{V}_{[-1,c]}$.

A modification of the $V_{[1,c]}$ shows that Berger's Theorem fails for $\|\widetilde{X}\|_\infty$.

**Theorem 3.** *For $X_c = V_{[1,c]} + V_{[1,-c]}$, we have $\|\widetilde{X}_c\|_\infty = 2(1 - |c|^2)$ and*

$$\|\widetilde{X_c^2}\|_\infty = 4\Big(\frac{(1 + |c|^4)}{(1 + |c|^2)^2}\Big)$$

*for all $c$ in $\mathbf{D}$. Thus, $\|\widetilde{X_c^2}\|_\infty > \|\widetilde{X}_c\|_\infty^2$ for all $c$ with $1 > |c| > 0$.*

*Proof.* This is a direct calculation using the facts that $V_{[1,c]}^2 = V_{[1, \frac{2c}{1+|c|^2}]}$ and that $\widetilde{V}_{[1,c]}(0) = 1 - |c|^2 = \|\widetilde{V}_{[1,c]}\|_\infty$. Note that $X_c^2 = V_{[1,c]}^2 + 2I + V_{[1,-c]}^2$. $\square$

## 3. Weyl-type unitary operators on the Segal-Bargmann space

We next briefly consider a space which is a model for Bergman spaces on bounded symmetric domains, even though the domain here is all of $\mathbf{C}^n$. The Segal-Bargmann space $H^2(\mathbf{C}^n, d\mu)$ is a Bergman space which consists of all entire functions which are square-integrable with respect to the normalized Gaussian measure $d\mu(z) = \exp[-|z|^2/2](2\pi)^{-n}dv(z)$. Here, $dv(z)$ is the standard Lebesgue volume measure on $\mathbf{C}^n$. The Bergman kernel for evaluation at $c$ is just $K(z, c) = \exp(z \cdot c/2)$, where we take $z \cdot c = z_1\overline{c}_1 + z_2\overline{c}_2 + \cdots + z_n\overline{c}_n$. Thus, $k_c(z) = \exp(z \cdot c/2 - |c|^2/4)$ is the normalized kernel function. We limit our attention to the analogs of the $V_{[1,c]}$ and $V_{[-1,c]}$. These are the Weyl unitary operators acting on $H^2(\mathbf{C}^n, d\mu)$ by

$$(W_c f)(z) = k_c(z) f(z - c)$$

and the involutive unitary operators

$$(U_c f)(z) = k_c(z) f(c - z).$$

It is well known [1] that the map $c \to W_c$ gives a strongly continuous projective irreducible representation of $(\mathbf{C}^n, +)$ which extends to a unitary representation of the Heisenberg group. For $\chi_c(z) = \exp(i\mathrm{Im}\{z \cdot c\})$, we have

$$W_a W_b = \chi_a(b/2) W_{a+b}.$$

It follows that $W_c^* = W_c^{-1} = W_{-c}$. It is also easy to check that

$$W_c k_a = \chi_c(a/2) k_{a+c}.$$

For $U_c$, it is easy to check that $U_c^{-1} = U_c^* = U_c$ but the multiplicative structure is not evident. A direct calculation shows that

$$U_c k_a = \chi_a(c/2) k_{c-a}.$$

We can now establish results analogous to Theorem 2.

**Theorem 4.** *We have* $\|\widetilde{W_c}\|_\infty = \exp(-|c|^2/4)$ *and* $\|\widetilde{U_c}\|_\infty = 1$ *for all* $c$ *in* $\mathbf{C}^n$.

*Proof.* We check first that

$$\langle W_c k_a, k_a \rangle = \chi_c(a) \exp(-|c|^2/4).$$

It follows immediately that $\|\widetilde{W_c}\|_\infty = \exp(-|c|^2/4)$. We also have

$$\langle U_c k_a, k_a \rangle = \exp(-|c - 2a|^2/4)$$

and, taking $a = c/2$, it follows that $\|\widetilde{U_c}\|_\infty = 1$. □

The method of Theorem 3 shows that Berger's Theorem fails for $\|\widetilde{X}\|_\infty$ with $X$ in $Op[H^2(\mathbf{C}^n, d\mu)]$.

**Theorem 5.** *For* $Y_c = W_c + W_{-c}$, *we have* $\|\widetilde{Y_c}\|_\infty = 2\exp(-|c|^2/4)$ *and* $\|\widetilde{Y_c^2}\|_\infty = 2\big(1 + \exp(-|c|^2)\big)$ *for all* $c$ *in* $\mathbf{C}^n$. *Thus,* $\|\widetilde{Y_c^2}\|_\infty > \|\widetilde{Y_c}\|_\infty^2$ *for all* $c$ *with* $|c| \neq 0$.

*Proof.* This is a direct calculation using the fact that $W_c^2 = W_{2c}$. □

## 4. Extensions to general bounded symmetric domains

It is natural to try to give general versions of our results for operators on the Bergman space $L_a^2(\Omega)$ of square-integrable holomorphic functions on $\Omega$, a general bounded symmetric domain (BSD) in $\mathbf{C}^n$. Here we use normalized Lebesgue measure on $\Omega$. We do not have a complete picture, but there is enough to justify a brief discussion.

BSD's are Hermitian symmetric spaces of the non-compact type [2], [7], [9]. There is a standard classification of BSD's going back to H. Cartan. We work in the Harish-Chandra realization of BSD's as bounded convex domains $\Omega$ containing the origin 0 of $\mathbf{C}^n$ and invariant under the map $z \to \lambda z$ for $\lambda$ in $\mathbf{C}$ and $|\lambda| = 1$. The group $Aut(\Omega)$ of biholomorphic automorphisms of $\Omega$ is transitive. In particular, for each $c$ in $\Omega$, there is an automorphism $\varphi_c$ so that: (1) $\varphi_c \circ \varphi_c =$ identity, (2) $\varphi_c(0) = c$, and (3) $\varphi_c(a(c)) = a(c)$, for $a(c)$ the midpoint, in the Bergman metric, of the unique geodesic segment joining $c$ to 0. Note that, on $\mathbf{D}$, $\varphi_c = [-1, c]$ and $a(c)$ is determined by (†).

For the Bergman kernel functions $K(z,a)$ on $\Omega$, we have $K(z,a) = \overline{K(a,z)}$ and $K(z,0) = 1$. It is also known that $K(z,a) \neq 0$; see [10]. For $k_a(z) = K(z,a)\{K(a,a)\}^{-1/2}$, we have $\|k_a\| = 1$ in $L^2_a(\Omega)$. It is known that $K(\lambda a, \lambda b) = K(a,b)$ for $|\lambda| = 1$ with $\lambda$ in $\mathbf{C}$ and there are transformation laws [3, pp. 926-928]

$$K(\varphi_c(z), \varphi_c(a)) k_c(z) \overline{k_c(a)} = K(z,a). \tag{††}$$

Considering the involutive unitary operators

$$(U_c f)(z) = k_c(z) f(\varphi_c(z))$$

on $L^2_a(\Omega)$, we know [3] that $U_c^* = U_c^{-1} = U_c$ and we can partially extend Theorem 2.

**Theorem 6.** *For arbitrary $a, c$ in $\Omega$, we have*

$$(U_c k_a)(z) = \lambda(c,a) k_{\varphi_c(a)}(z)$$

*for all $\lambda(c,a)$ in $\mathbf{C}$ with $|\lambda(c,a)| = 1$. Taking $a = a(c)$ to be the fixed point of $\varphi_c$ described above, we find that $\|\widetilde{U_c}\|_\infty = 1$.*

*Proof.* Using (††), we have

$$\begin{aligned} K(\varphi_c(z), a) &= K\Big(\varphi_c(z), \varphi_c(\varphi_c(a))\Big) \\ &= \frac{K(z, \varphi_c(a)) K(c,c)}{K(z,c) K(c, \varphi_c(a))}. \end{aligned} \tag{† † †}$$

It follows from the definition of $U_c$ that

$$\begin{aligned} (U_c k_a)(z) &= k_c(z) k_a(\varphi_c(z)) \\ &= \frac{K(z,c) K(\varphi_c(z), a)}{K(c,c)^{1/2} K(a,a)^{1/2}}. \end{aligned} \tag{† † † †}$$

Combining († † †) and († † † †) gives

$$(U_c k_a)(z) = \lambda(c,a) k_{\varphi_c(a)}(z) \tag{†∗}$$

and, since $U_c$ is unitary, we must have $|\lambda(c,a)| = 1$.

Now taking $a = a(c)$, the fixed point of $\varphi_c$ discussed above, we have

$$(U_c k_{a(c)})(z) = \lambda(c, a(c)) k_{a(c)}(z) \tag{† ∗ ∗}$$

so that $|\widetilde{U_c}(a(c))| = 1$ and $\|\widetilde{U_c}\|_\infty = 1$. □

*Remarks.* It is not hard to check that

$$\langle U_c k_a, k_a \rangle = \frac{K(\varphi_c(a), a))}{K(\varphi_c(a), c)} \frac{K(c,c)^{1/2}}{K(a,a)}.$$

Since $U_c^* = U_c$, it follows that

$$\frac{K(\varphi_c(a), a))}{K(\varphi_c(a), c)} \tag{† † ∗}$$

must be real-valued. In the case $\Omega = \mathbf{D}$, the expression in († † ∗) is always positive. We do not know whether positivity persists in general.

For a general BSD $\Omega$, the analysis of the Weyl-type operator

$$(\dagger * \dagger) \qquad (V_c f)(z) = k_c(z) f(-\varphi_c(z))$$

is non-trivial. We can check that $V_c$ is unitary, with

$$V_c k_a = \mu(c,a) k_{\varphi_c(-a)}$$

for $|\mu(c,a)| = 1$ and all $c, a$ in $\Omega$. It remains difficult to determine $\|\widetilde{V_c}\|_\infty$.

## 5. Problems

The most obvious problems left open are:

**Problem 1.** For $\Omega$ a BSD in $\mathbf{C}^n$ with boundary $\partial\Omega$, is

$$\lim_{c \to \partial\Omega} \|\widetilde{V_c}\|_\infty = 0$$

for $V_c$ defined by $(\dagger * \dagger)$?

**Problem 2.** Is there a bounded domain (perhaps not BSD) where $\|\widetilde{X}\|_\infty$ is an equivalent norm to $\|X\|$ on $Op\{L^2_a(\Omega)\}$?

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