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ℓ^2-LINEAR INDEPENDENCE FOR THE SYSTEM OF INTEGER TRANSLATES OF A SQUARE INTEGRABLE FUNCTION

SANDRA SALIANI

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ABSTRACT. We prove that if the system of integer translates of a square integrable function is ℓ^2-linear independent, then its periodization function is strictly positive almost everywhere. Indeed we show that the above inference holds for any square integrable function since the following statement on Fourier analysis is true: For any (Lebesgue) measurable subset A of $[0,1]$, with positive measure, there exists a nontrivial square summable function, with support in A, whose partial sums of Fourier series are uniformly bounded in the uniform norm. This answers a question posed by Guido Weiss.

1. INTRODUCTION

Given a square integrable function $\psi \in L^2(\mathbb{R})$, many properties of the system of integer translates

$$\mathscr{B}_\psi = \{T_k\psi, k \in \mathbb{Z}\}, \quad T_k\psi(x) = \psi(x-k), \quad x \in \mathbb{R},\, k \in \mathbb{Z},$$

can be completely described in terms of properties of the 1-periodic function

$$p_\psi(\xi) = \sum_{k\in\mathbb{Z}} |\hat{\psi}(\xi+k)|^2, \quad \xi \in \mathbb{R},$$

called the periodization function of ψ (note that $p_\psi \in L^1(\mathbb{T})$). Systems of integer translates arise in the context of wavelet analysis and, more generally, in the theory of shift-invariant spaces. We refer the reader to the work of Hernández, Šikić, Weiss, and Wilson [1] and to references contained therein for a comprehensive summary of these properties.

In this paper we focus on ℓ^2-linear independence. Let us recall different concepts of independence in great generality.

Definition 1.1. Let $(e_n)_{n\in\mathbb{N}}$ be a sequence in a Hilbert space H. We say that

(i) $(e_n)_{n\in\mathbb{N}}$ is *linearly independent* if every finite subsequence of $(e_n)_{n\in\mathbb{N}}$ is linearly independent.

(ii) $(e_n)_{n\in\mathbb{N}}$ is *ℓ^2-linearly independent* if whenever the series $\sum_{n=0}^{+\infty} c_n e_n$ is convergent and equal to zero for some coefficients $(c_n)_{n\in\mathbb{N}} \in \ell^2(\mathbb{N})$, then necessarily $c_n = 0$ for all $n \in \mathbb{N}$.

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(iii) $(e_n)_{n\in\mathbb{N}}$ is *ω-independent* if whenever the series $\sum_{n=0}^{+\infty} c_n e_n$ is convergent and equal to zero for some scalar coefficients $(c_n)_{n\in\mathbb{N}}$, then necessarily $c_n = 0$ for all $n \in \mathbb{N}$.

(iv) $(e_n)_{n\in\mathbb{N}}$ is *minimal* if for all $k \in \mathbb{N}$, $e_k \notin \overline{\operatorname{span}\{e_n,\, n \neq k\}}$.

Since we will not always be dealing with unconditionally convergent series, we order $\mathbb{Z} = \{0, 1, -1, 2, -2, \dots\}$ as is usually done with Fourier series.

Hence $\sum_{k\in\mathbb{Z}} c_k T_k \psi = 0$ means $\lim_{n\to+\infty} \sum_{|k|\le n} c_k T_k \psi = 0$.

So $\mathscr{B}_\psi$ is ℓ^2-linearly independent if and only if whenever $\{c_k\} \in \ell^2$ and

$$\lim_{n\to+\infty} \left\|\textstyle\sum_{|k|\le n} c_k\, T_k \psi\right\|_2 = 0,$$

then necessarily $c_k = 0$ for all $k \in \mathbb{Z}$.

Relations between the various types of independence for $\mathscr{B}_\psi$ and properties of the periodization function are summarized in the following scheme:

$\mathscr{B}_\psi$ is minimal	$\Longleftrightarrow$	$\dfrac{1}{p_\psi} \in L^1(\mathbb{T})$
$\Downarrow$		
$\mathscr{B}_\psi$ is ω-independent		
$\Downarrow$		
$\mathscr{B}_\psi$ is ℓ^2-linearly independent	$\Longleftarrow$	$p_\psi(\xi) > 0$ a.e.
$\Downarrow$		
$\mathscr{B}_\psi$ is linearly independent		Always true

In particular, it is known that

$$p_\psi(\xi) > 0 \text{ a.e. } \Rightarrow \mathscr{B}_\psi \text{ is } \ell^2\text{-linearly independent.}$$

A question raised in [1] is the following: Is the converse true?

Šikić and Speegle [5] have given a positive answer if p_ψ is bounded. See also the note by Nielsen and Šikić [3] for a condition characterizing $\mathscr{B}_\psi$ being a Schauder basis in terms of p_ψ. Recently Paluszyński [4] proved that $p_\psi(\xi) > 0$ a.e. is equivalent to L^2-Cesàro linear independence of $\mathscr{B}_\psi$. The latter means that if the Cesàro averages

$$\frac{1}{n}\sum_{h=0}^{n-1} S_h, \quad S_h = \sum_{|k|\le h} c_k T_k \psi, \quad \{c_k\} \in \ell^2,$$

tend to zero in the L^2-norm, then necessarily $c_k = 0$, for all $k \in \mathbb{Z}$.

The approach we have used in addressing this problem has been global in nature: rather than examining the assumptions to be put on a single ψ, we analyze the issue as a whole, for all $\psi \in L^2(\mathbb{R})$.

The result is that the converse holds for any ψ if the following statement on Fourier analysis is true: For any (Lebesgue) measurable subset A of $[0, 1]$, with positive measure $|A|$, we can find a nontrivial square summable function, with support in A, whose partial sums of Fourier series are uniformly bounded in the uniform norm.

By support of $f \in L^1(\mathbb{T})$ (denoted $\operatorname{supp} f$) we mean the smallest closed set S such that $f(\xi) = 0$ almost everywhere in the complement of S.

After the proof of the main result in Section 2, in Section 3 existence of such a good function for any measurable set $A \subset [0, 1]$ is shown.

Existence is obtained as a corollary of general results by Kislyakov and Vinogradov, although we realize that there may be other direct proofs that we are not aware of.

We end with some notation. If $A \subset [0,1]$ we set $A^c = [0,1] \setminus A$. For a Lebesgue measurable set E, χ_E is the characteristic function of E: $\chi_E(\xi) = 1$ if $\xi \in E$, zero otherwise; $|E|$ denotes its Lebesgue measure. For $f \in L^2(\mathbb{T})$ the symmetric partial sums of the Fourier series are

$$S_n(f)(\xi) = \sum_{|k|\le n} \hat{f}(k) e^{2\pi i k\xi}.$$

For $f \in L^1(\mathbb{R})$, the Fourier transform is

$$\hat{f}(\xi) = \int_{\mathbb{R}} f(x)\, e^{-2\pi i x\xi}\, dx.$$

2. Main result

In this section we prove the main result. We first show that we need only uniform boundedness of the uniform norm restricted to the complement of the set A.

Theorem 2.1. *Suppose the following statement is true:*

(S) For every measurable $A \subset [0,1]$, $|A| > 0$, $|A^c| > 0$, there exists $0 \neq f \in L^2(\mathbb{T})$, such that

(1) $\operatorname{supp} f \subset A$;

(2) $\operatorname*{ess\,sup}_{t\notin A} |S_n(f)(t)| = \|S_n(f)\|_{L^\infty(A^c)}$ *are uniformly bounded*

(i.e. there exists $M > 0$ such that $\|S_n(f)\|_{L^\infty(A^c)} \le M$, for all $n \in \mathbb{N}$).

Then

for any $0 \neq \psi \in L^2(\mathbb{R})$ the following statement is true:

$$\mathscr{B}_\psi \text{ is } \ell^2\text{-linearly independent } \Rightarrow p_\psi(\xi) > 0 \text{ a.e.}$$

Proof. Let $0 \neq \psi \in L^2(\mathbb{R})$. We shall show that if the set where $p_\psi(\xi) = 0$ has positive measure, then $\mathscr{B}_\psi$ is not ℓ^2-linearly independent.

Assume $|A| = |\{p_\psi(\xi) = 0\}| > 0$. Since $\psi \neq 0$, then $|A^c| > 0$.

By (S) there exists $0 \neq f \in L^2(\mathbb{T})$ such that both (1) and (2) hold. Now a simple calculation shows that

$$\|\sum_{|k|\le n} \hat{f}(-k)\, T_k\psi\|_2^2 = \int_0^1 |\sum_{|k|\le n} \hat{f}(k) e^{2\pi i k\xi}|^2\, \chi_{A^c}(\xi)\, p_\psi(\xi)\, d\xi. \tag{2.1}$$

By a.e. convergence of the partial sums to f and $\operatorname{supp} f \subset A$, we get a.e.

$$\lim_n |\sum_{|k|\le n} \hat{f}(k) e^{2\pi i k\xi}|^2\, \chi_{A^c}(\xi)\, p_\psi(\xi) = 0.$$

By uniform boundedness of $S_n(f)$ in A^c, $p_\psi \in L^1(\mathbb{T})$, and by the Lebesgue dominated convergence theorem we get

$$\lim_n \int_0^1 |\sum_{|k|\le n} \hat{f}(k) e^{2\pi i k\xi}|^2\, \chi_{A^c}(\xi)\, p_\psi(\xi) = 0.$$

So equality (2.1) gives us a nonzero sequence $c_k = \hat{f}(-k)$ in $\ell^2(\mathbb{Z})$ such that

$$\sum_{k\in\mathbb{Z}} c_k\, T_k\psi = 0,$$

and so $\mathscr{B}_\psi$ is not ℓ^2-linearly independent. □

Example 2.2. (1) If $A \subset [0,1]$ contains an interval I, the function $f = \chi_J$, for any interval J interior to I, verifies conditions (1) and (2) of the previous theorem. Indeed, by the localization principle, $S_n(f)$ converges uniformly (to zero) in every closed interval contained in J^c. But J^c consists of, at most, two intervals strictly containing A^c, hence the uniform boundedness.

(2) Let A be the set of irrationals in $[0,1/2]$. Then A is totally disconnected but the function $f = \chi_{A\cap[0,1/4]}$ has the same Fourier series of $\chi_{[0,1/4]}$ and works well as in the previous case.

(3) Let A be a Cantor-like set, in $[0,1]$, of positive measure. Let f be any function with support in A. Then $S_n(f)$ converges uniformly in any closed subinterval of the intervals (a_n, b_n) contiguous to A, but a priori nothing can be said for $\bigcup_n (a_n, b_n) = A^c$.

Remark 2.3. In light of Example 2.2(1), it is easy to prove that the statement (S) in Theorem 2.1 is equivalent to the apparent stronger requirement:

(S′) For every measurable $A \subset [0,1]$, $|A| > 0$, $|A^c| > 0$, there exists $0 \neq f \in L^2(\mathbb{T})$, such that

(1) $\operatorname{supp} f \subset A$,
(2) $\|S_n(f)\|_{L^\infty(\mathbb{T})}$ are uniformly bounded.

Indeed, to prove that (S) implies (S′), it is sufficient to assume that A does not contain an interval; otherwise statement (S′) is true regardless of (S).

We are going to show, by a density argument, that the same function f provided by (S) verifies (S′).

Let f and M be as in (S). Let $\xi_0 \in A$ and $n \in \mathbb{N}$. Since $S_n(f)$ is continuous in ξ_0, we can find an open neighborhood I_0 of ξ_0 such that

$$|S_n(f)(\xi) - S_n(f)(\xi_0)| < 1, \quad \text{for all } \xi \in I_0.$$

There exists at least one $\xi \in A^c \cap I_0$ (otherwise A contains an interval), and finally

$$|S_n(f)(\xi_0)| \le 1 + |S_n(f)(\xi)| < 1 + M.$$

3. Existence

The existence of a function f satisfying condition (S′), for a given set $A \subset [0,1]$, can be derived by the following theorem in Kislyakov's paper [2, Theorem 4], whose proof relies also upon a result by Vinogradov [6]. The latter makes use of the Carleson almost everywhere convergence theorem.

Let us recall some basic notation: U^∞ denotes the space of functions $f \in L^\infty(\mathbb{T})$ for which the following norm is finite:

$$\|f\|_{U^\infty} = \sup\left\{ \left| \sum_{n\le k\le m} \hat{f}(k)\xi^k \right|,\ n,m \in \mathbb{Z},\ n \le m,\ \xi \in \mathbb{T} \right\}.$$

Theorem 3.1. *For every $F \in L^\infty(\mathbb{T})$ with $\|F\|_\infty \leq 1$ and every $0 < \varepsilon \leq 1$ there exists a function $G \in U^\infty$ with the following properties: $|G| + |F - G| = |F|$; $|\{\xi \in \mathbb{T}, F(\xi) \neq G(\xi)\}| \leq \varepsilon \|F\|_1$; $\|G\|_{U^\infty} \leq \mathrm{const}(1 + \log(\varepsilon^{-1}))$.*

The application of Theorem 3.1 is clear: For any measurable set $A \subset [0,1]$, $|A| > 0$, $|A^c| > 0$, it is sufficient to take $F = \chi_A$ and $0 < \varepsilon < 1$. Then G provided above is not zero since otherwise $|\{\xi \in \mathbb{T}, \chi_A(\xi) \neq 0\}| = |A| \leq \varepsilon |A|$. The support of G is contained in A since $|G| \leq |F|$, and finally $\|G\|_{U^\infty} \leq \mathrm{const}(1 + \log(\varepsilon^{-1}))$ implies that the partial sums of Fourier series of G are uniformly bounded in the uniform norm.

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Dipartimento di Matematica e Informatica, Università degli Studi della Basilicata, 85100 Potenza, Italia

E-mail address: sandra.saliani@unibas.it