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NON-EXISTENCE OF QUADRATIC HARMONIC MAPS OF S^4 INTO S^5 OR S^6

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ABSTRACT. In this paper, we settle the last two open cases of non-existence of full quadratic harmonic maps from S^4 to S^5 or S^6. Assume that there exist full quadratic harmonic maps from S^4 to S^n for some integer n. As a consequence of our theorem we obtain that the sufficient and necessary condition of the existence of such maps is that n satisfy $4 \leq n \leq 13$ and $n \neq 5, 6$.

1. INTRODUCTION

Let S^n denote the unit sphere in the Euclidean space $\mathbb{R}^{n+1}$. A quadratic harmonic map $f : S^m \to S^n$ is the restriction of a map $F : \mathbb{R}^{m+1} \to \mathbb{R}^{n+1}$ whose components are harmonic polynomials of homogeneous degree 2. Such a map f is called *full* if the image of f spans $\mathbb{R}^{n+1}$. A spherical harmonic on S^m of order p is an eigenfunction of the spherical Laplacian with eigenvalue $\lambda_p = p(p + m - 1)$. It is well known that a spherical harmonic of order p is the restriction to S^m of a harmonic polynomial of homogeneous degree p in $\mathbb{R}^{m+1}$. So a quadratic harmonic map $f : S^m \to S^n$ is also called a λ_2-eigenmap, and generally one can investigate a λ_p-eigenmap. A λ_p-eigenmap is a harmonic map with constant energy density $\lambda_p/2$. Up to isometries on the domain and the range, how many equivalence classes are there for the given m, n and p? The range dimension n obviously depends on the given m and p. What are the possible values of n for the given m and p? These problems are far from being solved even for $p = 2$ [2, 3, 5, 10, 11]. Besides the classical examples such as the Hopf constructions and the Veronese maps, there are several available effective ways of constructing new eigenmaps out of the old ones [4, 7, 8, 10, 11]. Calabi proved that any full λ_2-eigenmap $f : S^2 \to S^n$ is rigid; that is, any such map is equivalent to the Veronese map $S^2 \to S^4$. In 1987, G. Toth [6] gave a complete classification of full λ_2-eigenmaps from S^3 to S^n. In 2003, after giving a rigidity result for a λ_2-eigenmap from S^4 to itself, Huixia He, Hui Ma, and Feng Xu [7] completely solved the existence problem of λ_2-eigenmaps from S^{2n-k} to S^n for $k = 1, \cdots, 5$. Gauchman, Toth, Lam, Tang, Ueno and Yiu have done much work on quadratic harmonic maps between spheres; see [5, 6, 7, 8, 9, 10, 11, 12] for more details.

The existence problem of λ_p-eigenmaps between the Euclidean spheres constantly generates great interest from researchers since many challenging problems are still open. In 1994, Gauchman and Toth [4] showed that full λ_2-eigenmaps $f : S^4 \to S^n$

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exist for $n = 4, 7$ and $9 \le n \le 13$. As a special case of their results, Huixia He et al. [7] showed that the existence result is also true for $n = 8$. As remarked in [7], the only unsettled range dimensions of full λ_2-eigenmaps $f : S^4 \to S^n$ are $n = 5$ and $n = 6$. In this paper we prove

Theorem 1.1. *There are no full λ_2-eigenmaps $f : S^4 \to S^n$ for $n = 5, 6$.*

Combining Theorem 1.1 with the known results [4, 7, 10] we claim immediately

Corollary 1.2. *There exist full λ_2-eigenmaps $f : S^4 \to S^n$ if and only if $4 \le n \le 13$ and $n \ne 5$ or 6.*

In section 2, we introduce some preliminaries which are needed in the proof of our main Theorem 1.1. Finally, in section 3 we give the proof of Theorem 1.1.

2. Preliminaries

Suppose that $F : \mathbb{R}^m \to \mathbb{R}^n$ is a quadratic form; that is, each component of F is a homogeneous polynomial of degree 2. We may assume that

$$F(x) = \sum_{i=1}^{m}\sum_{j=1}^{m} \boldsymbol{a}_{ij} x_i x_j, \boldsymbol{a}_{ij} = \boldsymbol{a}_{ji} \in \mathbb{R}^n, \boldsymbol{x} = (x_1, \cdots, x_m) \in \mathbb{R}^m.$$

Then a direct computation shows that

Lemma 2.1. *The notation is as above. Then the following identity holds:*

$$\begin{aligned} &|F(x)|^2 \\ &\quad= \sum_{i=1}^{m} |\boldsymbol{a}_{ii}|^2 x_i^4 + \sum_{1\le i<j\le m} [2(2|\boldsymbol{a}_{ij}|^2 + \boldsymbol{a}_{ii}\boldsymbol{a}_{jj})(x_i x_j)^2 \\ &\qquad + 4(\boldsymbol{a}_{ij}\boldsymbol{a}_{ii} x_i^3 x_j + \boldsymbol{a}_{ij}\boldsymbol{a}_{jj} x_i x_j^3)] \\ &\qquad + 4 \sum_{1\le i<j<k\le m} [(2\boldsymbol{a}_{ij}\boldsymbol{a}_{ik} + \boldsymbol{a}_{ii}\boldsymbol{a}_{jk}) x_i^2 x_j x_k + (2\boldsymbol{a}_{ij}\boldsymbol{a}_{jk} + \boldsymbol{a}_{jj}\boldsymbol{a}_{ik}) x_i x_j^2 x_k \\ &\qquad + (2\boldsymbol{a}_{ik}\boldsymbol{a}_{jk} + \boldsymbol{a}_{kk}\boldsymbol{a}_{ij}) x_i x_j x_k^2] \\ &\qquad + 8 \sum_{1\le i<j<k<l\le m} (\boldsymbol{a}_{ij}\boldsymbol{a}_{kl} + \boldsymbol{a}_{ik}\boldsymbol{a}_{jl} + \boldsymbol{a}_{il}\boldsymbol{a}_{jk}) x_i x_j x_k x_l. \end{aligned}$$

An orthogonal multiplication is a bilinear map $f : \mathbb{R}^p \times \mathbb{R}^q \to \mathbb{R}^r$ satisfying $|f(x, y)| = |x| \cdot |y|$ for $x \in \mathbb{R}^p, y \in \mathbb{R}^q$. Here $|x|$ is the standard Euclidean norm. If $p = q$, then the Hopf construction $\Phi(f)$ defined by

$$\Phi(f) : \mathbb{R}^{2p} \to \mathbb{R}^{r+1}, (x, y) \in \mathbb{R}^p \times \mathbb{R}^p \mapsto (|x|^2 - |y|^2, 2f(x, y))$$

induces a λ_2-eigenmap $f : S^{2p-1} \to S^r$. There is a close relationship between orthogonal multiplication and λ_2-eigenmaps. The following elegant result is due to Zi Zhou Tang [10]. The first part of it plays a crucial role in the present paper.

Theorem 2.2 (Tang)**.** *Let $g : S^m \to S^n$ be a non-constant λ_2-eigenmap. Then for each $p \in Im(g)$, $g^{-1}(p)$ is a great sphere with $dim(g^{-1}(p)) \le \frac{m-1}{2}$, and g is homotopic to the Hopf construction on the orthogonal multiplication $\mathbb{R}^{k+1} \times \mathbb{R}^{m-k} \to \mathbb{R}^n$, where $k = dim(g^{-1}(p))$. Furthermore, if the equality $dim(g^{-1}(p)) = \frac{1}{2}(m-1)$ is achieved, then g is exactly the Hopf construction on the orthogonal multiplication $\mathbb{R}^{\frac{m+1}{2}} \times \mathbb{R}^{\frac{m+1}{2}} \to \mathbb{R}^n$.*

The existence of a full λ_2-eigenmap $f : S^m \to S^n$ is equivalent to the existence of a map $F : \mathbb{R}^{m+1} \to \mathbb{R}^{n+1}, x \mapsto (F_1(x), \cdots, F_{n+1}(x)), x \in \mathbb{R}^{m+1}$, such that F is the extension of f, that is,

$$F(x) = |x|^2 f\left(\frac{x}{|x|}\right), x \in \mathbb{R}^{m+1}, x \neq 0.$$

The function $F(x)$ is also called a quadratic form. Each component $F_i(x)$ of F is a harmonic polynomial of homogeneous degree 2. By Theorem 2.2, there must be some point $p \in Im(f)$ with $dim(f^{-1}(p)) = k$. By suitable orthogonal transformations on $\mathbb{R}^{m+1}$ and $\mathbb{R}^{n+1}$ respectively, we may assume that $p = (1, 0, \cdots, 0) \in \mathbb{R}^{n+1}, f^{-1}(p) = (\xi_1, \cdots, \xi_{k+1}, 0, \cdots, 0) \in \mathbb{R}^{m+1}$, and

$$\sum_{i=1}^{k+1} \xi_i^2 = 1, k = dim(f^{-1}(p)). \tag{2.1}$$

Since $F_1(x)$ can always be diagonalized, from (2.1) and using the same arguments as in [7], we find that

$$F_1(x) = \sum_{i=1}^{m+1} \lambda_i x_i^2, \sum_{i=1}^{m+1} \lambda_i = 0, \tag{2.2}$$

where

$$\lambda_1 = \cdots = \lambda_{k+1} = 1, -1 \leq \lambda_i < 1, i = k+2, \cdots, m+1.$$

We can express

$$(F_2(x), \cdots, F_{n+1}(x)) \equiv (0, F_2(x), \cdots, F_{n+1}(x)) \in \mathbb{R}^n = span\{\boldsymbol{E}_2, \cdots, \boldsymbol{E}_{n+1}\}$$

as

$$(F_2(x), \cdots, F_{n+1}(x)) = \sum_{i=1}^{m+1} \sum_{j=1}^{m+1} \boldsymbol{b}_{ij} x_i x_j, \boldsymbol{b}_{ij} = \boldsymbol{b}_{ji} \in \mathbb{R}^{n+1}, \tag{2.4}$$

where $\boldsymbol{E}_2, \cdots, \boldsymbol{E}_{n+1}$ is the orthonormal basis of $\mathbb{R}^{n+1}$, but $\boldsymbol{E}_1$ is missing here.

Lemma 2.3. *Suppose that there exists a full λ_2-eigenmap $f : S^m \to S^n$ and $F : \mathbb{R}^{m+1} \to \mathbb{R}^{n+1}$ as its extension with components $F_i(x)$, $x \in \mathbb{R}^{m+1}$, $i = 1, 2, \cdots, n+1$. We can always assume that $F_1(x)$ satisfies* (2.2). *Define $\boldsymbol{b}_{ij}$ as in* (2.4). *Then we have the following identities:*

$$|\boldsymbol{b}_{ii}|^2 = 1 - \lambda_i^2, i = 1, \cdots, m+1. \tag{2.5}$$

$$\boldsymbol{b}_{ii}\boldsymbol{b}_{ij} = 0; \boldsymbol{b}_{jj}\boldsymbol{b}_{ij} = 0, 1 \leq i < j \leq m+1. \tag{2.6}$$

$$2|\boldsymbol{b}_{ij}|^2 + \boldsymbol{b}_{ii}\boldsymbol{b}_{jj} = 1 - \lambda_i\lambda_j, 1 \leq i < j \leq m+1. \tag{2.7}$$

$$2\boldsymbol{b}_{ij}\boldsymbol{b}_{ik} + \boldsymbol{b}_{ii}\boldsymbol{b}_{jk} = 0; 2\boldsymbol{b}_{ij}\boldsymbol{b}_{jk} + \boldsymbol{b}_{jj}\boldsymbol{b}_{ik} = 0; 2\boldsymbol{b}_{ik}\boldsymbol{b}_{jk} + \boldsymbol{b}_{kk}\boldsymbol{b}_{ij} = 0, \quad 1 \leq i < j < k \leq m+1. \tag{2.8}$$

$$\boldsymbol{b}_{ij}\boldsymbol{b}_{kl} + \boldsymbol{b}_{ik}\boldsymbol{b}_{jl} + \boldsymbol{b}_{il}\boldsymbol{b}_{jk} = 0, 1 \leq i < j < k < l \leq m+1. \tag{2.9}$$

Proof. From the discussion above, we see that the first part of Lemma 2.3 is true. By applying Lemma 2.1 and comparing the coefficients of the leftmost and rightmost terms of the following equality:

$$(\sum_{i=1}^{m+1}\sum_{j=1}^{m+1}\boldsymbol{b}_{ij}x_ix_j)^2=\sum_{i=2}^{n+1}F_i^2(x)=F^2(x)-F_1^2(x)$$
$$=\sum_{i=1}^{m+1}(1-\lambda_i^2)x_i^4+2\sum_{1\le i<j\le m+1}(1-\lambda_i\lambda_j)(x_ix_j)^2,$$

we obtain the five desired identities (2.5)–(2.9). □

3. Proof of Theorem 1.1

The main idea to prove Theorem 1.1 is different from that given in [7]. He, Ma, and Xu use a property of quaternion algebras and a result due to Casseles, while we try to diagonalize two vectors $\widetilde{N_1},\widetilde{N_2}$ simultaneously; see more details below.

We first consider the case $n=6$. That is, we suppose that there exists a full λ_2-eigenmap

$$f:S^4\to S^6$$

with the extension map $F:\mathbb{R}^5\to\mathbb{R}^7$. Since RP^4 cannot be immersed into S^6, both (2.1) and (2.2) hold true for $k=1$, $m=4$ and $n=6$.

For $i=1,2$, we can express $(F_2(x),\cdots,F_7(x))\in\mathbb{R}^6=span\{E_2,\cdots,E_7\}$ as the sum $L_i+M_i+N_i$ of three vector-valued functions, where $E_2,\cdots,E_7$ is the standard orthonormal basis of $\mathbb{R}^7$ but E_1 is missing here. Furthermore

$$L_i=\boldsymbol{b}_{ii}x_i^2,M_i=\sum_{1\le\alpha,\beta\le5;\alpha,\beta\ne i}\boldsymbol{b}_{\alpha\beta}x_\alpha x_\beta,\boldsymbol{b}_{\alpha,\beta}=\boldsymbol{b}_{\beta\alpha},$$

$$N_i=2\sum_{\alpha=1;\alpha\ne i}^{5}b_{i\alpha}x_ix_\alpha,\boldsymbol{b}_{i\alpha}=\boldsymbol{b}_{\alpha i};\boldsymbol{b}_{ii},\boldsymbol{b}_{\alpha\beta},b_{i\alpha}\in\mathbb{R}^6. \tag{3.1}$$

Define $\widetilde{N_i}$ by

$$N_i=\sqrt{2}x_i\widetilde{N_i}. \tag{3.2}$$

Then we have the following.

Lemma 3.1. *Both $\widetilde{N_1}$ and $\widetilde{N_2}$ can be diagonalized simultaneously. To be more precise, by performing an orthogonal transformation we may assume that*

$$\widetilde{N_1}=\sqrt{2}(c_3x_3,0,c_4x_4,c_5x_5,0,0),\widetilde{N_2}=\sqrt{2}(0,c_3x_3,c_5x_5,-c_4x_4,0,0), \tag{3.3}$$

where

$$c_\alpha=\sqrt{\frac{1-\lambda_\alpha}{2}},\alpha=1,2,\cdots,7,$$

and λ_α is determined by (2.2).

Proof. From (3.1) and (3.2) it is easy to see that

$$\widetilde{N_i}=\sqrt{2}\sum_{\alpha=1;\alpha\ne i}^{5}\boldsymbol{b}_{i\alpha}x_\alpha,i=1,2. \tag{3.4}$$

First note that $\boldsymbol{b}_{ij} = \boldsymbol{b}_{ji}$. It is easy to check that (2.9) still holds for all $1 \leq i, j, k, l \leq 5$. Since $k = dim(f^{-1}(p)) = 1$, from (2.5) and (2.7) we have $\boldsymbol{b}_{ij} = 0$ for $1 \leq i, j \leq 2$. Thus the inner product of $\widetilde{N_1}$ and $\widetilde{N_2}$ satisfies

$$\begin{aligned}
\widetilde{N_1}\widetilde{N_2} &= 2\sum_{\alpha=1;\alpha\neq 1}^{5} \boldsymbol{b}_{1\alpha}x_\alpha \sum_{\beta=1;\beta\neq 2}^{5} \boldsymbol{b}_{2\beta}x_\beta \\
&= -2\sum_{\alpha,\beta;\alpha\neq 1,\beta\neq 2} (\boldsymbol{b}_{12}\boldsymbol{b}_{\alpha\beta} + \boldsymbol{b}_{1\beta}\boldsymbol{b}_{2\alpha})x_\alpha x_\beta \\
&= -2\sum_{\alpha,\beta;\alpha\neq 1,\beta\neq 2} \boldsymbol{b}_{1\beta}\boldsymbol{b}_{2\alpha}x_\alpha x_\beta \\
&= -\widetilde{N_1}\widetilde{N_2}.
\end{aligned} \tag{3.5}$$

Consequently, $\widetilde{N_1}\widetilde{N_2} = 0$. On the other hand, for $i = 1, 2$, comparing the coefficients of the leftmost and rightmost terms of the following equality

$$\begin{aligned}
&x_i^4 + (\sum_{\alpha=1;\alpha\neq i}^{5} x_\alpha^2)^2 + 2x_i^2 \sum_{\alpha=1;\alpha\neq i}^{5} x_\alpha^2 = |x|^4 = |F|^4 \\
&= x_i^4 + (\sum_{\alpha=1;\alpha\neq i}^{5} x_\alpha^2)^2 + 2x_i^2 \sum_{\alpha=1;\alpha\neq i}^{5} \lambda_\alpha x_\alpha^2 + L_i^2 + M_i^2 + N_i^2 + 2(L_iM_i + L_iN_i + M_iN_i),
\end{aligned}$$

we obtain

$$L_i = 0,\ M_iN_i = 0, \tag{3.6}$$

and

$$|N_i|^2 = 2x_i^2 \sum_{\alpha=3}^{5} \mu_\alpha^2 x_\alpha^2, \mu_\alpha = \sqrt{1-\lambda_\alpha}. \tag{3.7}$$

We may have a similar formula for $|M_i|^2$, but we do not need it later. So we omit it here. From (3.2) and (3.7) we deduce that

$$|\widetilde{N_i}|^2 = \sum_{\alpha=3}^{5} \mu_\alpha^2 x_\alpha^2, i = 1, 2. \tag{3.8}$$

If we choose $\widetilde{N_1}$ and $\widetilde{N_2}$ as in (3.3), then it is easy to check that $\widetilde{N_1}\widetilde{N_2} = 0$ and furthermore (3.8) is also satisfied. Hence up to an orthogonal transformation on $\mathbb{R}^6$, $\widetilde{N_1}$ and $\widetilde{N_2}$ can take the form as in (3.3). This completes the proof of Lemma 3.1.

Now by using the results of previous lemmas and theorems, we continue to prove Theorem 1.1. From Lemma 3.1 and comparing the components of $\widetilde{N_i}$ in (3.3) and (3.4) we get

$$\begin{aligned}
&\boldsymbol{b}_{13} = c_3\boldsymbol{E}_2, \quad \boldsymbol{b}_{14} = c_4\boldsymbol{E}_4, \quad \boldsymbol{b}_{15} = c_5\boldsymbol{E}_5, \\
&\boldsymbol{b}_{23} = c_3\boldsymbol{E}_3, \quad \boldsymbol{b}_{24} = -c_4\boldsymbol{E}_5, \quad \boldsymbol{b}_{25} = c_5\boldsymbol{E}_4.
\end{aligned} \tag{3.9}$$

Applying Lemma 2.3, a direct computation shows that all components of vectors $\boldsymbol{b}_{ij}$ are shown as in Table 1. For example, if we assume that $\boldsymbol{b}_{44} \cdot \boldsymbol{E}_2 = r$, from (2.8) and (3.9) we have

$$\boldsymbol{b}_{34} \cdot \boldsymbol{E}_4 = \boldsymbol{b}_{34} \cdot \frac{1}{c_4}\boldsymbol{b}_{14} = -\frac{1}{2}\frac{1}{c_4}\boldsymbol{b}_{44} \cdot \boldsymbol{b}_{13} = -\frac{1}{2}\frac{c_3}{c_4}\boldsymbol{b}_{44} \cdot \boldsymbol{E}_2 = -\frac{1}{2}\frac{c_3}{c_4}r.$$

TABLE 1

	$\boldsymbol{b}_{33}$	$\boldsymbol{b}_{44}$	$\boldsymbol{b}_{55}$	$\boldsymbol{b}_{34}$	$\boldsymbol{b}_{35}$	$\boldsymbol{b}_{45}$
$\boldsymbol{E}_2$	0	r	$-r$	0	0	$-\frac{1}{2}(\frac{c_5}{c_4}+\frac{c_4}{c_5})s$
$\boldsymbol{E}_3$	0	s	$-s$	0	0	$\frac{1}{2}(\frac{c_5}{c_4}+\frac{c_4}{c_5})r$
$\boldsymbol{E}_4$	0	0	0	$-\frac{1}{2}\frac{c_3}{c_4}r$	$\frac{1}{2}\frac{c_3}{c_5}s$	0
$\boldsymbol{E}_5$	0	0	0	$\frac{1}{2}\frac{c_3}{c_4}s$	$\frac{1}{2}\frac{c_3}{c_5}r$	0
$\boldsymbol{E}_6$	t_1	u_1	$-(t_1+u_1)$	α_1	β_1	γ_1
$\boldsymbol{E}_7$	t_2	u_2	$-(t_2+u_2)$	α_2	β_2	γ_2

Other components can be calculated similarly. Let

$$\boldsymbol{t}=\begin{pmatrix} t_1 \\ t_2 \end{pmatrix}, \boldsymbol{u}=\begin{pmatrix} u_1 \\ u_2 \end{pmatrix}, \boldsymbol{\alpha}=\begin{pmatrix} \alpha_1 \\ \alpha_2 \end{pmatrix}, \boldsymbol{\beta}=\begin{pmatrix} \beta_1 \\ \beta_2 \end{pmatrix}, \boldsymbol{\gamma}=\begin{pmatrix} \gamma_1 \\ \gamma_2 \end{pmatrix}.$$

These are all vectors in $\mathbb{R}^2 = span\{\boldsymbol{E}_6, \boldsymbol{E}_7\}$. From (2.6) and Table 1 we obtain

$$\boldsymbol{t}\cdot\boldsymbol{\alpha} = \boldsymbol{t}\cdot\boldsymbol{\beta} = \boldsymbol{t}\cdot\boldsymbol{\gamma} = \boldsymbol{u}\cdot\boldsymbol{\alpha} = \boldsymbol{u}\cdot\boldsymbol{\beta} = \boldsymbol{u}\cdot\boldsymbol{\gamma} = 0, \tag{3.10}$$

where $\boldsymbol{t}\cdot\boldsymbol{\alpha}$ denotes the inner product of two vectors. From Table 1 and (3.10) we also have $\boldsymbol{\alpha}\cdot\boldsymbol{\beta} = \boldsymbol{b_{34}}\cdot\boldsymbol{b_{35}} = -\frac{1}{2}\boldsymbol{b_{33}}\cdot\boldsymbol{b_{45}} = -\frac{1}{2}\boldsymbol{t}\cdot\boldsymbol{\gamma} = 0$. Similarly, we have

$$\boldsymbol{\alpha}\cdot\boldsymbol{\gamma} = \boldsymbol{\beta}\cdot\boldsymbol{\gamma} = 0. \tag{3.11}$$

From (2.5) and Table 1 we have

$$|\boldsymbol{t}|^2 = 1-\lambda_3^2, \tag{3.12}$$

$$r^2+s^2+|\boldsymbol{u}|^2 = 1-\lambda_4^2, \tag{3.13}$$

$$r^2+s^2+|\boldsymbol{t}|^2+|\boldsymbol{u}|^2+2\boldsymbol{t}\cdot\boldsymbol{u} = 1-\lambda_5^2. \tag{3.14}$$

These three equalities imply that

$$2\boldsymbol{t}\cdot\boldsymbol{u} = -1+\lambda_3^2+\lambda_4^2-\lambda_5^2. \tag{3.15}$$

From (2.7), (3.15) and Table 1 we have

$$\frac{c_3^2}{c_4^2}(r^2+s^2)+4|\boldsymbol{\alpha}|^2 = -4\lambda_5-1, \tag{3.16}$$

$$\frac{c_3^2}{c_5^2}(r^2+s^2)+4|\boldsymbol{\beta}|^2 = -4\lambda_4-1. \tag{3.17}$$

To derive a contradiction from the above equations, we discuss case by case according to the rank of the system of two-dimensional vectors $\{\boldsymbol{\alpha}, \boldsymbol{\beta}, \boldsymbol{\gamma}\}$.

Case 1: $rank\{\boldsymbol{\alpha}, \boldsymbol{\beta}, \boldsymbol{\gamma}\} = 2$. Without loss of generality, we may assume that $\boldsymbol{\alpha} \neq 0, \boldsymbol{\beta} \neq 0$. From (3.10) and (3.11) we have $\boldsymbol{t} = \boldsymbol{u} = \boldsymbol{\gamma} = 0$. From (3.12), $\lambda_3 = \pm 1$. On the other hand, by Theorem 2.2, the preimage of each point in the image of the map is a great sphere with maximum dimension 1. From (2.2) we deduce that $\lambda_3 \neq 1$ and consequently $\lambda_3 = -1$ and $\lambda_4 = \lambda_5 = -\frac{1}{2}$. From (3.13) we have $r^2+s^2 = \frac{3}{4}$ and from (3.16) $\boldsymbol{\alpha} = 0$. This is a contradiction. If we assume that $\boldsymbol{\alpha} \neq 0, \boldsymbol{\gamma} \neq 0$ or $\boldsymbol{\beta} \neq 0, \boldsymbol{\gamma} \neq 0$, a similar contradiction can be derived.

Case 2: $rank\{\boldsymbol{\alpha},\boldsymbol{\beta},\boldsymbol{\gamma}\}=1$. Since $\boldsymbol{\alpha},\boldsymbol{\beta},\boldsymbol{\gamma}$ are pairwise orthogonal, two of them must be zero and the third one is non-zero. We assume first that

$$\boldsymbol{\alpha}=0,\boldsymbol{\beta}=0,\boldsymbol{\gamma}\neq 0.$$

From (3.16) and (3.17) we have $\lambda_4=\lambda_5$. Letting $(i,j)=(4,5)$ in (2.7), we obtain

$$r^2+s^2-|\boldsymbol{u}|^2=\lambda_4^2+4\lambda_4+\frac{5}{2}-2|\boldsymbol{\gamma}|^2. \tag{3.18}$$

We also obtain from (3.16) in return

$$r^2+s^2=\frac{c_4^2}{c_3^2}(-4\lambda_4-1). \tag{3.19}$$

Combining (3.13) and (3.18) we solve

$$r^2+s^2=2\lambda_4+\frac{7}{4}-|\boldsymbol{\gamma}|^2. \tag{3.20}$$

Noting that $\lambda_3+\lambda_4+\lambda_5=-2,\lambda_4=\lambda_5$, from (3.18) and (3.20) we have

$$2(4|\gamma|^2-25)\lambda_4=-12|\boldsymbol{\gamma}|^2+25.$$

Obviously $4|\boldsymbol{\gamma}|^2-25\neq 0$, and we have

$$\lambda_4=\lambda_5=\frac{1}{2}\frac{-12|\gamma|^2+25}{4|\boldsymbol{\gamma}|^2-25}.$$

So we get

$$\lambda_3=\frac{4|\gamma|^2+25}{4|\boldsymbol{\gamma}|^2-25}. \tag{3.21}$$

If $4|\boldsymbol{\gamma}|^2-25>0$, then $\lambda_3>1$, contradicting the fact that $-1\leq\lambda_3<1$. If $4|\boldsymbol{\gamma}|^2-25<0$, then $\lambda_3\leq -1$ and only if $\boldsymbol{\gamma}=0$, then $\lambda_3=-1$. This contradicts the assumption that $\boldsymbol{\gamma}\neq 0$. Next we consider the subcase $\boldsymbol{\alpha}=0,\boldsymbol{\gamma}=0$, but $\boldsymbol{\beta}\neq 0$. From (3.10) we know that $\boldsymbol{t}\perp\boldsymbol{\beta},\boldsymbol{u}\perp\boldsymbol{\beta}$. Since all the vectors are two-dimensional, we conclude that $\boldsymbol{t}$ and $\boldsymbol{u}$ are parallel. Letting $(i,j)=(4,5)$ in (2.7) we have

$$[(\frac{c_4}{c_5})^2+(\frac{c_5}{c_4})^2](r^2+s^2)-2|\boldsymbol{u}|^2=2\lambda_4^2+4\lambda_4+4\lambda_5+5. \tag{3.22}$$

Combining (3.22) with (3.13) we get

$$(\frac{c_4}{c_5}+\frac{c_5}{c_4})^2(r^2+s^2)=-4\lambda_3-1. \tag{3.23}$$

From (3.23) and (3.16) and noting that $\boldsymbol{\alpha}=0$, we get

$$\lambda_3^2+7\lambda_3\lambda_5+\lambda_3+13\lambda_5+3=0.$$

Hence we have

$$\lambda_5=-\frac{\lambda_3^2+\lambda_3+3}{7\lambda_3+13},\lambda_4=-(\lambda_3+\lambda_5+2)=-\frac{6\lambda_3^2+26\lambda_3+23}{7\lambda_3+13}.$$

From (3.13) and (3.23) we deduce that

$$|\boldsymbol{u}|^2=-\frac{3(1-\lambda_4)(\lambda_3+1)(2\lambda_3+3)}{7\lambda_3+13}.$$

Since $\boldsymbol{t}$ and $\boldsymbol{u}$ are parallel, (3.15) becomes

$$4|\boldsymbol{t}|^2\cdot|\boldsymbol{u}|^2=(-1+\lambda_3^2+\lambda_4^2-\lambda_5^2)^2.$$

Substituting the expressions for $|\boldsymbol{t}|^2$ and $|\boldsymbol{u}|^2$ into the above equality, we obtain $-16 = 9$. This is impossible. For the subcase $\boldsymbol{\beta} = 0, \boldsymbol{\gamma} = 0, \boldsymbol{\alpha} \neq 0$, a similar contradiction can be derived.

Case 3: $rank\{\boldsymbol{\alpha}, \boldsymbol{\beta}, \boldsymbol{\gamma}\} = 0$. In this case, we deduce from (3.16) and (3.17) that $\lambda_4 = \lambda_5$. Letting $(i, j) = (4, 5)$ in (2.7) we get

$$r^2 + s^2 - \boldsymbol{t} \cdot \boldsymbol{u} - |\boldsymbol{u}|^2 = 1 - \lambda_4^2. \tag{3.24}$$

Together with (3.13) we solve

$$|\boldsymbol{u}|^2 = \frac{1}{4}(1 - \lambda_3^2), r^2 + s^2 = 1 - \lambda_4^2 - \frac{1}{4}(1 - \lambda_3^2). \tag{3.25}$$

Substituting these into (3.16) we have $\lambda_4 = -\frac{1}{2}$. Thus $\lambda_5 = \lambda_4 = -\frac{1}{2}, \lambda_3 = -1$. Therefore $\boldsymbol{t} = \boldsymbol{u} = 0$ and the last two components of $\boldsymbol{b_{ij}}$ vanish identically. This contradicts the assumption that the map f is full.

For the case $n = 5$, only two possible cases of the rank can occur. In fact, vectors $\{\boldsymbol{\alpha}, \boldsymbol{\beta}, \boldsymbol{\gamma}\}$ now reduce to one-dimensional, that is, they are scalars. The proof in this case is much simpler than the previous one. We omit the details here. This completes the proof of Theorem 1.1.

Remark. We can give another proof of Theorem 1.1 by using a similar idea as in [7] and the property of Octernion. However, this method involves discussing 35 possible cases and the proof is complicated.

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References

1. J.W.S. Cassels, *On the representation of rational functions as sums of squares*, Act. Arith. **9** (1964), 79-82. MR0162791 (29:95)
2. J. Eells, L. Lemaire, *Selected topics in harmonic maps*, CBMS Regional Conf. Ser. in Math. 50, Amer. Math. Soc., Providence, RI, 1983. MR703510 (85g:58030)
3. J. Eells, A. Ratto, *Harmonic maps and minimal immersions with symmetries*, Ann. of Math. Stud., vol. 130, Princeton University Press, 1993. MR1242555 (94k:58033)
4. H. Gauchman, G. Tóth, *Constructions of harmonic polynomial maps between spheres,* Geom. Dedicata **50** (1994), 57-79. MR1280796 (95m:58043)
5. G. Tóth, *Harmonic maps and minimal immersions through representation theory*, Academic Press, Boston, 1990. MR1042100 (91a:58050)
6. G. Tóth, *Classification of quadratic harmonic maps of S^3 into spheres,* Indiana University Math. J. **36** (1987), 231-269. MR891772 (89b:58058)
7. H.X. He, H. Ma, F. Xu, *On eigenmaps between spheres*, Bull. London Math. Soc. **35** (2003), 344-354. MR1960944 (2003m:53109)
8. K.Y. Lam, *Some new results on composition of quadratic forms,* Invent. Math. **79** (1985), 467-474. MR782229 (86e:11027)
9. Z.Z. Tang, *Harmonic polynomial morphisms between Euclidean spaces,* Science China, Ser. A. **42** (1999), 570-576. MR1717002 (2000i:53095)
10. Z.Z. Tang, *New constructions of eigenmaps between spheres*, Internat. J. Math. **12** (2001), 277-288. MR1841516 (2002j:58027)

11. K. Ueno, *Some new examples of eigenmaps from S^m into S^n,* Proc. Japan Academy, Ser. A, **69** (1993), 205-208. MR1232826 (94d:57061)
12. P. Yiu, *Quadratic forms between spheres and the nonexistence of sums of squares formulae,* Math. Proc. Camb. Phil. Soc. **100** (1986), 493-504. MR857724 (87m:55019)

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