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WEAK TOPOLOGIES IN COMPLETE $CAT(0)$ METRIC SPACES

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ABSTRACT. In this paper we consider some open questions concerning Δ-convergence in complete $CAT(0)$ metric spaces (i.e. Hadamard spaces). Suppose (X, d) is a Hadamard space such that the sets $\{z \in X |\, d(x, z) \leq d(z, y)\}$ are convex for each $x, y \in X$. We introduce a so-called half-space topology such that convergence in this topology is equivalent to Δ-convergence for any sequence in X. For a major class of Hadamard spaces, our results answer positively open questions nos. 1, 2 and 3 in [W. A. Kirk and B. Panyanak, *A concept of convergence in geodesic spaces*, Nonlinear Anal., 68 (2008) 3689-3696]. Moreover, we give a new characterization of Δ-convergence and a new topology that we call the weak topology via a concept of a dual metric space. The relations between these topologies and the topology which is induced by the distance function have been studied. The paper concludes with some examples.

1. INTRODUCTION AND PRELIMINARIES

A $CAT(0)$-space is a metric space (X, d) which satisfies the following condition.

CAT(0)-INEQUALITY: For every two points $x_0, x_1 \in X$ and for every $0 < t < 1$ there exists some $z \in X$ such that

$$d^2(y, z) \leq (1-t)d^2(y, x_0) + td^2(y, x_1) - t(1-t)d^2(x_0, x_1) \quad (y \in X). \tag{1.1}$$

It can be easily verified that such a point 'z' is unique, so we can define $x_t := z$. A complete $CAT(0)$-space is called *Hadamard space*. These spaces are well-studied by many authors; we refer the reader to the standard texts such as [3], [4], [6] and [7]. The following are the main examples:

Hilbert spaces, Hadamard manifolds (i.e. simply connected complete Riemannian manifolds with nonpositive sectional curvature which can be of infinite dimension), $\mathbb{R}$-trees (see Definition 4.6 below) as well as examples that have been built out of given Hadamard spaces such as: closed convex subsets, direct products, warped products, L^2-spaces, direct limits and Reshetnyak's gluing (see [12], Section 3).

A notion of convergence in Hadamard spaces, Δ-convergence introduced by Lim [9], has been studied by many authors (e.g. [5] and [8]). For more details about Δ-convergence, see Section 2 below. But it is not known yet whether there is a topology τ on X such that a sequence in X converges relative to τ if and only if it Δ-converges (see [8], problem 1, p. 3696). We introduce such a topology (we call it half-space topology; see Definition 3.5) provided that the, sets $F(x, y) = \{z \in X | d(x, z) \leq d(z, y)\}$ are convex for any x, y in X (Theorem 3.6). We call this

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the $(\overline{Q_4})$-condition, and it is satisfied in many Hadamard spaces including Hilbert spaces, $\mathbb{R}$-trees, Hadamard spaces with constant curvature and their closed convex subsets.

On the other hand, we have introduced [1] a *dual space* (X^*, D) for a Hadamard space (X, d), based on a work of Berg and Nikolaev [2]. The main idea is as follows. It is well-known that a normed linear space satisfies the $CAT(0)$-inequality if and only if it satisfies the parallelogram identity, i.e. is a pre-Hilbert space; hence it is not so unusual to have an inner product-like notion in Hadamard spaces. Berg and Nikolaev [2] have introduced the concept of *quasilinearization* along these lines. Let us formally denote a pair $(a, b) \in X \times X$ by $\overrightarrow{ab}$ and call it a *vector*. Then the quasilinearization map $\langle\, ,\, \rangle : (X \times X) \times (X \times X) \to \mathbb{R}$ is defined by

$$\langle \overrightarrow{ab}\,,\, \overrightarrow{uv} \rangle = \frac{1}{2}(d^2(a,v) + d^2(b,u) - d^2(a,u) - d^2(b,v)) \quad (a,b,u,v \in X). \tag{1.2}$$

We say that (X, d) satisfies the *Cauchy-Schwarz* inequality if

$$\langle \overrightarrow{ab}\,,\, \overrightarrow{uv} \rangle \le d(a,b)d(u,v) \quad (a,b,u,v \in X). \tag{1.3}$$

Berg and Nikolaev have then proved the following result.

Theorem 1.1 ([2], Corollary 3). *A geodesically connected metric space is a $CAT(0)$-space if and only if it satisfies the Cauchy-Schwarz inequality.*

Also, we can formally add *compatible* vectors, more precisely $\overrightarrow{xy} + \overrightarrow{yz} = \overrightarrow{xz}$, for all $x, y, z \in X$. Now, consider the map $\Theta : \mathbb{R} \times X \times X \to C(X; \mathbb{R})$ defined by

$$\Theta(t,a,b)(x) = t\langle \overrightarrow{ab}\,,\, \overrightarrow{ax} \rangle \quad (t \in \mathbb{R}, a, b, x \in X), \tag{1.4}$$

where $C(X; \mathbb{R})$ is the space of all continuous real-valued functions on X. Then the Cauchy-Schwarz inequality implies that $\Theta(t, a, b)$ is a Lipschitz function with Lipschitz semi-norm $L(\Theta(t, a, b)) = t\, d(a, b) \quad (t \in \mathbb{R}\,,\, a, b \in X)$, where $L(\varphi) = \sup\{\frac{\varphi(x)-\varphi(y)}{d(x,y)} \,:\, x, y \in X\,,\, x \neq y\}$ is the Lipschitz semi-norm for any function $\varphi : X \to \mathbb{R}$. Now, we introduce a pseudometric D on $\mathbb{R} \times X \times X$ by

$$D\big((t,a,b)\,,\,(s,u,v)\big) = L\big(\Theta(t,a,b) - \Theta(s,u,v)\big) \quad (t,s \in \mathbb{R}, a,b,u,v \in X). \tag{1.5}$$

Lemma 1.2 ([1], Lemma 2.1). $D\big((t,a,b)\,,\,(s,u,v)\big) = 0$ *if and only if* $t\,\langle \overrightarrow{ab}\,,\, \overrightarrow{xy} \rangle = s\,\langle \overrightarrow{uv}\,,\, \overrightarrow{xy} \rangle$*, for all* $x, y \in X$.

For a Hadamard space (X, d), the pseudometric space $(\mathbb{R} \times X \times X\,,\, D)$ can be considered as a subspace of the pseudometric space of all real-valued Lipschitz functions $(Lip(X, \mathbb{R})\,,\, L)$. Also, D imposes an equivalence relation on $\mathbb{R} \times X \times X$, where the equivalence class of (t, a, b) is

$$[t\overrightarrow{ab}] = \{\, s\overrightarrow{uv} \,:\, t\,\langle \overrightarrow{ab}\,,\, \overrightarrow{xy} \rangle = s\,\langle \overrightarrow{uv}\,,\, \overrightarrow{xy} \rangle \quad (x, y \in X)\}.$$

The set $X^* := \{[t\,\overrightarrow{ab}]\,;\, (t, a, b) \in \mathbb{R} \times X \times X\,\}$ is a metric space with metric D, which is called the *dual metric space* of (X, d).

For example, if X is a closed and convex subset of a Hilbert space $\mathcal{H}$ with nonempty interior, then $X^* = \mathcal{H}$ (see [1], p. 3451). Note that, in general, X^* acts on $X \times X$ by

$$\langle\, x^*\,,\, \overrightarrow{xy}\,\rangle := t\,\langle \overrightarrow{ab}\,,\, \overrightarrow{xy} \rangle \quad (x^* = [t\,\overrightarrow{ab}] \in X^*, (x,y) \in X \times X). \tag{1.6}$$

This action is well-defined by Lemma 1.2. Also, we use the following notation:
(1.7)
$$\langle\, \alpha x^* + \beta y^* \,,\, \overrightarrow{xy}\,\rangle := \alpha\langle x^* \,,\, \overrightarrow{xy}\rangle + \beta\langle y^* \,,\, \overrightarrow{xy}\rangle \quad (\alpha, \beta \in \mathbb{R}, x, y \in X, x^*, y^* \in X^*).$$

Among other properties, we have a separation property of a dual metric space (see [1], Proposition 2.3). For proofs of the above statements about dual metric spaces and more details, see [1].

Thanks to the dual space concept, we can introduce another topology, called weak topology (Definition 3.1) which lies between metric topology and half-space topology and coincides with half-space topology when (X, d) is flat (see section 4 below). Moreover, we prove a new characterization for Δ-convergence (Theorem 2.6), i.e. $\{x_n\}$ Δ-converges to $x \in X$ if and only if $\limsup_{n\to\infty}\langle\overrightarrow{xx_n}, \overrightarrow{xy}\rangle \le 0 \quad (y \in X)$ and it is interesting to compare this with the definition of w-convergence (Definition 2.5) which is a natural generalization of weak convergence in Hilbert spaces; i.e. $\{x_n\}$ w-converges to $x \in X$ if and only if $\lim_{n\to\infty}\langle\overrightarrow{xx_n}, \overrightarrow{xy}\rangle = 0 \quad (y \in X)$.

2. w-convergence and Δ-convergence

Let (X, d) be a Hadamard space, $\{x_n\}$ be a bounded sequence in X and $x \in X$. Let $r(x, \{x_n\}) = \limsup_{n\to\infty} d(x, x_n)$. The asymptotic ratio of $\{x_n\}$ is given by $r(\{x_n\}) = \inf\{r(x, \{x_n\}) | x \in X\}$ and the asymptotic center of $r(\{x_n\})$ is the set $A(\{x_n\}) = \{x \in X | r(x, \{x_n\}) = r(\{x_n\})\}$. It is known that in the Hadamard spaces, $A(\{x_n\})$ consists of exactly one point (see e.g. [8], p. 3690).

Definition 2.1 ([9], p. 180)**.** A sequence $\{x_n\}$ in the Hadamard space (X, d) Δ-*converges* to $x \in X$ if $A(\{x_{n_k}\}) = \{x\}$ for every subsequence $\{x_{n_k}\}$ of $\{x_n\}$.

Among various properties of Δ-convergence, we mention only the following three propositions. The first proposition states that every closed convex subset of X is Δ-closed, the second is analogous to the classical Banach-Alaoglo theorem and the last is a useful characterization of Δ-convergence. This characterization is based on a concept which was introduced by Sosov [11].

Proposition 2.2 ([8], Proposition 5.2)**.** *If a sequence $\{x_n\}$ in a Hadamard space (X, d) Δ-converges to $x \in X$, then*
$$x \in \bigcap_{k=1}^{\infty} \overline{conv}\{x_k, x_{k+1}, ...\},$$
where $\overline{conv}(A) = \bigcap\{B : B \supseteq A$, where B is closed and convex$\}$ for any $A \subset X$.

Proposition 2.3 ([9], Theorem 5.2)**.** *Every bounded sequence $\{x_n\}$ in a Hadamard space (X, d) has a Δ-convergent subsequence.*

Proposition 2.4. *A sequence $\{x_n\}$ in a Hadamard space (X, d) Δ-converges to $x \in X$ if and only if $\lim_{n\to\infty} d(x, P_I x_n) = 0$ for all geodesic I issuing from the point x, where $P_I : X \longrightarrow I$ is the projection map.*

Proof. This is essentially Proposition 5.2 of [5], and the same proof remains valid. □

Having the notion of quasilinearization, we have introduced the following notion of convergence.

Definition 2.5 ([1], p. 3452). A sequence $\{x_n\}$ in the Hadamard space (X, d) *w-converges* to $x \in X$ if $\lim_{n\to\infty}\langle \overrightarrow{xx_n}, \overrightarrow{xy}\rangle = 0 \quad (y \in X)$, i.e. $\lim_{n\to\infty}(d^2(x_n, x) - d^2(x_n, y) + d^2(x, y)) = 0 \quad (y \in X)$.

It is obvious that convergence in the metric implies w-convergence, and it is easy to check that w-convergence implies Δ-convergence ([1], Proposition 2.5), but we will show that the converse is not valid (see Example 4.7 below). However the following theorem shows another characterization of Δ-convergence as well as, more explicitly, a relation between w-convergence and Δ-convergence.

Theorem 2.6. *Let (X, d) be a Hadamard space, $\{x_n\}$ be a sequence in X and $x \in X$. Then $\{x_n\}$ Δ-converges to x if and only if $\limsup_{n\to\infty}\langle \overrightarrow{xx_n}, \overrightarrow{xy}\rangle \le 0$ for all $y \in X$.*

Proof. Let $\{x_n\}$ Δ-converge to x, I be the geodesic segment between x and an arbitrary point $y \in X$ and $P_I : X \longrightarrow I$ be the projection map (see [3], p. 176, Proposition 2.4). By a property of the projection map ([3], p. 176, Proposition 2.4(1)) we have

$$\begin{aligned} d^2(x_n, y) &\ge d^2(x_n, P_I x_n) + d^2(P_I x_n, y) \\ &\ge (d(x_n, x) - d(x, P_I x_n))^2 + (d(P_I x_n, x) - d(x, y))^2, \end{aligned}$$

and by Proposition 2.4, $\lim_{n\to\infty} d(P_I x_n, x) = 0$. Hence $\limsup_{n\to\infty}(d^2(x_n, x) - d^2(x_n, y) + d^2(x, y)) \le 0$, i.e. $\limsup_{n\to\infty}\langle \overrightarrow{xx_n}, \overrightarrow{xy}\rangle \le 0$.

On the other hand, suppose

$$\limsup_{n\to\infty}\,(d^2(x_n, x) - d^2(x_n, y) + d^2(x, y)) \le 0 \quad (y \in X), \tag{2.1}$$

$\{x_{n_k}\}$ is an arbitrary subsequence of $\{x_n\}$ and $\{x_{n_{k_l}}\}$ is a subsequence of $\{x_{n_k}\}$ such that

$$\lim_{l\to\infty} d(x_{n_{k_l}}, x) = \limsup_{k\to\infty} d(x_{n_k}, x). \tag{2.2}$$

Now by (2.1) and (2.2) we have

$$\begin{aligned} d^2(x, y) &\le -\limsup_{l\to\infty}(d^2(x_{n_{k_l}}, x) - d^2(x_{n_{k_l}}, y)) \\ &= \liminf_{l\to\infty}(d^2(x_{n_{k_l}}, y) - d^2(x_{n_{k_l}}, x)) \\ &\le \liminf_{l\to\infty} d^2(x_{n_{k_l}}, y) - \liminf_{l\to\infty} d^2(x_{n_{k_l}}, x) \\ &= \liminf_{l\to\infty} d^2(x_{n_{k_l}}, y) - \lim_{l\to\infty} d^2(x_{n_{k_l}}, x) \\ &\le \limsup_{k\to\infty} d^2(x_{n_k}, y) - \limsup_{k\to\infty} d^2(x_{n_k}, x). \end{aligned}$$

Therefore $\limsup_{k\to\infty} d^2(x_{n_k}, x) < \limsup_{k\to\infty} d^2(x_{n_k}, y)$ for all $y \in X$ with $y \ne x$, and this means that x is the asymptotic center of $\{x_{n_k}\}$; i.e. $\{x_n\}$ Δ-converges to x. $\square$

Definition 2.7. We say that the Hadamard space (X, d) satisfies the $(\mathcal{S})$ property if for any $(x, y) \in X \times X$ there exists a point $y_x \in X$ such that $[\overrightarrow{xy}] = [\overrightarrow{y_x x}]$.

It is obvious that, for example, any Hilbert space enjoys the $(\mathcal{S})$ property (let $y_x := 2x - y$ and then $[\overrightarrow{xy}] = [\overrightarrow{y_x x}] = [y - x]$), and it is not hard to check that any *symmetric* Hadamard manifold satisfies the $(\mathcal{S})$ property ($\exp_x^{-1} y$ acts in the role of $[\overrightarrow{xy}]$ in Hadamard manifolds; see [1], p. 3455).

Lemma 2.8. *If a Hadamard space (X, d) satisfies the $(\mathcal{S})$ property and $x \in X$, then a sequence $\{x_n\}$ in X w-converges to x if and only if it Δ-converges to x.*

Proof. We should prove only the 'if' part. Let $\{x_n\}$ Δ-converge to x and let y be an arbitrary point of X. By Theorem 2.6 we have simultaneously $\limsup_{n\to\infty}\langle\overrightarrow{xx_n}, \overrightarrow{xy}\rangle \leq 0$ and $\limsup_{n\to\infty}\langle\overrightarrow{xx_n}, \overrightarrow{xy_x}\rangle \leq 0$. Therefore

$$\liminf_{n\to\infty}\langle\overrightarrow{xx_n}, \overrightarrow{xy}\rangle = \liminf_{n\to\infty}(-\langle\overrightarrow{xx_n}, \overrightarrow{yx}\rangle) = -\limsup_{n\to\infty}\langle\overrightarrow{xx_n}, \overrightarrow{xy_x}\rangle \geq 0.$$

Hence $\limsup_{n\to\infty}\langle\overrightarrow{xx_n}, \overrightarrow{xy}\rangle = 0$; i.e. $\{x_n\}$ w-converges to $x \in X$. □

Remark 2.9. The term 'sequence' can be replaced by the term 'net' throughout this paper. In other words, the concepts Δ-convergence and w-convergence can be defined for any net $\{x_\alpha\}$ completely similar to Definition 2.1 and Definition 2.4 for any sequence $\{x_n\}$. By a small modification, all the arguments remain valid too (e.g. see [8], Definition 3.3 and Proposition 3.5).

3. Weak topology and half-space topology

At first, we show that w-convergence is induced by a topology in the Hadamard space (X, d).

Definition 3.1. Let (X, d) be a Hadamard space and $W(x, y; \epsilon) = \{a \in X |\ |\langle\overrightarrow{xa}, \overrightarrow{xy}\rangle| < \epsilon\}$ for any $x, y \in X$ and $\epsilon > 0$. We call the topology which is generated by the family $\{W(x, y; \epsilon) | x, y \in X \text{ and } \epsilon > 0\}$ the *weak* topology (or briefly w-topology) on X.

More precisely, the family $\{\bigcap_{i=1}^n W(x_i, y_i; \epsilon_i) | n \in \mathbb{N}, x_1, ..., x_n, y_1, ..., y_n \in X$ and $\epsilon_1, ..., \epsilon_n > 0\}$ is the basis of the w-topology. This topology is completely similar to the *weak topology* in Hilbert spaces and coincides with it in this case.

Theorem 3.2. *Let (X, d) be a Hadamard space, $\{x_n\}$ be a sequence in X and $x \in X$. Then $\{x_n\}$ w-converges to x if and only if $\{x_n\}$ converges to x in the w-topology.*

Proof. Let $\{x_n\}$ converge to the point x in the w-topology. Then for any $y \in X$, any $\epsilon > 0$ and for sufficiently large n we have $x_n \in W(x, y; \epsilon)$ or equivalently $|\langle\overrightarrow{xx_n}, \overrightarrow{xy}\rangle| < \epsilon$, and this means that $\lim_{n\to\infty}\langle\overrightarrow{xx_n}, \overrightarrow{xy}\rangle = 0$; i.e. $\{x_n\}$ w-converges to x.

For the converse, let $\{x_n\}$ w-converge to x and $W(a, b; \epsilon)$ be an arbitrary element of the subbasis of the w-topology such that $x \in W(a, b; \epsilon)$. Hence $\delta := \frac{1}{2}(\epsilon - |\langle\overrightarrow{ax}, \overrightarrow{ab}\rangle|) > 0$ and by w-convergence there exists some integer N such that $|\langle\overrightarrow{xx_n}, \overrightarrow{xa}\rangle| < \delta$ and $|\langle\overrightarrow{xx_n}, \overrightarrow{xb}\rangle| < \delta$ for all $n \geq N$. Therefore

$$\begin{aligned} |\langle\overrightarrow{ax_n}, \overrightarrow{ab}\rangle| &= |\langle\overrightarrow{ax}, \overrightarrow{ab}\rangle + \langle\overrightarrow{xx_n}, \overrightarrow{xb}\rangle - \langle\overrightarrow{xx_n}, \overrightarrow{xa}\rangle| \\ &< |\langle\overrightarrow{ax}, \overrightarrow{ab}\rangle| + 2\delta = \epsilon \quad (n \geq N); \end{aligned}$$

i.e. $x_n \in W(a, b; \epsilon)$ for $n \geq N$. Hence $\{x_n\}$ converges to x in the w-topology. □

The following geometric condition on a $CAT(0)$-space (X,d) has been introduced by W.A. Kirk and B. Panyanak ([8], p. 3693). The geodesic segment between $a,b\in X$ is denoted by $[a,b]$.

($\mathbf{Q_4}$) For points $x,y,p,q\in X$,

$$(\,d(p,x)<d(x,q)\,\&\,d(p,y)<d(y,q)\,)\Rightarrow d(p,m)\le d(m,q)$$

for any point m in the segment $[x,y]$.

As they have mentioned, this condition holds in many $CAT(0)$-spaces including Hilbert spaces and $\mathbb{R}$-trees. This condition has been studied more deeply by R. Espínola and A. Fernández-León, who proved that any $CAT(0)$-space of constant curvature satisfies the (Q_4) condition ([5], Theorem 5.7), but any $CAT(0)$ gluing space containing two spaces of constant but different curvatures does not ([5], Theorem 5.11). Now consider the following modification of the (Q_4) condition.

($\overline{\mathbf{Q_4}}$) For points $x,y,p,q\in X$,

$$(\,d(p,x)\le d(x,q)\,\&\,d(p,y)\le d(y,q)\,)\Rightarrow d(p,m)\le d(m,q)$$

for any point m in the segment $[x,y]$.

Since $(\overline{Q_4})$ implies (Q_4), there are some Hadamard spaces that do not satisfy $(\overline{Q_4})$. On the other hand, Hilbert spaces, $\mathbb{R}$-trees and any $CAT(0)$-space of constant curvature satisfy the $(\overline{Q_4})$ condition. One can prove the latter case, completely similarly to Theorem 5.7 of [5].

We will use some notation in a metric space (X,d).

Let

$$F(x,y):=\{z\in X|\,d(x,z)\le d(z,y)\},$$

$$H(x,y):=\{z\in X|\,d(x,z)<d(z,y)\}$$

and $M(x,y):=\{z\in X|\,d(x,z)=d(z,y)\}$ for any $x,y\in X$. Note that $H(x,y)=F(y,x)^c$, $M(x,y)=F(x,y)\cap F(y,x)$ and the $(\overline{Q_4})$ condition just says that $F(x,y)$ is convex for any $x,y\in X$.

Lemma 3.3. *Let (X,d) be a Hadamard space which satisfies the $(\overline{Q_4})$ condition, $a,b\in X$ with $a\neq b$, m be the midpoint of $[a,b]$ and $P:X\to[a,b]$ be the point projection map. Then*

i) $Px=m$ when $x\in M(a,b)$,

ii) $Px\in F(a,b)$ when $x\in F(a,b)$.

Proof. i) Since $M(a,b)\cap[a,b]=\{m\}$, it is sufficient to suppose that $x\in M(a,b)\setminus[a,b]$. By the $(\overline{Q_4})$ condition, $M(a,b)=F(a,b)\cap F(b,a)$ is a convex subset of X; hence $[x,m]\subset M(a,b)$. Now, let $\overline{\Delta}(\overline{a},\overline{b},\overline{x})\subset\mathbb{R}^2$ be a comparison triangle of the geodesic triangle $\Delta(a,b,x)\subset X$ and $\overline{m}$ be the midpoint of $[\overline{a},\overline{b}]\subset\mathbb{R}^2$. Let $\theta_1:=\angle_m(a,x)$ and $\theta_2:=\angle_m(b,x)$ be the Alexandrov angles at the point m ([3], p. 184, Proposition 3.1). It is known that $\theta_1+\theta_2=\pi$ ([4], Lemma 4.3.7) and by properties of angles in $CAT(0)$-spaces ([4], Theorem 4.3.5) we have $\theta_1\le\angle_{\overline{m}}(\overline{a},\overline{x})=\frac{\pi}{2}$ and $\theta_2\le\angle_{\overline{m}}(\overline{b},\overline{x})=\frac{\pi}{2}$; hence $\theta_1=\theta_2=\frac{\pi}{2}$.

On the other hand, by [3], p. 176, Proposition 2.4(3), we have $\angle_{Px}(a,x)\ge\frac{\pi}{2}$ and $\angle_{Px}(b,x)\ge\frac{\pi}{2}$. So if $Px\neq m$, then in the geodesic triangle $\Delta(m,Px,x)$ there are two angles in which their sum is greater than or equal to π and this is not possible (see [4], Theorem 4.3.5 and Definition 4.1.15). Therefore $Px=m$.

ii) Suppose $x \in F(a,b)$ but $Px \notin F(a,b)$ and define the real-valued map $\varphi : [x, Px] \longrightarrow \mathbb{R}$ by $\varphi(y) = d(a,y) - d(b,y) \quad (y \in [x, Px])$. Since $[x, Px]$ is connected, φ is continuous, $\varphi(x) \leq 0$ and $\varphi(Px) > 0$, so there exists a point $z \in [x, Px]$ such that $\varphi(z) = 0$, i.e. $z \in M(a,b)$. Now by (i) we have $Pz \in M(a,b)$, i.e. $\varphi(Pz) = 0$. But by [3], p. 176, Proposition 2.4(2), $Pz = Px$. This contradiction implies that $Px \in F(a,b)$. □

Lemma 3.4. *Let (X,d) be a Hadamard space, $a, c \in X$, $b \in [a,c]$, $a \neq b$, $x \in F(a,b)$ and $P_{[a,b]}x \in F(a,b)$. Then $P_{[a,b]}x = P_{[a,c]}x$, where $P_{[a,b]} : X \to [a,b]$ and $P_{[a,c]} : X \to [a,c]$ are projection maps.*

Proof. The proof is based on a useful characterization of the projection map, i.e. [3], p. 176, Proposition 2.4(3). By this, we know that $x' = P_{[a,b]}x$ if and only if

$$x' \in [a,b] \ \ \& \ \ \forall y \in [a,b] \quad \angle_{x'}(x,y) \geq \frac{\pi}{2} \tag{3.1}$$

as well as $x' = P_{[a,c]}x$ if and only if

$$x' \in [a,c] \ \ \& \ \ \forall y \in [a,c] \quad \angle_{x'}(x,y) \geq \frac{\pi}{2}. \tag{3.2}$$

We have to show that (3.1) implies (3.2). Since $[a,b] \subset [a,c]$, it is sufficient to show that

$$\forall y \in [b,c] \quad \angle_{x'}(x,y) \geq \frac{\pi}{2} \tag{3.3}$$

when (3.1) holds. Note that by the monotonicity property of the Alexandrov angle ([3], p. 161, Proposition 1.7(3)) we have

$$\forall y \in [b,c] \quad \angle_{x'}(x,y) \geq \angle_{x'}(x,b)$$

and by (3.1) we have $\angle_{x'}(x,b) \geq \frac{\pi}{2}$. Therefore (3.3) is concluded. □

Definition 3.5. Let (X,d) be a Hadamard space. We call the topology which is generated by the family $\{H(x,y) | x, y \in X\}$ the *half-space* topology (or briefly h-topology) on X.

More precisely, the family $\{\bigcap_{i=1}^{n} H(x_i, y_i) | \, n \in \mathbb{N}, \, x_1, ..., x_n, y_1, ..., y_n \in X\}$ is the basis of the h-topology. The following theorems are the main results of this paper. These theorems give affirmative answers respectively to the open problems nos. 1, 2 and 3 of [8], p. 3696, when the $(\overline{Q_4})$ condition holds.

Theorem 3.6. *Let (X,d) be a Hadamard space that satisfies the $(\overline{Q_4})$ condition, $\{x_n\}$ be a sequence in X and $x \in X$. Then $\{x_n\}$ Δ-converges to x if and only if $\{x_n\}$ converges to x in the half-space topology.*

Proof. Suppose $\{x_n\}$ Δ-converges to x but does not converge to x in the h-topology, so there are some $a, b \in X$ and a subsequence $\{x_{n_k}\}$ of $\{x_n\}$ such that $x \in H(a,b)$ and $\{x_{n_k}\} \subset H(a,b)^c = F(b,a)$. But by Proposition 2.2, $x \in \overline{conv}\{x_{n_1}, x_{n_2}, ...\} \subset F(b,a) = H(a,b)^c$. Recall that $F(b,a)$ is a closed convex subset of X by the $(\overline{Q_4})$ condition.

For the converse, let $\{x_n\}$ converge to the point x in the h-topology and I be a geodesic issuing from x; for any $\epsilon > 0$, choose $y \in I \setminus \{x\}$ such that $d(x,y) < 2\epsilon$.

For sufficiently large n we have $x_n \in H(x,y) \subset F(x,y)$. Therefore $p_n := P_{[x,y]}x_n \in F(x,y)$ by Lemma 3.3(ii). But $p_n = P_I x_n$ by Lemma 3.4. Now

$$d(x,p_n) \leq \frac{1}{2}(d(x,p_n) + d(p_n,y)) = \frac{1}{2}d(x,y) < \epsilon;$$

i.e. $\lim_{n\to\infty} d(x,p_n) = 0$. Finally the assertion follows from Proposition 2.4. $\square$

Theorem 3.7. *Let (X,d) be a Hadamard space that satisfies the $(\overline{Q_4})$ condition, let $\{x_n\}$ and $\{y_n\}$ be two sequences in X, let $\{x_n\}$ Δ-converge to $x \in X$ and let $y_n \in \overline{conv}\{x_n, x_{n+1}, ...\}$ for any $n \in \mathbb{N}$. Then $\{y_n\}$ Δ-converges to x.*

Proof. Since $\{x_n\}$ is a Δ-convergent sequence, $\{x_n\}$ and so $\{y_n\}$ are bounded sequences. If $\{y_n\}$ does not Δ-converge to x, then there exist a point $y \neq x$ and a subsequence $\{y_{n_k}\}$ of $\{y_n\}$ such that $\{y_{n_k}\}$ Δ-converges to y by Proposition 2.3. Now by Theorem 3.6 there exists some $N \in \mathbb{N}$ such that $x_n \in H(x,y) \subset F(x,y)$ for all $n \geq N$. Since $F(x,y)$ is closed and convex by the $(\overline{Q_4})$ condition, we have

$$y_n \in \overline{conv}\{x_n, x_{n+1}, ...\} \subset F(x,y) = H(y,x)^c \quad (n \geq N).$$

But this contradicts Theorem 3.6 because $\{y_{n_k}\}$ Δ-converges to y by assumption. $\square$

Theorem 3.8. *Let (X,d) be a Hadamard space that satisfies the $(\overline{Q_4})$ condition and $\{x_n\}$ be a sequence in X that Δ-converges to $x \in X$. Then $\{x\} = \bigcap_{n=1}^{\infty} \overline{conv}\{x_n, x_{n+1}, ...\}$.*

Proof. By Proposition 2.2 we have $x \in \bigcap_{n=1}^{\infty} \overline{conv}\{x_n, x_{n+1}, ...\}$. Let $y \in X$ with $y \neq x$. Then by Theorem 3.6 there exists some $N \in \mathbb{N}$ such that $x_n \in H(x,y) \subset F(x,y)$ for all $n \geq N$. Therefore by the $(\overline{Q_4})$ condition, we have

$$\bigcap_{n=1}^{\infty} \overline{conv}\{x_n, x_{n+1}, ...\} \subset \bigcap_{n=N}^{\infty} \overline{conv}\{x_n, x_{n+1}, ...\} \subset F(x,y),$$

so $y \notin \bigcap_{n=1}^{\infty} \overline{conv}\{x_n, x_{n+1}, ...\}$. $\square$

4. Comparing between half-space, weak and metric topologies in Hadamard spaces

Let (X,d) be a Hadamard space. Our aim is to prove that if the $(\overline{Q_4})$ condition holds, then in general we have

$$\textit{half space topology} \subsetneqq \textit{weak topology} \subsetneqq \textit{metric topology}. \tag{4.1}$$

First, we begin by a technical lemma.

Lemma 4.1. *i) $W(x,y;\epsilon) \subset H(x,y)$, where $x,y \in X$ and $\epsilon = \frac{1}{2}d^2(x,y)$.*

ii) If the $(\overline{Q_4})$ condition holds, $a,b \in X$, $x \in H(a,b)$ and $y := P_{F(b,a)}x$, then $H(x,y) \subset H(a,b)$, where $P_{F(b,a)} : X \longrightarrow F(b,a)$ is the projection map.

Proof. i) If $z \in W(x,y;\epsilon)$, then $|\langle \overrightarrow{xz}, \overrightarrow{xy} \rangle| < \epsilon$; hence

$$d^2(x,y) + d^2(x,z) - d^2(y,z) < 2\epsilon = d^2(x,y),$$

so $d(x,z) < d(y,z)$, i.e. $z \in H(x,y)$.

ii) Since the $(\overline{Q_4})$ condition holds, $F(b,a)$ is a closed convex subset of X. Therefore the map $P_{F(b,a)}$ is well-defined. Let $z \in H(a,b)^c = F(b,a)$. By a property of the projection map ([4], p. 176, Proposition 2.4(3)) we have

$$d^2(x,z) \geq d^2(x,y) + d^2(y,z) \geq d^2(y,z);$$

i.e. $z \in F(y,x) = H(x,y)^c$. □

Now, we are able to compare these topologies.

Proposition 4.2. *i) w-topology is weaker than metric topology.*
ii) h-topology is weaker than w-topology when the $(\overline{Q_4})$ condition holds.
iii) w-topology and h-topology are both Hausdorff topologies.

Proof. i) This is obvious, because of the continuity of the distance function. The sets

$$W(x,y;\epsilon) = \{z \in X|\ |d^2(x,y) + d^2(z,x) - d^2(z,y)| < 2\epsilon\} \quad (x,y \in X,\ \epsilon > 0)$$

are open in the metric topology and this yields the assertion.

ii) Suppose the $(\overline{Q_4})$ condition holds. It is sufficient to prove that for any $a,b \in X$ the set $H(a,b)$ is open in the w-topology. To this end let $x \in H(a,b)$. By Lemma 4.1(ii) we have $H(x,y) \subset H(a,b)$ for some $y \in X$, and by Lemma 4.1(i) we have $W(x,y;\epsilon) \subset H(x,y)$ for some $\epsilon > 0$. Therefore $x \in W(x,y;\epsilon) \subset H(a,b)$, which completes the proof.

iii) Let $x,y \in X$ with $x \neq y$ and $\epsilon := \frac{1}{2}d^2(x,y) > 0$. Then $x \in W(x,y;\epsilon) \subset H(x,y)$, $y \in W(y,x;\epsilon) \subset H(y,x)$ and $H(x,y) \cap H(y,x) = \varnothing$. □

It is known that in any finite-dimensional Hilbert space, the weak topology and the strong topology coincide. For Hadamard spaces, this happens in the so-called proper Hadamard spaces.

Proposition 4.3. *For any Hadamard space (X,d) the following are equivalent.*
i) (X,d) is a locally compact space.
ii) Every closed bounded subset of (X,d) is compact.
iii) Every bounded sequence $\{x_n\}$ in (X,d) has a convergent subsequence.

Such spaces are called *proper* Hadamard spaces.

Proof. This is a consequence of the Hopf-Rinow Theorem; see [3], pp. 35-36, Proposition 3.7, Corollary 3.8 and Remark 3.9. □

Proposition 4.4. *In any proper Hadamard space (X,d), Δ-convergence, w-convergence and strong convergence (i.e. convergence in the metric topology) for any sequence are equivalent.*

Proof. Let a sequence $\{x_n\}$ Δ-converge to $x \in X$. It is enough to show that every subsequence $\{x_{n_k}\}$ of $\{x_n\}$ has a subsequence which strongly converges to x. By Proposition 4.3(iii) there exists a subsequence $\{x_{n_{k_l}}\}$ of the bounded sequence $\{x_{n_k}\}$ which converges strongly to some point $x' \in X$ and so $\{x_{n_{k_l}}\}$ Δ-converges to x' too. But since $\{x_n\}$ Δ-converges to x by assumption, we have $x' = x$, as required. □

Recall that Δ-convergence is equivalent to convergence in the half-space topology when the $(\overline{Q_4})$ condition is satisfied (Theorem 3.6).

Proposition 4.5. *i) In any proper Hadamard space (X,d) with the $(\overline{Q_4})$ condition, half-space topology, weak topology and metric topology coincide.*

ii) In any Hilbert space, half-space topology and weak topology coincide.

Proof. i) This is deduced from Remark 2.9, Theorem 3.6, Proposition 4.2 and Proposition 4.4.

ii) This is just a reformulation of the well-known Opial property (see [10], Lemma 1). □

It is known that in many infinite-dimensional Hilbert spaces, there exist weakly convergent sequences such that they do not converge strongly; hence *weak topology* $\subsetneqq$ *metric topology.*

Definition 4.6 ([3], p. 167, Example 1.15(5))**.** An $\mathbb{R}$-*tree* is a metric space (X,d) such that

i) there is a unique geodesic segment (denoted $[x,y]$) joining each pair of points $x,y \in X$;

ii) if $[x,y] \cap [y,z] = \{y\}$, then $[x,y] \cup [y,z] = [x,z]$.

It is well-known that every $\mathbb{R}$-tree is a $CAT(0)$-space (in fact $CAT(\kappa)$-space for any $\kappa \in \mathbb{R}$; see [3], p. 167, Example 1.15(5)).

The following counterexample shows that Δ-convergence does not imply w-convergence, i.e. *half space topology* $\subsetneqq$ *weak topology*, and this completes the proof of (4.1).

Example 4.7. Consider the following equivalence relation $\mathcal{R}$ on the set $\mathbb{N} \times [0,1]$:

$$(n,0)\mathcal{R}(m,0) \quad and \quad (n,t)\mathcal{R}(n,t) \quad (n,m \in \mathbb{N}\,,\, t \in [0,1]).$$

Now, define the following metric on $X := (\mathbb{N} \times [0,1])\diagup\mathcal{R}$:

$$d((n,t),(n,s)) = |t-s| \quad and \quad d((n,t),(m,s)) = t+s, \quad when\ n \neq m.$$

It is easy to see that (X,d) is a complete $\mathbb{R}$-tree and satisfies the $(\overline{Q_4})$ condition.

Let $\mathbf{0} = [(1,0)]$, $x_n = [(n,1)]$ and $y = [(m,t)]$ all be in (X,d). Since

$$\begin{aligned}\langle \overrightarrow{\mathbf{0}x_n}, \overrightarrow{\mathbf{0}y} \rangle &= \frac{1}{2}(d^2(\mathbf{0},y) + d^2(\mathbf{0},x_n) - d^2(x_n,y)) \\ &= \frac{1}{2}(t^2 + 1 - (t+1)^2) = -t \leq 0,\end{aligned}$$

$\{x_n\}$ Δ-converges to $\mathbf{0}$ (i.e. convergence in the h-topology) but has no w-convergent subsequence (i.e. convergence in the w-topology).

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