

RIGIDITY OF BI-LIPSCHITZ EQUIVALENCE OF WEIGHTED HOMOGENEOUS FUNCTION-GERMS IN THE PLANE

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ABSTRACT. The main goal of this work is to show that if two weighted homogeneous (but not homogeneous) function-germs $(\mathbb{C}^2, 0) \rightarrow (\mathbb{C}, 0)$ are bi-Lipschitz equivalent, in the sense that these function-germs can be included in a strongly bi-Lipschitz trivial family of weighted homogeneous function-germs, then they are analytically equivalent.

1. INTRODUCTION

The restriction of the complex polynomial

$$f_t(x, y) = xy(x - y)(x - ty) ; 0 < |t| < 1$$

defines a family of function-germs $(\mathbb{C}^2, 0) \rightarrow (\mathbb{C}, 0)$ with isolated singularity. In 1965, H. Whitney justified the rigidity of the analytic classification of function-germs by proving that: given $t \neq s$, there is no $\phi: (\mathbb{C}^2, 0) \rightarrow (\mathbb{C}^2, 0)$ germ of a bi-analytic map such that $f_t = f_s \circ \phi$; i.e. f_t is not analytically equivalent to f_s . On the other hand, with respect to the topological point of view, this family is not so interesting, since for any $t \neq s$ there exists a germ of homeomorphism $\phi: (\mathbb{C}^2, 0) \rightarrow (\mathbb{C}^2, 0)$ such that $f_t = f_s \circ \phi$; i.e. f_t is topologically equivalent to f_s . In fact, the topological classification of reduced polynomial function-germs $(\mathbb{C}^2, 0) \rightarrow (\mathbb{C}, 0)$ is well-understood, as was shown in [6]. In the seminal paper [3], Henry and Parusinski considered the bi-Lipschitz equivalence, which is between the analytic and the topological equivalence, of function-germs. This paper motivated other papers about the problem of bi-Lipschitz classification of function-germs. For instance, [4] and [1] showed that, in some sense, for weighted homogeneous real function-germs in two variables the problem of bi-Lipschitz classification is quite close to the problem of analytic classification. The results presented in [3] point out a rigidity of the bi-Lipschitz classification of function-germs. More precisely, they considered the family

$$f_t(x, y) = x^3 + y^6 - 3t^2xy^4 ; 0 < |t| < \frac{1}{2}$$

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and proved that given $t \neq s$, there is no $\phi: (\mathbb{C}^2, 0) \rightarrow (\mathbb{C}^2, 0)$ germ of a bi-Lipschitz map such that $f_t = f_s \circ \phi$; i.e. f_t is not bi-Lipschitz equivalent to f_s . The strategy used by them was to introduce a new invariant based on the observation that the bi-Lipschitz homeomorphism does not move much the regions around the relative polar curves. For a single germ f the invariant is given in terms of the leading coefficients of the asymptotic expansions of f along the branches of its generic polar curve. In the case that the bi-Lipschitz trivality of a family of function-germs comes by integrating a Lipschitz vector field, here called strong bi-Lipschitz trivality, the calculations are much easier and very illustrative. The main goal of this work is to show that if two weighted homogeneous (but not homogeneous) function-germs $(\mathbb{C}^2, 0) \rightarrow (\mathbb{C}, 0)$ are bi-Lipschitz equivalent, in the sense that these function-germs can be included in a strongly bi-Lipschitz trivial family of weighted homogeneous function-germs, then they are analytically equivalent.

2. PRELIMINARIES

2.1. Analytic and bi-Lipschitz equivalences. Two germs of analytic functions $f, g: (\mathbb{C}^n, 0) \rightarrow (\mathbb{C}, 0)$ are called *analytically equivalent* if there exists a germ of a bi-analytic map $\phi: (\mathbb{C}^n, 0) \rightarrow (\mathbb{C}^n, 0)$ such that $f = g \circ \phi$.

The following two results play an important role in studying the analytic equivalence of function-germs $(\mathbb{C}^n, 0) \rightarrow (\mathbb{C}, 0)$.

Theorem 2.1 (Mather's Lemma). *Let G be a Lie group, M a smooth manifold and $\alpha: G \times M \rightarrow M$ a smooth group action. Let V be a connected smooth submanifold of M . Then V is contained in an orbit of α if and only if*

- a) *For all $v \in V$, the tangent space of V at v is contained in the tangent space of the orbit $G \cdot v$ at v and*
- b) *The dimension of the tangent space of $G \cdot v$ does not depend on $v \in V$.*

For a reference to Mather's Lemma, for example, the reader can see [7].

Theorem 2.2 (Thom-Levine). *Let U be a domain in \mathbb{C} , W a neighborhood of 0 in \mathbb{C}^n and $F: W \times U \rightarrow \mathbb{C}$ such that $F(0, t) = 0$. Let us denote $f_t(x) = F(x, t) \forall t \in U, \forall x \in W$. If there is a family of analytic vector fields $v: W \times U \rightarrow \mathbb{C}^n, v(0, t) = 0$ for all $t \in U$ and*

$$\frac{\partial f_t}{\partial t}(x) = (df_t)(x)(v(x, t))$$

$\forall t \in U, \forall x \in W$, then f_t is analytically equivalent to $f_{t'}$ for any $t, t' \in U$.

For a reference to Thom-Levine's Theorem, the reader can see Proposition 2.2 of [5].

Two function-germs $f, g: (\mathbb{C}^n, 0) \rightarrow (\mathbb{C}, 0)$ are called *bi-Lipschitz equivalent* if there exists a bi-Lipschitz map-germ $\phi: (\mathbb{C}^n, 0) \rightarrow (\mathbb{C}^n, 0)$ such that $f = g \circ \phi$. In order to study the bi-Lipschitz equivalence of analytic function-germs and motivated by Thom-Levine's Theorem for analytic equivalence of function-germs, we define the following notion of bi-Lipschitz trivality of a family of function-germs. Let f_t ($t \in U$, a domain in \mathbb{C}) be a family of analytic function-germs. That is, there is a neighborhood W of 0 in \mathbb{C}^n and an analytic function $F: W \times U \rightarrow \mathbb{C}$ such that $F(0, t) = 0$ and $f_t(x) = F(x, t) \forall t \in U, \forall x \in W$. We call f_t *strongly bi-Lipschitz*

trivial when there is a continuous family of Lipschitz vector fields $v_t : (\mathbb{C}^n, 0) \rightarrow (\mathbb{C}^n, 0)$, where the Lipschitz constant of v_t does not depend on t , such that

$$\frac{\partial f_t}{\partial t}(x) = (df_t)(x)(v_t(x))$$

$\forall t \in U, \forall x \in W$.

The following result shows that if we include two function-germs f and g into a strongly bi-Lipschitz trivial family of function-germs, then the two initial function-germs f and g are bi-Lipschitz equivalent.

Theorem 2.3. *If f_t is strongly bi-Lipschitz trivial, then f_t is bi-Lipschitz equivalent to $f_{t'}$ for any $t \neq t' \in U$.*

To prove Theorem 2.3 it is enough to follow the proof of Thom-Levine’s Theorem and observe that the flow of a Lipschitz vector field defines a family of bi-Lipschitz homeomorphisms.

2.2. Weighted homogeneous functions. Let $w = (w_1, \dots, w_n)$ be an n -tuple of positive integer numbers. We say that a polynomial function $f(x_1, \dots, x_n)$ is w -homogeneous of degree d if $f(s^{w_1}x_1, \dots, s^{w_n}x_n) = s^d f(x_1, \dots, x_n)$ for all $s \in \mathbb{C}^*$. Let $H_w^d(n, 1)$ denote the space of w -homogeneous n -variable polynomials of degree d . Let \mathcal{O}_n be the ring of analytic function-germs at the origin $0 \in \mathbb{C}^n$ and let \mathcal{M}_n be the maximal ideal of \mathcal{O}_n .

Proposition 2.4. *Let $F(x_1, \dots, x_n, t)$ be a polynomial function such that for each $t \in U$, the function $f_t(x_1, \dots, x_n) = F(x_1, \dots, x_n, t)$ is w -homogeneous with an isolated singularity at $0 \in \mathbb{C}^n$, where U is a domain of \mathbb{C} . If $\frac{\partial F}{\partial t}$ belongs to the ideal of \mathcal{O}_n generated by $\{x_i \frac{\partial F}{\partial x_j} : i, j = 1, \dots, n\}$ for each fixed $t \in U$, then f_{t_1} is analytically equivalent to f_{t_2} for any $t_1, t_2 \in U$.*

Proof. Let us denote by TF the ideal of \mathcal{O}_n generated by $\{x_i \frac{\partial F}{\partial x_j} : i, j = 1, \dots, n\}$. It is clear that $H_w^d(n, 1)$ can be considered a subset of the space of m -jets $J^m(n, 1)$, for m large enough. The set

$$A_w^d(n, 1) = \{f \in H_w^d(n, 1) : f \text{ has an isolated singularity at the origin}\}$$

is a Zariski open subset of $H_w^d(n, 1)$. In particular, $A_w^d(n, 1)$ can be seen as a connected submanifold of the m -jets space $J^m(n, 1)$. Let $\mathcal{R}(n, n)$ be the group of analytic diffeomorphism-germs $(\mathbb{C}^n, 0) \rightarrow (\mathbb{C}^n, 0)$. We consider the natural action of $G = j^m(\mathcal{R}(n, n))$ on the manifold $M = J^m(n, 1)$ given by

$$(j^m(\phi), j^m(f)) \mapsto j^m(f \circ \phi).$$

So, given $f \in A_w^d(n, 1)$, we get

$$\begin{aligned} \text{codim}(G \cdot f) &= \dim(M) - \dim(G \cdot f) \\ &= \dim(M) - \dim(T_f(G \cdot f)) \\ &= \dim\left(\frac{\mathcal{M}_n}{\mathcal{M}_n^{m+1}}\right) - \dim\left(\frac{TF}{\mathcal{M}_n^{m+1}}\right) \\ &= \dim\left(\frac{\mathcal{M}_n}{TF}\right) \\ &= n + \mu(f), \end{aligned}$$

where $\mu(f)$ is the Milnor number of f . Since the Milnor number of $f \in A_w^d(n, 1)$ does not depend on f , we get that the dimension of $T_f(G \cdot f)$ does not depend on $f \in A_w^d(n, 1)$. Let

$$P = \{f_t : t \in U\}.$$

Then,

- (1) by the assumptions, $T_f P \subset T_f(G \cdot f)$, for any $f \in P$, and
- (2) $\dim(T_f(G \cdot f))$ is constant for $f \in P$, because $P \subset A_w^d(n, 1)$.

It follows from Mather’s Lemma that P is included in a single G -orbit. □

Remark 2.5. The family of polynomial function-germs $f_t(x, y) = F(x, y, t) = x^3 + (1 + t)xy^5 + t^2y^6$ gives an example where the Milnor number of f_0 is 13 and the Milnor number of f_t is 10 for any $t \neq 0$; hence f_0 is not analytically equivalent to f_t for any $t \neq 0$. On the other hand, we show that, $\forall t$, $\frac{\partial F}{\partial t}$ belongs to the ideal of \mathcal{O}_2 generated by $\{x\frac{\partial F}{\partial x}, x\frac{\partial F}{\partial y}, y\frac{\partial F}{\partial x}, y\frac{\partial F}{\partial y}\}$. In fact, given t , let us denote by J_t the ideal of \mathcal{O}_2 generated by

$$\left\{x\frac{\partial F}{\partial x}(x, y, t), x\frac{\partial F}{\partial y}(x, y, t), y\frac{\partial F}{\partial x}(x, y, t), y\frac{\partial F}{\partial y}(x, y, t)\right\}.$$

For $t = 0$, it is easy to see that $\frac{\partial F}{\partial t} = xy^5$ belongs to J_0 . Now, for $t \neq 0$, we have the following system of equations:

$$\begin{cases} 3x^3 + (1+t)xy^5 + 0y^6 + 0x^2y^4 & \equiv 0 \pmod{J_t + \mathcal{M}_2^9} \\ 0x^3 + 5(1+t)xy^5 + 6t^2y^6 + 0x^2y^4 & \equiv 0 \pmod{J_t + \mathcal{M}_2^9} \\ 0x^3 + 6t^2xy^5 + 0y^6 + 5(1+t)x^2y^4 & \equiv 0 \pmod{J_t + \mathcal{M}_2^9} \\ 0x^3 + 0xy^5 + 0y^6 + 3x^2y^4 & \equiv 0 \pmod{J_t + \mathcal{M}_2^9} \end{cases}$$

For $t \neq 0$, the following determinant is different from 0:

$$\begin{vmatrix} 3 & (1+t) & 0 & 0 \\ 0 & 5(1+t) & 6t^2 & 0 \\ 0 & 6t^2 & 0 & 5(1+t) \\ 0 & 0 & 0 & 3 \end{vmatrix};$$

hence we get $x^3, xy^5, y^6, x^2y^4 \in J_t + \mathcal{M}_2^9$. Now, it is easy to see that $\mathcal{M}_2^6 \subset J_t + \mathcal{M}_2^9$ and, by Nakayama’s Lemma, $\mathcal{M}_2^6 \subset J_t$. Finally, as $\frac{\partial F}{\partial t}$ belongs to \mathcal{M}_2^6 , it follows that $\frac{\partial F}{\partial t}$ belongs to $J_t, \forall t \neq 0$.

Thus, this example shows that the hypothesis of weighted homogeneity is essential in Proposition 2.4.

3. RESULTS

Proposition 3.1. *Let $F(x, y, t)$ be a polynomial function such that: for each $t \in U$, the function $f_t(x, y) = F(x, y, t)$ is w -homogeneous ($w_1 > w_2$) with an isolated singularity at $(0, 0) \in \mathbb{C}^2$, where U is an open subset of \mathbb{C} . If f_t defines a strongly bi-Lipschitz trivial family of function-germs at the origin $(0, 0) \in \mathbb{C}^2$, then there exists a function $k : U \rightarrow \mathbb{C}$ such that*

$$\frac{\partial F}{\partial t}(x, y, t) - k(t)y\frac{\partial F}{\partial y}(x, y, t)$$

is identically null on the polar set $\{(x, y, t) : \frac{\partial F}{\partial x}(x, y, t) = 0\}$.

Before we start the proof of Proposition 3.1 we need the following result.

Lemma 3.2. *For each $t \in U$, let $\Gamma_t = \{(x, y, t) : \frac{\partial F}{\partial x}(x, y, t) = 0\}$. If $y = 0$ is a component of Γ_{t_0} for some t_0 , then $y = 0$ is a component of Γ_t for all $t \in U$. Moreover, $F_t(x, y) = y\bar{F}_t(x, y)$, where $\bar{F}_t(x, y)$ is a weighted homogeneous polynomial and $\bar{F}_t(x, 0) \neq 0$.*

Proof of Lemma 3.2. If $y = 0$ is a component of Γ_{t_0} , we can write

$$F_t(x, y) = H_t(x, y)\bar{F}_t(x, y)$$

such that H_t and \bar{F}_t are weighted homogeneous of degree a and b respectively and $H_{t_0}(x, y) = y^m$. Since the weighted degree of F_t does not depend on t , a and b do not depend on t . We have $m = 1$, because if $m \geq 2$, then f_{t_0} does not have an isolated singularity at the origin. So, H_t is a weighted homogeneous polynomial of degree w_2 for all t . Since, $w_1 > w_2$, it follows that $H_t(x, y) = a(t)y$ and $a(t) \neq 0$ for all $t \in U$. Hence, we have proved the lemma. \square

Proof of Proposition 3.1. By the assumptions, there exists a Lipschitz vector field

$$v(x, y, t) = (v_1(x, y, t), v_2(x, y, t), 1)$$

such that $\frac{\partial F}{\partial v} = 0$. Let us denote

$$\Gamma_t = \{(x, y, t) : \frac{\partial F}{\partial x}(x, y, t) = 0\}.$$

Let $a_1(t), \dots, a_r(t)$ be functions defined in the following way:

Case $\{y = 0\}$ is not a component of Γ_t .

In this case, $a_1(t), \dots, a_r(t)$ are the roots of the polynomial equation

$$\frac{\partial F}{\partial x}(x, 1, t) = 0.$$

So,

$$\gamma_i(s) = (a_i(t)s^{w_1}, s^{w_2}, t), \quad i = 1, \dots, r$$

parameterize the branches of Γ_t .

Case $\{y = 0\}$ is a component of Γ_t .

In this case, $a_1(t), \dots, a_r(t)$ are the roots of the polynomial equation

$$\frac{\partial \bar{F}}{\partial x}(x, 1, t) = 0.$$

So,

$$\gamma_i(s) = (a_i(t)s^{w_1}, s^{w_2}, t), \quad i = 1, \dots, r$$

parameterize the branches of Γ_t other than $\{y = 0\}$.

Let us define the functions $k_1(t), \dots, k_r(t)$ by

$$k_i(t) = \frac{\frac{\partial F}{\partial t}(a_i(t), 1, t)}{\frac{\partial F}{\partial y}(a_i(t), 1, t)}.$$

We claim that $k_i(t) = k_j(t)$ for any $i \neq j$. In fact, since $v_2(x, y, t)$ is a Lipschitz function, there exists a constant $\lambda > 0$ such that

$$|v_2(a_i(t)s^{w_1}, s^{w_2}, t) - v_2(a_j(t)s^{w_1}, s^{w_2}, t)| \leq \lambda|a_i(t) - a_j(t)||s|^{w_1}.$$

On the other hand,

$$\begin{aligned} &|v_2(a_i(t)s^{w_1}, s^{w_2}, t) - v_2(a_j(t)s^{w_1}, s^{w_2}, t)| \\ &= \left| \frac{\frac{\partial F}{\partial t}(a_i(t)s^{w_1}, s^{w_2}, t)}{\frac{\partial F}{\partial y}(a_i(t)s^{w_1}, s^{w_2}, t)} - \frac{\frac{\partial F}{\partial t}(a_j(t)s^{w_1}, s^{w_2}, t)}{\frac{\partial F}{\partial y}(a_j(t)s^{w_1}, s^{w_2}, t)} \right| \\ &= |k_i(t) - k_j(t)||s|^{w_2}. \end{aligned}$$

Since $w_1 > w_2$, we have $k_i(t) = k_j(t)$.

Let us denote $k(t) = k_1(t) = \dots = k_r(t)$. Now, let us fix t . We see that the function

$$\frac{\partial F}{\partial t}(x, y, t) - k(t)y \frac{\partial F}{\partial y}(x, y, t)$$

is identically null on each branch of Γ_t at $(0, 0, t)$, even when $\{y = 0\}$ is a component of Γ_t . □

Remark 3.3. As a consequence of the proof of Proposition 3.1, we see that $k(t)$ satisfies additional properties. Since $k(t)$ is an algebraic expression of the functions $a_i(t)$ and, since we can develop $a_i(t)$ as a Puiseux series in a neighborhood of each t_0 , we get that for any $t_0 \in U$ there exist two open sets $0 \in V \subset \mathbb{C}$ and $t_0 \in U' \subset U$ such that

$$s \mapsto s^N + t_0$$

maps V onto U' and the function

$$s \mapsto k(s^N + t_0)$$

is analytic for some $N \in \mathbb{N}$.

Theorem 3.4. *Let $F(x, y, t)$ be a polynomial function such that: for each $t \in U$, the function $f_t(x, y) = F(x, y, t)$ is w -homogeneous ($w_1 > w_2$) with an isolated singularity at $(0, 0) \in \mathbb{C}^2$, where $U \subset \mathbb{C}$ is a domain. If $\{f_t : t \in U\}$, as a family of function-germs at $(0, 0) \in \mathbb{C}^2$, is strongly bi-Lipschitz trivial, then f_{t_1} is analytically equivalent to f_{t_2} for any $t_1, t_2 \in U$.*

Proof. Let $k(t)$ be given by Proposition 3.1; hence we have a function $k: U \rightarrow \mathbb{C}$ such that

$$\frac{\partial F}{\partial t}(x, y, t) - k(t)y \frac{\partial F}{\partial y}(x, y, t)$$

is identically null on the polar set $\{(x, y, t) : \frac{\partial F}{\partial x}(x, y, t) = 0\}$.

Let $t_0 \in U$. According to Remark 3.3, there exist two open sets $0 \in V \subset \mathbb{C}$ and $t_0 \in U' \subset U$ such that

$$s \mapsto s^N + t_0$$

maps V onto U' and the function

$$s \mapsto k(s^N + t_0)$$

is analytic for some $N \in \mathbb{N}$. Let us denote $G(x, y, s) = F(x, y, s^N + t_0)$, $g_s(x, y) = G(x, y, s)$ and $\tilde{k}(s) = k(s^N + t_0)$. Thus, we have an analytic function $\tilde{k}: V \rightarrow \mathbb{C}$ such that

$$\frac{\partial G}{\partial s}(x, y, s) - Ns^{N-1}\tilde{k}(s)y \frac{\partial G}{\partial y}(x, y, s)$$

is identically null on the polar set $\{(x, y, s) : \frac{\partial G}{\partial x}(x, y, s) = 0\}$. Let

$$P_1(x, y, s), \dots, P_r(x, y, s)$$

be such that they define the analytic irreducible factors of $\frac{\partial G}{\partial x}(x, y, s)$ in \mathcal{O}_3 . Let $j(s) = Ns^{N-1}\tilde{k}(s)$. Let

$$\alpha_i = \max\{\alpha \in \mathbb{N} : P_i^{\alpha_i} \text{ divides } \frac{\partial G}{\partial x} \text{ in } \mathcal{O}_3\}.$$

By hypothesis, we have integer positive numbers β_1, \dots, β_r such that

$$\frac{\partial G}{\partial s}(x, y, s) = u(x, y, s)P_1^{\beta_1}(x, y, s) \cdots P_r^{\beta_r}(x, y, s) + j(s)y\frac{\partial G}{\partial y}(x, y, s)$$

with $u = u(x, y, s) \in \mathcal{O}_3$. Moreover, we can suppose that P_i does not divide u in \mathcal{O}_3 . Let us denote

$$u_i(x, y, s) = u(x, y, s)P_1^{\beta_1}(x, y, s) \cdots \widehat{P_i^{\beta_i}(x, y, s)} \cdots P_r^{\beta_r}(x, y, s).$$

Thus, we have the equation

$$\frac{\partial G}{\partial s}(x, y, s) = u_i(x, y, s)P_i^{\beta_i}(x, y, s) + j(s)y\frac{\partial G}{\partial y}(x, y, s)m,$$

where $u_i = u_i(x, y, s) \in \mathcal{O}_3$ and P_i does not divide u_i in \mathcal{O}_3 .

We should show that $\beta_i \geq \alpha_i$. If $\alpha_i = 1$, we have nothing to do. Thus, let us consider $\alpha_i > 1$. It follows from

$$\frac{\partial G}{\partial s}(x, y, s) = u_i(x, y, s)P_i^{\beta_i}(x, y, s) + j(s)y\frac{\partial G}{\partial y}(x, y, s)$$

that

$$(3.1) \quad \frac{\partial^2 G}{\partial x \partial s} = \frac{\partial u_i}{\partial x} P_i^{\beta_i} + \beta_i u_i P_i^{\beta_i - 1} \frac{\partial P_i}{\partial x} + j y \frac{\partial^2 G}{\partial x \partial y}.$$

Now, since $P_i^{\alpha_i}$ divides $\frac{\partial G}{\partial x}$, we have that $P_i^{\alpha_i - 1}$ divides $\frac{\partial^2 G}{\partial s \partial x}$ and $\frac{\partial^2 G}{\partial y \partial x}$; hence, by using (3.1), we get that

$$(3.2) \quad P_i^{\alpha_i - 1} \text{ divides } P_i^{\beta_i - 1} \left(\frac{\partial u_i}{\partial x} P_i + \beta_i u_i \frac{\partial P_i}{\partial x} \right).$$

Since P_i does not divide either u_i or $\frac{\partial P_i}{\partial x}$, it follows from (3.2) that $\beta_i \geq \alpha_i$.

Once $\beta_i \geq \alpha_i$, we have that $\frac{\partial G}{\partial s}$ can be written in the following way:

$$\frac{\partial G}{\partial s}(x, y, s) = q(x, y, s)\frac{\partial G}{\partial x}(x, y, s) + j(s)y\frac{\partial G}{\partial y}(x, y, s).$$

Now, $\frac{\partial G}{\partial s}(x, y, s)$ and $y\frac{\partial G}{\partial y}(x, y, s)$, fixed s , are w -homogeneous polynomial functions of degree d , in the variables x and y , and $\frac{\partial G}{\partial x}(x, y, s)$ is w -homogeneous of degree $d - w_1$; hence $q(x, y, s)$ should be w -homogeneous of degree w_1 . In particular, $w(0, 0, s) = 0$ and $\frac{\partial G}{\partial s}$ belongs to the ideal of \mathcal{O}_3 generated by

$$\left\{ x \frac{\partial G}{\partial x}, x \frac{\partial G}{\partial y}, y \frac{\partial G}{\partial x}, y \frac{\partial G}{\partial y} \right\}.$$

Thus, it comes from Mather's Lemma or the analytic version of the Thom-Levine result that G_{s_1} is analytically equivalent to G_{s_2} for any $s_1, s_2 \in V$. This means that f_{t_1} is analytically equivalent to f_{t_2} for any $t_1, t_2 \in U^l$. Finally, it follows from the connectivity of U that f_{t_1} is analytically equivalent to f_{t_2} for any $t_1, t_2 \in U$. \square

If $\frac{\partial F}{\partial x}$ were analytically reduced, this theorem could be proved in the following simpler way. Let $k(t)$ be given by Proposition 3.1. We fix $t \in U$, and since $\frac{\partial F}{\partial x}(x, y, t)$ is analytically reduced, there exists an analytic function $u(x, y)$ such that

$$\frac{\partial F}{\partial t}(x, y, t) - k(t)y \frac{\partial F}{\partial y}(x, y, t) = u(x, y) \frac{\partial F}{\partial x}(x, y, t).$$

It follows from this equation that $u(x, y)$ is w -homogeneous of degree w_1 ; in particular $u(0, 0) = 0$. Thus, $\frac{\partial F}{\partial t}$ belongs to the ideal of \mathcal{O}_2 generated by

$$\left\{ x \frac{\partial F}{\partial x}(x, y, t), x \frac{\partial F}{\partial y}(x, y, t), y \frac{\partial F}{\partial x}(x, y, t), y \frac{\partial F}{\partial y}(x, y, t) \right\}.$$

Finally, since $f_t(x, y) = F(x, y, t)$ is w -homogeneous for all $t \in U$, it follows from Proposition 2.4 that $\{f_t : t \in U\}$ defines a family of function-germs at the origin $(0, 0) \in \mathbb{C}^2$ such that f_{t_1} is analytically equivalent to f_{t_2} for any $t_1, t_2 \in U$.

The above argument allows us to extend Theorem 3.4 to w -homogeneous polynomials in n variables ($w_1 > \dots > w_n$) with the additional hypothesis that the ideal

$$\left\{ \frac{\partial F}{\partial x_1}, \dots, \frac{\partial F}{\partial x_{n-1}} \right\}$$

is radical. However, the following example shows that, for $n \geq 3$ variables, if we remove some hypothesis above this result is not true at all.

Example 3.5. $f_t(x, y, z) = x^4 + y^4 + z^k + tx^2y^2$ is strongly bi-Lipschitz trivial (for t close to 0), but f_{t_1} is not analytically equivalent to f_{t_2} when $t_1 \neq t_2$.

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