

**SINGULAR INTEGRALS  
ON CARLESON MEASURE SPACES  $\text{CMO}^p$   
ON PRODUCT SPACES OF HOMOGENEOUS TYPE**

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**ABSTRACT.** In the setting of product spaces  $\widetilde{M}$  of homogeneous type, we prove that every product non-isotropic smooth (NIS) operator  $T$  is bounded on the generalized Carleson measure space  $\text{CMO}^p(\widetilde{M})$  of Han, Li and Lu for  $p_0 < p < 1$ . Here  $p_0$  depends on the homogeneous dimensions of the measures on factors of the product space  $\widetilde{M}$  and on the regularity of the quasi-metrics on factors of  $\widetilde{M}$ . The  $L^p$  boundedness for  $1 < p < \infty$  of the class of NIS operators was developed in both the one-parameter case and the multiparameter case by Nagel and Stein, and the  $H^p$  boundedness was established in the multiparameter case by Han, Li and Lu.

1. INTRODUCTION

The purpose of this paper is to prove that all singular integral operators  $T$  that are non-isotropic smooth on an appropriate test function space extend to bounded operators on the space  $\text{CMO}^p(\widetilde{M})$  of generalized Carleson measures for a range of  $p$  given by  $p_0 < p < 1$  (see Theorem 1.1 below). On a product space  $\widetilde{M}$  of homogeneous type, we now introduce the required definitions, notation, background and earlier results. We begin with the function spaces  $\text{CMO}^p(\widetilde{M})$  on product spaces  $\widetilde{M}$  of homogeneous type, established in [HLL2]. Then we describe the non-isotropic smooth (NIS) operators of Nagel and Stein on both a single space  $M$  and product spaces  $\widetilde{M}$  of homogeneous type as developed in [NS01b], [NS04] and [NS06]. Next we mention the results of Han, Li and Lu [HLL2] on the duality of  $H^p(\widetilde{M})$  and  $\text{CMO}^p(\widetilde{M})$  for  $p_0 < p \leq 1$  and the boundedness of a class of NIS singular integrals that generalizes the class treated by Nagel and Stein. Finally, our Theorems 1.1 and 1.2 complete the picture of the boundedness of these operators. For the product space  $\widetilde{M}$ , each NIS singular integral operator satisfying the properties (II-1) to (II-6) or properties (II-1)' to (II-6)' below, extends to a bounded operator on the generalized Carleson measure space  $\text{CMO}^p(\widetilde{M})$  for an appropriate range  $p_0 < p < 1$ .

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It is well known that in the classical one-parameter case, the dual space BMO of the Hardy space  $H^1$  can be characterized by Carleson measures. Moreover, Chang and Fefferman proved that the dual of the product  $H^1$  space is characterized by the product Carleson measure [CF]. The generalized Carleson measure space  $\text{CMO}^p$  was first introduced in the work of Han and Lu in the multiparameter setting  $\mathbb{R}^n \times \mathbb{R}^m$  with the implicit flag structure [HL1]. Later, in [HLL1], Han, Lu and the first author further proved that the space  $\text{CMO}^p$  is the dual of the product  $H^p$  space for  $0 < p_0 < p \leq 1$  for some  $p_0$  on products  $M_1 \times M_2$  of spaces of homogeneous type, where  $M_1$  and  $M_2$  satisfy properties (1.2) and (1.3).

Turning to properties (1.2) and (1.3), we recall that  $(M, \rho, \mu)$  is a space of homogeneous type in the sense of Coifman and Weiss [CW] if  $\rho$  is a quasi-metric: that is, (i)  $\rho(x, y) = 0$  iff  $x = y$ ; (ii)  $\rho(x, y) = \rho(y, x)$ ; and (iii)  $\rho(x, z) \leq A[\rho(x, y) + \rho(y, z)]$  with the *quasi-metric constant*  $A \geq 1$ ; and if  $\mu$  is a non-negative measure satisfying the following doubling property:

$$(1.1) \quad \mu(B(x, 2\delta)) \leq C\mu(B(x, \delta)) \quad \text{for all } \delta > 0 \text{ and some constant } C,$$

where  $B(x, \delta) = \{y : \rho(x, y) < \delta\}$  is the metric ball centered at  $x$  with radius  $\delta$ . In [MS], Macias and Segovia proved that one can replace the quasi-metric  $\rho$  by another quasi-metric  $d \approx \rho$  such that  $d$  yields the same topology on  $M$  as  $\rho$  and, moreover,

$$(1.2) \quad \mu(B(x, r)) \sim r,$$

where  $B(x, r) = \{y \in M : d(y, x) < r\}$  and  $d$  has the following regularity property:

$$(1.3) \quad |d(x, y) - d(x', y)| \leq C_0 d(x, x')^\theta [d(x, y) + d(x', y)]^{1-\theta}$$

for some *regularity exponent*  $\theta$  where  $0 < \theta < 1$ ,  $0 < r < \infty$  and all  $x, x', y \in M$ . Under these additional properties (1.2) and (1.3), the Littlewood–Paley–Stein theory and the Hardy space on  $(M, d, \mu)$  have been established. We refer to [DH] and the references therein for more details. The definition of the Hardy space  $H^p$  on  $M_1 \times M_2$  is stated in [HLL1] but first appeared in [HLY].

Recently, Han, Lu and the first author [HLL2] developed the multiparameter Hardy space theory on the so-called product Carnot–Carathéodory spaces  $\widetilde{M} = M_1 \times \cdots \times M_n$  formed by vector fields satisfying Hörmander’s *finite rank* condition. Their theory includes the product Hardy space  $H^1$  and its dual the product BMO space (which serve as the endpoint estimates for those singular integral operators considered by Nagel and Stein in [NS04]), the boundedness of singular integral operators, the Calderón–Zygmund decomposition, and interpolation of operators. More specifically, each  $M_i$  is a space of homogeneous type in the sense of Coifman and Weiss, satisfying property (1.2) with the regularity exponent  $\theta_i$  and satisfying the following property on the scaling of the measure  $\mu_i$  for  $i = 1, 2, \dots, n$ : there exist positive constants  $Q_i$  such that for all  $x \in M_i$  and  $\lambda \geq 1$ ,

$$(1.4) \quad \mu_i(B(x, \lambda r)) \approx \lambda^{Q_i} \mu_i(B(x, r)),$$

where the implicit constants are independent of  $x$  and  $r$ . The constant  $Q_i$  is called the *homogeneous dimension* of the measure  $\mu_i$  for  $i = 1, 2, \dots, n$ . This property (1.4) is satisfied by the earlier example of Nagel and Stein, when  $M$  arises as the boundary of an unbounded model polynomial domain in  $\mathbb{C}^2$ . For instance, this is the case when  $\Omega = \{(z, w) \in \mathbb{C}^2 : \text{Im}(w) > P(z)\}$ , where  $P$  is a real, subharmonic, non-harmonic polynomial of degree  $m$ .

To state the singular integral operators on  $M$  studied by Nagel and Stein, we first recall that  $\varphi$  is a *bump function associated to a ball*  $B(x_0, \delta)$  if  $\varphi$  is supported on this ball and satisfies the differential inequalities  $|\partial_X^a \varphi| \lesssim \delta^{-a}$  for all monomials  $\partial_X$  of degree  $a \geq 0$  in the vector fields  $X_1, \dots, X_k$ .

A class of *non-isotropic smooth operators*  $T$  is initially given as mappings from  $C_0^\infty(M)$  to  $C^\infty(M)$  with a distribution kernel  $K(x, y)$  which is  $C^\infty$  away from the diagonal of  $M \times M$  such that the following properties hold:

(I-1) If  $\varphi, \psi \in C_0^\infty(M)$  have disjoint supports, then

$$\langle T\varphi, \psi \rangle := \int_{M \times M} K(x, y)\varphi(y)\psi(x) dy dx.$$

(I-2) If  $\varphi$  is a normalized bump function associated to a ball of radius  $r$ , then  $|\partial_X^a T\varphi| \lesssim r^{-a}$  for each integer  $a \geq 0$ .

(I-3) If  $x \neq y$ , then for every integer  $a \geq 0$  and letting  $V(x, y)$  denote the volume of the ball  $B(x, d(x, y))$ , we have  $|\partial_{X,Y}^a K(x, y)| \lesssim d(x, y)^{-a} V(x, y)^{-1}$ .

(I-4) Properties (I-1) through (I-3) also hold with  $x$  and  $y$  interchanged. That is, these properties also hold for the adjoint operator  $T^t$  defined by  $\langle T^t \varphi, \psi \rangle := \langle T\psi, \varphi \rangle$ .

Nagel and Stein proved the following result.

**Theorem A** ([NS04]). *Each singular integral  $T$  satisfying properties (I-1) through (I-4) extends to a bounded operator on  $L^p(M)$  whenever  $1 < p < \infty$ .*

An important example of such operators is the class of non-isotropic smooth (NIS) operators introduced in [NRSW]. See also [K] and [NS01a] for further examples.

Turning to the product setting, for the sake of simplicity we focus on the case  $\widetilde{M} = M_1 \times M_2$  of two factors. The same definitions and results extend to the case of finitely many factors. Similarly to the single-factor case, the operator  $T$  is initially defined from  $C_0^\infty(\widetilde{M})$  to  $C^\infty(\widetilde{M})$ . The distribution kernel  $K(x_1, y_1, x_2, y_2)$  of  $T$  is a  $C^\infty$  function away from the “cross”  $\{(x, y) : x_1 = y_1 \text{ and } x_2 = y_2; x = (x_1, x_2), y = (y_1, y_2)\}$  and satisfies the following additional properties:

(II-1) If  $\varphi_i, \psi_i \in C_0^\infty(M_i)$  have disjoint supports for  $i = 1, 2$ , then  $\langle T(\varphi_1 \otimes \varphi_2), \psi_1 \otimes \psi_2 \rangle := \int K(x_1, y_1, x_2, y_2)\varphi_1(y_1)\varphi_2(y_2)\psi_1(x_1)\psi_2(x_2) dy dx$ .

(II-2) For each bump function  $\varphi_2$  on  $M_2$  and each  $x_2 \in M_2$ , there exists a singular integral operator  $T^{\varphi_2, x_2}$  (of one parameter) on  $M_1$  such that

$$\langle T(\varphi_1 \otimes \varphi_2), \psi_1 \otimes \psi_2 \rangle = \int_{M_2} \langle T^{\varphi_2, x_2} \varphi_1, \psi_1 \rangle \psi_2(x_2) dx_2.$$

Moreover,  $x_2 \mapsto T^{\varphi_2, x_2}$  is smooth and uniform in the sense that  $T^{\varphi_2, x_2}$ , as well as  $\rho_2^L \partial_{X_2}^L (T^{\varphi_2, x_2})$  for each  $L \geq 0$ , satisfy the conditions (I-1) to (I-4) uniformly, where  $\rho_2$  is the radius of the ball associated to  $\varphi_2$ .

(II-3) If  $\varphi_i$  is a bump function on a ball  $B^i(r_i)$  in  $M_i$  with radius  $r_i$  for  $i = 1, 2$ , then for all integers  $a_1, a_2 \geq 0$ ,  $|\partial_{X_1}^{a_1} \partial_{X_2}^{a_2} T(\varphi_1 \otimes \varphi_2)| \lesssim r_1^{-a_1} r_2^{-a_2}$ . In (II-2) and (II-3), both inequalities are taken in the sense of (I-2) whenever  $\varphi_2$  is a bump function for a ball  $B^2(r_2)$  in  $M_2$ .

(II-4) For all integers  $a_1, a_2 \geq 0$ ,

$$\left| \partial_{X_1, Y_1}^{a_1} \partial_{X_2, Y_2}^{a_2} K(x_1, y_1; x_2, y_2) \right| \lesssim \frac{d_1(x_1, y_1)^{-a_1} d_2(x_2, y_2)^{-a_2}}{V_1(x_1, y_1) V_2(x_2, y_2)}.$$

(II-5) The properties (II-1) through (II-4) hold when the indices 1 and 2 are interchanged, that is, when the roles of  $M_1$  and  $M_2$  are interchanged.

(II-6) The properties (II-1) through (II-4) are assumed to hold for the three “transposes” of  $T$ , i.e. those operators which arise by interchanging  $x_1$  and  $y_1$ , or interchanging  $x_2$  and  $y_2$ , or doing both interchanges.

The main result in Nagel and Stein’s paper [NS04] is the following.

**Theorem B** ([NS04]). *For  $1 < p < \infty$ , each product singular integral satisfying conditions (II-1) to (II-6) extends to a bounded operator on  $L^p(\widetilde{M})$ .*

Building on the results in [HLL1] and [HLLW], Han, Lu and the first author proved the following theorem.

**Theorem C** ([HLL2]). *Suppose  $\widetilde{M} = M_1 \times M_2$ . Let  $\theta_1, \theta_2$  be the regularity exponents of the quasi-metrics of  $d_1$  and  $d_2$  and let  $Q_1, Q_2$  be the homogeneous dimension of the measures  $\mu_1$  and  $\mu_2$ , as in properties (1.2) and (1.4). For the number*

$$p_0 := \max\left(\frac{2Q_1}{2Q_1 + \theta_1}, \frac{2Q_2}{2Q_2 + \theta_2}\right),$$

*the following results hold:*

- (a) *The dual of  $H^p(\widetilde{M})$  is  $\text{CMO}^p(\widetilde{M})$  for  $p_0 < p \leq 1$ .*
- (b) *For  $p_0 < p \leq 1$ , each product singular integral  $T$  satisfying properties (II-1) to (II-6) extends to be a bounded operator on  $H^p(\widetilde{M})$  and from  $H^p(\widetilde{M})$  to  $L^p(\widetilde{M})$ .*
- (c) *Each product singular integral  $T$  satisfying properties (II-1) to (II-6) extends to be a bounded operator on  $\text{BMO}(\widetilde{M})$  and from  $L^\infty(\widetilde{M})$  to  $\text{BMO}(\widetilde{M})$ .*

Note: The conclusions of Theorem C also hold if  $T$  satisfies the weaker conditions (II-1)’ to (II-6)’ , which are stated in Section 3.

However, their result does not address boundedness of the product singular integral  $T$  on the Carleson measure space  $\text{CMO}^p(\widetilde{M})$  for  $p_0 < p < 1$ .

In this note, we give a direct proof of boundedness of these product singular integrals  $T$  on the Carleson measure spaces  $\text{CMO}^p(\widetilde{M})$ . More precisely, we have the following main result.

**Theorem 1.1.** *Suppose  $\widetilde{M} = M_1 \times M_2$ . Then for  $p_0 := \max\left(\frac{2Q_1}{2Q_1 + \theta_1}, \frac{2Q_2}{2Q_2 + \theta_2}\right) < p < 1$ , each singular integral  $T$  satisfying properties (II-1) through (II-6) extends to a bounded operator on  $\text{CMO}^p(\widetilde{M})$ .*

We will obtain this result as a special case of our more general Theorem 1.2 as follows.

**Theorem 1.2.** *Suppose  $\widetilde{M} = M_1 \times M_2$ . Then for  $p_0 := \max\left(\frac{2Q_1}{2Q_1 + \theta_1}, \frac{2Q_2}{2Q_2 + \theta_2}\right) < p < 1$ , each singular integral  $T$  satisfying the properties (II-1)’ through (II-6)’ listed in Section 3 extends to a bounded operator on  $\text{CMO}^p(\widetilde{M})$ .*

As mentioned in [HLL2], the NIS operators (i.e. the singular integral operators that satisfy properties (II-1) to (II-6)) also satisfy properties (II-1)’ to (II-6)’ listed at the beginning of Section 3. Hence, Theorem 1.1 follows directly from Theorem 1.2.

The paper is organized as follows. In Section 2, we review for a single space  $M$  of homogeneous type the approximation to the identity and the dyadic cubes,

and then for product spaces  $\widetilde{M}$  of homogeneous type the relevant test function spaces, the distributions, the discrete version of Calderón's reproducing formula, the Hardy spaces  $H^p(\widetilde{M})$  and the Carleson measure spaces  $\text{CMO}^p(\widetilde{M})$ . In Section 3, we present the boundedness and endpoint estimates for NIS operators on  $\text{CMO}^p(\widetilde{M})$ , first reviewing the definition from [HLL2] of a more general class of NIS operators than those defined by Nagel and Stein in [NS04], and then establishing our Theorem 1.2, which directly implies Theorem 1.1.

## 2. PRELIMINARIES

In this section, for simplicity we consider the case of two factors. The underlying space is  $\widetilde{M} = M_1 \times M_2$ , where each  $M_i$ ,  $i = 1, 2$ , is a space of homogeneous type satisfying the additional scaling condition (1.4). We shall also suppose that  $\mu_i(M_i) = \infty$ ,  $\mu_i(\{x_i\}) = 0$  and  $0 < V_r(x_i) < \infty$  for all  $r > 0$ ,  $x_i \in M_i$ ,  $i = 1, 2$ . Denote by  $Q_i$  the homogeneous dimension of  $M_i$  as in condition (1.4) for  $i = 1, 2$ , where  $V_r(x_i)$  is the volume of the ball  $B(x_i, r)$ . These hypotheses allow us to construct an approximation to the identity; see Definition 2.1.

**2.1. Approximations to the identity on  $M$  and dyadic cubes on  $M$ .** Let  $(M, d, \mu)$  be a space of homogeneous type. We begin by recalling the definition of an approximation to the identity, which plays the same role for us as the heat kernel  $H(s, x, y)$  does in Nagel–Stein's product theory [NS04].

**Definition 2.1.** Let  $M$  satisfy the regularity condition (1.2) with regularity exponent  $\theta$ . A sequence  $\{S_k\}_{k \in \mathbb{Z}}$  of operators is said to be an *approximation to the identity* on  $M$  if there exists a positive constant  $C$  such that for all  $k \in \mathbb{Z}$  and all  $x, x', y$  and  $y' \in M$ , the kernels  $S_k(x, y)$  of the operators  $S_k$  satisfy the following conditions (i) to (v):

$$(2.1) \quad (i) \quad S_k(x, y) = 0 \text{ if } d(x, y) \geq C2^{-k}, \text{ and } |S_k(x, y)| \leq C \frac{1}{V_{2^{-k}}(x) + V_{2^{-k}}(y)};$$

$$(2.2) \quad (ii) \quad |S_k(x, y) - S_k(x', y)| \leq C2^{k\theta} d(x, x')^\theta \frac{1}{V_{2^{-k}}(x) + V_{2^{-k}}(y)};$$

$$(2.3) \quad (iii) \quad \text{property (ii) also holds with } x \text{ and } y \text{ interchanged};$$

$$(2.4) \quad (iv) \quad |[S_k(x, y) - S_k(x, y')] - [S_k(x', y) - S_k(x', y')]| \\ \leq C2^{2k\theta} d(x, x')^\theta d(y, y')^\theta \frac{1}{V_{2^{-k}}(x) + V_{2^{-k}}(y)}; \text{ and}$$

$$(2.5) \quad (v) \quad \int_M S_k(x, y) d\mu(y) = \int_M S_k(x, y) d\mu(x) = 1.$$

We remark that the existence of such an approximation to the identity follows from Coifman's construction, which first appeared in [DJS], on spaces of homogeneous type satisfying conditions (1.2) and (1.3). See also [HMY] for more details on  $M$ .

For each  $k \in \mathbb{Z}$ , define the difference operator  $D_k$  by

$$(2.6) \quad D_k := S_k - S_{k-1}.$$

Then it is easy to verify that  $\{D_k\}_{k \in \mathbb{Z}}$  satisfies the same properties (i) to (iv) as  $S_k$  with a comparable constant  $C$ . However, property (v) is replaced by

$$(2.7) \quad (v)' \quad \int_M D_k(x, y) d\mu(y) = \int_M D_k(x, y) d\mu(x) = 0.$$

We now recall M. Christ’s construction of generalized dyadic cubes on spaces of homogeneous type.

**Theorem 2.2** ([C]). *Let  $(M, \rho, \mu)$  be a space of homogeneous type. Then there exists a collection  $\{I_\alpha^k \subset M : k \in \mathbb{Z}, \alpha \in I^k\}$  of open subsets of  $M$ , where  $I^k$  is some index set, and positive constants  $C_1, C_2$  such that:*

- (i)  $\mu(M \setminus \bigcup_\alpha I_\alpha^k) = 0$  for each fixed  $k$  and  $I_\alpha^k \cap I_\beta^k = \emptyset$  if  $\alpha \neq \beta$ ;
- (ii) for any  $\alpha, \beta, k, l$  with  $l \geq k$ , either  $I_\beta^l \subset I_\alpha^k$  or  $I_\beta^l \cap I_\alpha^k = \emptyset$ ;
- (iii) for each  $(k, \alpha)$  and each  $l < k$  there is a unique  $\beta$  such that  $I_\alpha^k \subset I_\beta^l$ ;
- (iv)  $\text{diam}(I_\alpha^k) \leq C_1 2^{-k}$ ; and
- (v) each  $I_\alpha^k$  contains some ball  $B(z_\alpha^k, C_2 2^{-k})$ , where  $z_\alpha^k \in M$ .

We can think of  $I_\alpha^k$  as being a dyadic cube with diameter roughly  $2^{-k}$ , centered at  $z_\alpha^k$ . Similarly, we think of  $CI_\alpha^k$  as the cube with the same center as  $I_\alpha^k$  and diameter  $C \text{diam}(I_\alpha^k)$ . To simplify the notation, we will call the set  $I_\alpha^k$  a dyadic cube and denote the diameter of  $I_\alpha^k$  by  $\ell(I_\alpha^k)$ .

**2.2. Test functions, distributions and the discrete Calderón identity on**

$\widetilde{M} = M_1 \times M_2$ . We first recall the space of test functions on each  $M_i, i = 1, 2$ . For simplicity, we drop the subscript  $i$  here.

**Definition 2.3** ([HLL2]). Suppose  $0 < \gamma, \beta \leq \theta$  where  $\theta$  is the regularity exponent on  $M$  given in property (1.3). Take  $r > 0$ . A function  $f$  defined on  $M$  is said to be a *test function of type  $(x_0, r, \beta, \gamma)$  centered at  $x_0 \in M$*  if  $f$  satisfies the following conditions:

- (i)  $|f(x)| \leq C \frac{1}{V_r(x_0) + V(x, x_0)} \left( \frac{r}{r + d(x, x_0)} \right)^\gamma$ ;
- (ii)  $|f(x) - f(y)| \leq C \left( \frac{d(x, y)}{r + d(x, x_0)} \right)^\beta \frac{1}{V_r(x_0) + V(x, x_0)} \left( \frac{r}{r + d(x, x_0)} \right)^\gamma$  for all  $x, y \in M$  with  $d(x, y) < (r + d(x, x_0))/(2A)$ , where  $A$  is the quasi-metric constant.

We now recall the space of test functions and the associated distributions on  $\widetilde{M}$ .

**Definition 2.4** ([HLL2]). Suppose  $(x_0, y_0) \in \widetilde{M}, 0 < \gamma_1, \gamma_2, \beta_1, \beta_2 \leq \theta$  and  $r_1, r_2 > 0$ . A function  $f(x, y)$  defined on  $\widetilde{M}$  is said to be a *test function of type  $(x_0, y_0; r_1, r_2; \beta_1, \beta_2; \gamma_1, \gamma_2)$*  if (a) for each fixed  $y \in M_2, f(\cdot, y)$  is a test function in  $G(x_0, r_1, \beta_1, \gamma_1)$  on  $M_1$ ; (b) similarly, for any fixed  $x \in M_1, f(x, \cdot)$  is a test function in  $G(y_0, r_2, \beta_2, \gamma_2)$  on  $M_2$ ; and (c) the following conditions are satisfied:

- (i)  $\|f(\cdot, y)\|_{G(x_0, r_1, \beta_1, \gamma_1)} \leq C \frac{1}{V_{r_2}(y_0) + V(y_0, y)} \left( \frac{r_2}{r_2 + d(y, y_0)} \right)^{\gamma_2}$ ,

(ii)

$$\begin{aligned} & \|f(\cdot, y) - f(\cdot, y')\|_{G(x_0, r_1, \beta_1, \gamma_1)} \\ & \leq C \left( \frac{d(y, y')}{r_2 + d(y, y_0)} \right)^{\beta_2} \frac{1}{V_{r_2}(y_0) + V(y_0, y)} \left( \frac{r_2}{r_2 + d(y, y_0)} \right)^{\gamma_2}, \end{aligned}$$

for all  $y, y' \in M_2$  with  $d(y, y') \leq (r_2 + d(y, y_0))/(2A)$ , where  $A$  is the quasi-metric constant, and

(iii) properties (i) and (ii) also hold with  $x$  and  $y$  interchanged.

If  $f$  is a test function of type  $(x_0, y_0; r_1, r_2; \beta_1, \beta_2; \gamma_1, \gamma_2)$ , we write

$$f \in G(x_0, y_0; r_1, r_2; \beta_1, \beta_2; \gamma_1, \gamma_2),$$

and the norm of  $f$  is defined by

$$\|f\|_{G(x_0, y_0; r_1, r_2; \beta_1, \beta_2; \gamma_1, \gamma_2)} := \inf\{C : \text{properties (i), (ii) and (iii) hold}\}.$$

We denote by  $G(\beta_1, \beta_2; \gamma_1, \gamma_2)$  the class  $G(x_0, y_0; 1, 1; \beta_1, \beta_2; \gamma_1, \gamma_2)$  for any fixed  $(x_0, y_0) \in \widetilde{M}$ .

Let  $G_0(\beta_1, \beta_2; \gamma_1, \gamma_2)$  consist of those functions  $f(x, y)$  in  $G(\beta_1, \beta_2; \gamma_1, \gamma_2)$  that satisfy the cancellation property  $\int_{M_1} f(x, y) dx = \int_{M_2} f(x, y) dy = 0$ .

From [HLL2] we know that  $G(x_0, y_0; r_1, r_2; \beta_1, \beta_2; \gamma_1, \gamma_2) = G(\beta_1, \beta_2; \gamma_1, \gamma_2)$  with equivalent norms, for all  $(x_0, y_0) \in \widetilde{M}$  and  $r_1, r_2 > 0$ . Furthermore,  $G(\beta_1, \beta_2; \gamma_1, \gamma_2)$  is a Banach space with respect to the norm on  $G(\beta_1, \beta_2; \gamma_1, \gamma_2)$ .

Let  $\overset{\circ}{G}_{\theta_1, \theta_2}(\beta_1, \beta_2; \gamma_1, \gamma_2)$  be the completion of the space  $G_0(\theta_1, \theta_2; \theta_1, \theta_2)$  in  $G(\beta_1, \beta_2; \gamma_1, \gamma_2)$  with  $0 < \beta_i, \gamma_i < \theta_i$ , where  $\theta_i$  is the regularity exponent on  $M_i, i = 1, 2$ . For  $f \in \overset{\circ}{G}_{\theta_1, \theta_2}(\beta_1, \beta_2; \gamma_1, \gamma_2)$  we define  $\|f\|_{\overset{\circ}{G}_{\theta_1, \theta_2}(\beta_1, \beta_2; \gamma_1, \gamma_2)} := \|f\|_{G(\beta_1, \beta_2; \gamma_1, \gamma_2)}$ .

We define the distribution space  $(\overset{\circ}{G}_{\theta_1, \theta_2}(\beta_1, \beta_2; \gamma_1, \gamma_2))'$  to be the collection of all linear functionals  $L$  from  $\overset{\circ}{G}_{\theta_1, \theta_2}(\beta_1, \beta_2; \gamma_1, \gamma_2)$  to  $\mathbb{C}$  with the property that there exists  $C \geq 0$  such that for all  $f \in \overset{\circ}{G}_{\theta_1, \theta_2}(\beta_1, \beta_2; \gamma_1, \gamma_2), |L(f)| \leq C\|f\|_{\overset{\circ}{G}_{\theta_1, \theta_2}(\beta_1, \beta_2; \gamma_1, \gamma_2)}$ .

We now recall the discrete Calderón identity on  $\widetilde{M}$ .

**Theorem 2.5** ([HLL2]). *For  $i = 1, 2$ , let  $D_{k_i}$  be the operators given in equation (2.6) on each factor  $M_i$ . Then*

$$\begin{aligned} (2.8) \quad f(x, y) &= \sum_{k_1=-\infty}^{\infty} \sum_{k_2=-\infty}^{\infty} \sum_{\tau_1 \in I_{k_1}} \sum_{\tau_2 \in I_{k_2}} \mu_1(I_{\tau_1}^{k_1+N_1}) \mu_2(I_{\tau_2}^{k_2+N_2}) \\ &\quad \times D_{k_1}(x, y_{\tau_1}^{k_1+N_1}) D_{k_2}(y, y_{\tau_2}^{k_2+N_2}) \widetilde{\widetilde{D}}_{k_1} \widetilde{\widetilde{D}}_{k_2}(f)(y_{\tau_1}^{k_1+N_1}, y_{\tau_2}^{k_2+N_2}) \end{aligned}$$

where the series converges in both the norm of  $\overset{\circ}{G}_{\theta_1, \theta_2}(\beta'_1, \beta'_2; \gamma'_1, \gamma'_2)$  with  $0 < \beta'_i < \beta_i < \theta_i, 0 < \gamma'_i < \gamma_i < \theta_i$  for  $i = 1, 2$ , and the norm of  $L^p(\widetilde{M}), 1 < p < \infty$ . The kernels  $\widetilde{\widetilde{D}}_{k_i}(x, y)$  of the operators  $\widetilde{\widetilde{D}}_{k_i}$  satisfy the following estimates: for  $0 < \varepsilon_i < \theta_i,$

$$(2.9) \quad (i) \quad |\widetilde{\widetilde{D}}_{k_i}(x, y)| \leq C \frac{1}{V_{2^{-k_i}}(x) + V_{2^{-k_i}}(y) + V(x, y)} \frac{2^{-k\varepsilon_i}}{(2^{-k_i} + d(x, y))^{\varepsilon_i}};$$

(2.10)

$$(ii) \quad |\widetilde{D}_{k_i}(x, y) - \widetilde{D}_{k_i}(x', y)| \leq C \left( \frac{d(x, x')}{2^{-k} + d(x, y)} \right)^{\varepsilon_i} \frac{1}{V_{2^{-k}}(x) + V_{2^{-k}}(y) + V(x, y)} \\ \times \frac{2^{-k_i \varepsilon_i}}{(2^{-k_i} + d(x, y))^{\varepsilon_i}}, \quad \text{for } d(x, x') \leq (2^{-k} + d(x, y))/(2A);$$

(2.11)

$$(iii) \quad \int_{M_i} \widetilde{D}_{k_i}(x, y) d\mu_i(y) = \int_{M_i} \widetilde{D}_{k_i}(x, y) d\mu_i(x) = 0.$$

Here the operators  $\widetilde{D}_{k_i}$  are certain inverse operators acting on  $D_{k_i}$ ; see [HLL2].

**2.3. Hardy spaces and Carleson measure spaces on  $\widetilde{M}$ .** We first recall the following discrete Littlewood–Paley–Stein square function.

**Definition 2.6** ([HLL2]). For  $i = 1, 2$ , let  $\{S_{k_i}\}_{k_i \in \mathbb{Z}}$  be approximations to the identity on  $M_i$  and let  $D_{k_i} = S_{k_i} - S_{k_i-1}$ . For  $f \in (\mathring{G}_{\theta_1, \theta_2}(\beta_1, \beta_2; \gamma_1, \gamma_2))'$  with  $0 < \beta_i, \gamma_i < \theta_i$ ,  $i = 1, 2$ , the discrete Littlewood–Paley–Stein square function  $\widetilde{S}_d(f)$  of  $f$  is defined by

$$\widetilde{S}_d(f)(x, y) := \left\{ \sum_{k_1=-\infty}^{\infty} \sum_{k_2=-\infty}^{\infty} \sum_{\tau_1 \in I_{k_1}} \sum_{\tau_2 \in I_{k_2}} |D_{k_1} D_{k_2}(f)(x, y)|^2 \right. \\ \left. \times \chi_{I_{\tau_1}^{k_1+N_1}}(x) \chi_{I_{\tau_2}^{k_2+N_2}}(y) \right\}^{1/2}.$$

**Definition 2.7** ([HLL2]). Suppose  $\max\left(\frac{Q_1}{Q_1+\theta_1}, \frac{Q_2}{Q_2+\theta_2}\right) < p \leq 1$ . We define the

Hardy spaces  $H^p(\widetilde{M})$  by  $H^p(\widetilde{M}) := \{f \in (\mathring{G}_{\theta_1, \theta_2}(\beta_1, \beta_2; \gamma_1, \gamma_2))' : 0 < \beta_i, \gamma_i < \theta_i, i = 1, 2, \widetilde{S}_d(f) \in L^p(\widetilde{M})\}$ . For  $f \in H^p(\widetilde{M})$ , the norm of  $f$  is defined by  $\|f\|_{H^p(\widetilde{M})} := \|\widetilde{S}_d(f)\|_p$ .

We now recall the generalized Carleson measure space  $\text{CMO}^p(\widetilde{M})$ .

**Definition 2.8** ([HLL2]). Suppose  $p_0 := \max\left(\frac{2Q_1}{2Q_1+\theta_1}, \frac{2Q_2}{2Q_2+\theta_2}\right) < p \leq 1$  and  $0 < \beta_i, \gamma_i < \theta_i$  for  $i = 1, 2$ . Let  $\{S_{k_i}\}_{k_i \in \mathbb{Z}}$  be an approximation to the identity on  $M_i$ , and for  $k_i \in \mathbb{Z}$ , set  $D_{k_i} = S_{k_i} - S_{k_i-1}$ ,  $i = 1, 2$ . The generalized Carleson measure space  $\text{CMO}^p(\widetilde{M})$  is defined to be the set of all  $f \in (\mathring{G}_{\theta_1, \theta_2}(\beta_1, \beta_2; \gamma_1, \gamma_2))'$  such that

(2.12)

$$\|f\|_{\text{CMO}^p(\widetilde{M})} := \sup_{\Omega} \left\{ \frac{1}{|\Omega|^{\frac{2}{p}-1}} \int_{\Omega} \sum_{k_1, k_2} \sum_{I \times J \subseteq \Omega} |D_{k_1} D_{k_2}(f)(x, y)|^2 \chi_I(x) \chi_J(y) dx dy \right\}^{\frac{1}{2}} \\ < \infty,$$

where  $\Omega$  ranges over all open sets in  $\widetilde{M}$  with finite measure, and for each  $k_1$  and  $k_2$ ,  $I, J$  range over all the dyadic cubes in  $M_1$  and  $M_2$  with diameters  $\ell(I) = 2^{-k_1-N_1}$  and  $\ell(J) = 2^{-k_2-N_2}$ , respectively.

*Remark.* We point out that in [HLL2] it is shown that the definitions of  $H^p(\widetilde{M})$  and  $\text{CMO}^p(\widetilde{M})$  are independent of the choice of approximations to the identity of order  $\theta$  and that  $G_0(\theta_1, \theta_2, \theta_1, \theta_2)$  is dense in  $H^p(\widetilde{M})$ .

3. THE BOUNDEDNESS OF SINGULAR INTEGRAL OPERATORS ON CMO<sup>p</sup>( $\widetilde{M}$ )

In this section, we recall the singular integral operators on  $\widetilde{M} = M_1 \times M_2$ , where each factor  $M_i$  satisfies the doubling condition (1.1) and the regularity assumption (1.3) on the metric. Then we prove that such singular integral operators extend to bounded operators on CMO<sup>p</sup>( $\widetilde{M}$ ); see Theorem 1.2. For this purpose, we recall that  $\varphi$  is a bump function associated to a ball  $B(x_0, \delta)$  if  $\varphi$  is supported in this ball,  $|\varphi(x)| \leq 1$  and for some  $\eta \leq \theta$ ,  $\varphi$  belongs to the collection  $C^\eta(M)$  of all continuous functions  $f$  on  $M$  satisfying

$$\|f\|_\eta := \sup_{x \neq y} \frac{|f(x) - f(y)|}{(d(x, y))^\eta} < \infty,$$

with  $\|\varphi\|_\eta \leq \delta^{-\eta}$ .

We consider a class of *generalized non-isotropic smooth operators*  $T$  which are initially defined from  $C_0^\eta(M)$  ( $C^\eta$  functions with compact support,  $0 < \eta \leq \theta$ ) to  $C^\eta(M)$  with a distribution kernel  $K(x, y)$ . We require that the following properties hold:

(I-1)' If  $\varphi, \psi \in C_0^\eta(M)$  have disjoint supports, then

$$\langle T\varphi, \psi \rangle = \int_{M \times M} K(x, y)\varphi(y)\psi(x) dy dx.$$

(I-2)' If  $\varphi$  is a normalized bump function associated to a ball of radius  $r$ , then  $\|T\varphi\|_\infty \lesssim 1$  and  $\|T\varphi\|_\varepsilon \lesssim r^{-\varepsilon}$ ,  $\varepsilon \leq \eta$ .

(I-3)' If  $x \neq y$ , then  $|K(x, y)| \lesssim V(x, y)^{-1}$  and

$$|K(x, y) - K(x, y')| \lesssim \left(\frac{d(y, y')}{d(x, y)}\right)^\varepsilon V(x, y)^{-1}$$

for  $d(y, y') \leq d(x, y)/(2A)$ , where  $A$  is the quasi-metric constant.

(I-4)' Properties (I-1)' through (I-3)' also hold with  $x$  and  $y$  interchanged. That is, these properties also hold for the adjoint operator  $T^t$  defined by  $\langle T^t\varphi, \psi \rangle := \langle \varphi, T\psi \rangle$ .

To pass the above boundedness to the product case  $\widetilde{M} = M_1 \times M_2$ , again similarly to [NS04], we will consider the *generalized non-isotropic smooth operators*  $T$  initially defined from  $C_0^\eta(\widetilde{M})$  to  $C^\eta(\widetilde{M})$  associated with a distribution kernel  $K(x_1, y_1, x_2, y_2)$ , which is a  $C^\varepsilon$  function away from the “cross”  $\{(x, y) : x_1 = y_1 \text{ and } x_2 = y_2; x = (x_1, x_2), y = (y_1, y_2)\}$  and satisfies the following additional properties:

(II-1)' If  $\varphi_i, \psi_i \in C_0^\eta(M_i)$  have disjoint supports for  $i = 1, 2$ , then  $\langle T(\varphi_1 \otimes \varphi_2), \psi_1 \otimes \psi_2 \rangle = \int K(x_1, y_1, x_2, y_2)\varphi_1(y_1)\varphi_2(y_2)\psi_1(x_1)\psi_2(x_2) dy dx$ .

(II-2)' For each bump function  $\varphi_2$  on  $M_2$  and each point  $x_2 \in M_2$ , there exists a singular integral operator  $T^{\varphi_2, x_2}$  (of one parameter) on  $M_1$  so that

$$\langle T(\varphi_1 \otimes \varphi_2), \psi_1 \otimes \psi_2 \rangle = \int_{M_2} \langle T^{\varphi_2, x_2} \varphi_1, \psi_1 \rangle \psi_2(x_2) dx_2.$$

Moreover,  $x_2 \mapsto T^{\varphi_2, x_2}$  is in  $C^\eta(M_2)$  and is uniform in the sense that  $T^{\varphi_2, x_2}(x_1, y_1)$ , as a kernel on  $M_1$ , satisfies conditions (I-1)' to (I-4)' uniformly.

(II-3)' If  $\varphi_i$  is a bump function on a ball  $B^i(r_i)$  in  $M_i$ , then  $\|T(\varphi_1 \otimes \varphi_2)\|_{\eta,\eta} \lesssim r_1^{-a_1} r_2^{-a_2}$ , where

$$\|f(x_1, x_2)\|_{\eta,\eta} := \sup_{x_1 \neq x'_1, x_2 \neq x'_2} \frac{|[f(x_1, x_2) - f(x'_1, x_2)] - [f(x_1, x'_2) - f(x'_1, x'_2)]|}{|x_1 - x'_1|^\eta |x_2 - x'_2|^\eta}.$$

In (II-2)' and (II-3)', both inequalities are taken in the sense of (I-2)' whenever  $\varphi_2$  is a bump function for  $B^2(r_2)$  in  $M_2$ .

(II-4)'  $|K(x_1, y_1, x_2, y_2)| \lesssim V(x_1, y_1)^{-1} V(x_2, y_2)^{-1}$ ,

$$|K(x_1, y_1, x_2, y_2) - K(x'_1, y_1, x_2, y_2)| \lesssim \left(\frac{d(x_1, x'_1)}{d(x_1, y_1)}\right)^\varepsilon V(x_1, y_1)^{-1} V(x_2, y_2)^{-1}$$

for  $d(x_1, x'_1) \leq d(x_1, y_1)/(2A)$ , and

$$\begin{aligned} &|[K(x_1, y_1, x_2, y_2) - K(x'_1, y_1, x_2, y_2)] - [K(x_1, y_1, x'_2, y_2) - K(x'_1, y_1, x'_2, y_2)]| \\ &\lesssim \left(\frac{d(x_1, x'_1)}{d(x_1, y_1)}\right)^\varepsilon \left(\frac{d(x_2, x'_2)}{d(x_2, y_2)}\right)^\varepsilon V(x_1, y_1)^{-1} V(x_2, y_2)^{-1} \end{aligned}$$

for  $d(x_1, x'_1) \leq d(x_1, y_1)/(2A)$  and  $d(x_2, x'_2) \leq d(x_2, y_2)/(2A)$ , where  $A$  is the quasicmetric constant.

(II-5)' The same conditions hold when the indices 1 and 2 are interchanged, that is, whenever the roles of  $M_1$  and  $M_2$  are interchanged.

(II-6)' The same properties are assumed to hold for the three “transposes” of  $T$ , i.e. those operators which arise by interchanging  $x_1$  and  $y_1$ , or interchanging  $x_2$  and  $y_2$ , or doing both interchanges.

The main result in this section is the proof of Theorem 1.2.

We first point out that  $\text{CMO}^p(\widetilde{M})$  is a subset of the distributions  $(\overset{\circ}{G}_{\theta_1, \theta_2}(\beta_1, \beta_2; \gamma_1, \gamma_2))'$ , which means that  $T$  may not be defined directly on  $\text{CMO}^p(\widetilde{M})$ . The same problem appears in (b) of Theorem C. The key fact used in the proof of Theorem C is that  $L^2(\widetilde{M}) \cap H^p(\widetilde{M})$  is dense in  $H^p(\widetilde{M})$ ; thus it suffices to establish the boundedness of  $T$  on  $L^2(\widetilde{M}) \cap H^p(\widetilde{M})$ . However, this method does not work for Theorem 1.2 because  $L^2(\widetilde{M}) \cap \text{CMO}^p(\widetilde{M})$  is not dense in  $\text{CMO}^p(\widetilde{M})$ . Thus, we need the following result as a substitute for the method used above.

**Lemma 3.1.** *For  $p_0 := \max\left(\frac{2Q_1}{2Q_1 + \theta_1}, \frac{2Q_2}{2Q_2 + \theta_2}\right) < p \leq 1$ ,  $L^2(\widetilde{M}) \cap \text{CMO}^p(\widetilde{M})$  is dense in  $\text{CMO}^p(\widetilde{M})$  in the weak topology  $(H^p(\widetilde{M}), \text{CMO}^p(\widetilde{M}))$ . More precisely, for each  $f \in \text{CMO}^p(\widetilde{M})$ , there exists a sequence  $\{f_n\} \subset L^2(\widetilde{M}) \cap \text{CMO}^p(\widetilde{M})$  such that  $\|f_n\|_{\text{CMO}^p(\widetilde{M})} \leq C\|f\|_{\text{CMO}^p(\widetilde{M})}$ , where  $C$  is a positive constant independent of  $n$  and  $f$ , and, moreover, for each  $g \in H^p(\widetilde{M})$ ,  $\langle f_n, g \rangle \rightarrow \langle f, g \rangle$  as  $n$  tends to infinity.*

*Proof.* We first recall the discrete Calderón identity (2.8), namely,

$$(3.1) \quad f(x, y) = \sum_{k_1, k_2} \sum_{I, J} |I| |J| D_{k_1}(x, x_I) D_{k_2}(y, y_J) \widetilde{D}_{k_1} \widetilde{D}_{k_2}(f)(x_I, y_J),$$

where we write  $|\cdot|$  for  $\mu_1(\cdot)$  and  $\mu_2(\cdot)$ , and for each  $k_1$  and  $k_2$ ,  $I, J$  range over all the dyadic cubes in  $M_1$  and  $M_2$  with diameter  $\ell(I) = 2^{-k_1 - N_1}$  and  $\ell(J) = 2^{-k_2 - N_2}$ .

Moreover, the series converges in both the norm of  $\mathring{G}_{\theta_1, \theta_2}(\beta'_1, \beta'_2, \gamma'_1, \gamma'_2)$  with  $0 < \beta'_i < \beta_i < \theta_i, \gamma'_i, \gamma_i < \theta_i, i = 1, 2$ , and the norm of  $L^p(M_1 \times M_2)$ ,  $1 < p < \infty$ . We note that  $D_{k_1}(x, x_I)$  and  $D_{k_2}(y, y_J)$  as functions of  $x$  and  $y$  have compact support.

Now suppose that  $f \in \text{CMO}^p(\widetilde{M})$ . Set

$$f_n(x, y) := \sum_{|k_1| \leq n, |k_2| \leq n} \sum_{I, J: I \times J \subset B_n} |I||J| D_{k_1}(x, x_I) D_{k_2}(y, y_J) \widetilde{D}_{k_1} \widetilde{D}_{k_2}(f)(x_I, y_J),$$

where  $B_n = \{(x, y) : d(x, x_0) \leq n, d(y, y_0) \leq n\}$ .

It is easy to see that  $f_n \in L^2(\widetilde{M})$ . Next we show that  $f_n \in \text{CMO}^p(\widetilde{M})$ . In fact, we need to verify that for any open set  $\Omega \subset \widetilde{M}$  with finite measure,

$$(3.2) \quad \frac{1}{|\Omega|^{\frac{2}{p}-1}} \int_{\Omega} \sum_{k'_1, k'_2} \sum_{I' \times J' \subseteq \Omega} |D_{k'_1} D_{k'_2}(f_n)(x, y)|^2 \chi_{I'}(x) \chi_{J'}(y) dx dy \leq C \|f\|_{\text{CMO}^p(\widetilde{M})}^2.$$

We recall the almost orthogonality argument which appears as Lemma 2.11 in [HLL2]. Here and in the rest of the paper, for  $a, b \in \mathbb{R}$  we use  $a \wedge b, a \vee b$  to denote  $\min(a, b), \max(a, b)$ , respectively.

**Lemma 3.2** ([HLL2]). *Let  $\{S_{k_i}\}_{k_i \in \mathbb{Z}}$  and  $\{P_{k_i}\}_{k_i \in \mathbb{Z}}$  be two approximations to the identity with regularity exponent  $\theta_i$  and  $D_{k_i} = S_{k_i} - S_{k_i-1}$ ,  $E_{k_i} = P_{k_i} - P_{k_i-1}$ ,  $i = 1, 2$ . Then for each  $\varepsilon \in (0, \theta_1 \wedge \theta_2)$ , there exist positive constants  $C$  depending only on  $\varepsilon$  such that the kernel  $D_{l_1} D_{l_2} E_{k_1} E_{k_2}(x_1, x_2, y_1, y_2)$  of  $D_{l_1} D_{l_2} E_{k_1} E_{k_2}$  satisfies the following estimate:*

$$(3.3) \quad \begin{aligned} & |D_{l_1} D_{l_2} E_{k_1} E_{k_2}(x_1, x_2, y_1, y_2)| \leq C 2^{-|k_1-l_1|\varepsilon} 2^{-|k_2-l_2|\varepsilon} \\ & \times \frac{1}{V_{2^{-(k_1 \wedge l_1)}}(x_1) + V_{2^{-(k_1 \wedge l_1)}}(y_1) + V(x_1, y_1)} \frac{2^{-(k_1 \wedge l_1)\varepsilon}}{(2^{-(k_1 \wedge l_1)} + d(x_1, y_1))^\varepsilon} \\ & \times \frac{1}{V_{2^{-(k_2 \wedge l_2)}}(x_2) + V_{2^{-(k_2 \wedge l_2)}}(y_2) + V(x_2, y_2)} \frac{2^{-(k_2 \wedge l_2)\varepsilon}}{(2^{-(k_2 \wedge l_2)} + d(x_2, y_2))^\varepsilon}. \end{aligned}$$

Now back to the proof of Lemma 3.1. Substituting the definition of  $f_n$  into the left-hand side of the following inequality and then using Lemma 3.2 and Hölder's inequality, we have

$$\begin{aligned} & \sup_{u \in I', v \in J'} |D_{k'_1} D_{k'_2}(f_n)(u, v)|^2 \\ & \lesssim \sum_{k_1, k_2} 2^{-|k_1-k'_1|\varepsilon_1} 2^{-|k_2-k'_2|\varepsilon_2} \sum_{I, J} |I||J| \frac{1}{V(x_I, x_{I'}) + V_{2^{-(k_1 \wedge k'_1)}}(x_I) + V_{2^{-(k_1 \wedge k'_1)}}(x_{I'})} \\ & \times \left( \frac{2^{-(k_1 \wedge k'_1)}}{2^{-(k_1 \wedge k'_1)} + d(x_I, x_{I'})} \right)^{\varepsilon_1} \frac{1}{V(y_J, y_{J'}) + V_{2^{-(k_2 \wedge k'_2)}}(y_J) + V_{2^{-(k_2 \wedge k'_2)}}(y_{J'})} \\ & \times \left( \frac{2^{-(k_2 \wedge k'_2)}}{2^{-(k_2 \wedge k'_2)} + d(y_J, y_{J'})} \right)^{\varepsilon_2} |\widetilde{D}_{k_1} \widetilde{D}_{k_2}[f](x_I, y_J)|^2. \end{aligned}$$

As a consequence, we find that

$$\begin{aligned}
 (3.4) \quad & \frac{1}{|\Omega|^{\frac{2}{p}-1}} \sum_{k'_1, k'_2} \sum_{I' \times J' \subset \Omega} |I'| |J'| \sup_{u \in I', v \in J'} |D_{k'_1} D_{k'_2} [f_n](u, v)|^2 \\
 & \lesssim \frac{1}{|\Omega|^{\frac{2}{p}-1}} \sum_{k'_1, k'_2} \sum_{I' \times J' \subset \Omega} \sum_{k_1, k_2} \sum_{I, J} 2^{-|k_1 - k'_1| \varepsilon_1} 2^{-|k_2 - k'_2| \varepsilon_2} |I| |J| |I'| |J'| \\
 & \quad \times \left( \frac{2^{-(k_1 \wedge k'_1)}}{2^{-(k_1 \wedge k'_1)} + d(x_I, x_{I'})} \right)^{\varepsilon_1} \frac{1}{V(x_I, x_{I'}) + V_{2^{-(k_1 \wedge k'_1)}}(x_I) + V_{2^{-(k_1 \wedge k'_1)}}(x_{I'})} \\
 & \quad \times \left( \frac{2^{-(k_2 \wedge k'_2)}}{2^{-(k_2 \wedge k'_2)} + d(y_J, y_{J'})} \right)^{\varepsilon_2} \frac{1}{V(y_J, y_{J'}) + V_{2^{-(k_2 \wedge k'_2)}}(y_J) + V_{2^{-(k_2 \wedge k'_2)}}(y_{J'})} \\
 & \quad \times |\widetilde{D}_{k_1} \widetilde{D}_{k_2} [f](x_I, y_J)|^2.
 \end{aligned}$$

Note that

$$\begin{aligned}
 2^{-|k_1 - k'_1|} & \approx \frac{\text{diam}(I)}{\text{diam}(I')} \wedge \frac{\text{diam}(I')}{\text{diam}(I)}, \\
 2^{-(k_1 \wedge k'_1)} & \approx \text{diam}(I) \vee \text{diam}(I'),
 \end{aligned}$$

and  $d(x_I, x_{I'}) \geq \text{dist}(I, I')$ . Similar results hold for  $k_2, k'_2$  and  $J, J'$ . Applying the above estimate with any arbitrary points  $x_{I'}$  and  $y_{J'}$  in  $I'$  and  $J'$ , respectively, and the fact that  $ab = (a \vee b)^2 \left(\frac{a}{b} \wedge \frac{b}{a}\right)$  for all  $a, b > 0$ , we obtain that

$$\begin{aligned}
 (3.5) \quad & \frac{1}{|\Omega|^{\frac{2}{p}-1}} \sum_{k'_1, k'_2} \sum_{I' \times J' \subset \Omega} |I'| |J'| \sup_{u \in I', v \in J'} |D_{k'_1} D_{k'_2} [f_n](u, v)|^2 \\
 & \lesssim \frac{1}{|\Omega|^{\frac{2}{p}-1}} \sum_{k'_1, k'_2} \sum_{I' \times J' \subset \Omega} \sum_{k_1, k_2} \sum_{I, J} \left[ \frac{|I|}{|I'|} \wedge \frac{|I'|}{|I|} \right] \left[ \frac{|J|}{|J'|} \wedge \frac{|J'|}{|J|} \right] \\
 & \quad \times \left[ \frac{\text{diam}(I)}{\text{diam}(I')} \wedge \frac{\text{diam}(I')}{\text{diam}(I)} \right]^{\varepsilon_1} \left[ \frac{\text{diam}(J)}{\text{diam}(J')} \wedge \frac{\text{diam}(J')}{\text{diam}(J)} \right]^{\varepsilon_2} \cdot (|I| \vee |I'|) (|J| \vee |J'|) \\
 & \quad \times \frac{|I| \vee |I'|}{V_{\text{dist}(I, I')}(x_I) + |I| \vee |I'|} \left( \frac{\text{diam}(I) \vee \text{diam}(I')}{\text{diam}(I) \vee \text{diam}(I') + \text{dist}(I, I')} \right)^{\varepsilon_1} \\
 & \quad \times \frac{|J| \vee |J'|}{V_{\text{dist}(J, J')}(y_J) + |J| \vee |J'|} \left( \frac{\text{diam}(J) \vee \text{diam}(J')}{\text{diam}(J) \vee \text{diam}(J') + \text{dist}(J, J')} \right)^{\varepsilon_2} \\
 & \quad \times \inf_{u \in I', v \in J'} |D_{k_1} D_{k_2} [f](u, v)|^2.
 \end{aligned}$$

Now following the same steps as in the proof of Theorem 3.2 in [HLL2], we see that the right-hand side of the inequality (3.5) is bounded by  $\|f\|_{\text{CMO}^p(\widetilde{M})}^2$ , which implies that (3.2) holds. Hence  $f_n \in \text{CMO}^p(\widetilde{M})$  for each  $n$ , which implies that  $f_n \in L^2(\widetilde{M}) \cap \text{CMO}^p(\widetilde{M})$ .

For any  $h \in G_0(\theta_1, \theta_2, \theta_1, \theta_2)$ , by the discrete Calderón identity, we have

$$\begin{aligned} \langle f - f_n, h \rangle &= \left\langle \sum_{|k_1| > n, \text{ or } |k_2| > n, \text{ or } I \times J \not\subseteq B_n} |I||J|D_{k_1}(\cdot, x_I)D_{k_2}(\cdot, y_J) \right. \\ &\quad \left. \times \widetilde{D}_{k_1}\widetilde{D}_{k_2}(f)(x_I, y_J), h \right\rangle \\ &= \sum_{|k_1| > n, \text{ or } |k_2| > n, \text{ or } I \times J \not\subseteq B_n} |I||J|D_{k_1}D_{k_2}(h)(x_I, y_J) \\ &\quad \times \widetilde{D}_{k_1}\widetilde{D}_{k_2}(f)(x_I, y_J). \end{aligned}$$

Following the proof of Theorem 2.5, we can see that the function

$$H_n := \sum_{|k_1| > n, \text{ or } |k_2| > n, \text{ or } I \times J \not\subseteq B_n} |I||J|D_{k_1}D_{k_2}(h)(x_I, y_J)\widetilde{D}_{k_1}\widetilde{D}_{k_2}(f)(x_I, y_J)$$

belongs to  $H^p(\widetilde{M})$  and, moreover,  $\|H_n\|_{H^p(\widetilde{M})}$  tends to 0 as  $n$  tends to infinity. Hence  $\langle f - f_n, h \rangle$  tends to 0 as  $n$  tends to infinity.

Note that Theorem C shows the duality of  $\widetilde{\text{CMO}}^p(\widetilde{M})$  with  $H^p(\widetilde{M})$ . Also note that since  $G_0(\theta_1, \theta_2, \theta_1, \theta_2)$  is dense in  $H^p(\widetilde{M})$  (we refer to the remark in Section 2.3), we can extend the “inner product”  $\langle \cdot, \cdot \rangle$  from  $G_0(\theta_1, \theta_2, \theta_1, \theta_2)$  to  $H^p(\widetilde{M})$ , and hence for any  $g \in H^p(\widetilde{M})$ ,  $\langle f - f_n, g \rangle$  tends to 0 as  $n$  tends to infinity.

More precisely, for any  $g \in H^p(\widetilde{M})$  and for any  $\varepsilon > 0$ , there exists a function  $h \in G_0(\theta_1, \theta_2, \theta_1, \theta_2)$  such that  $\|g - h\|_{H^p(\widetilde{M})} < \varepsilon$ . Now by duality of  $\widetilde{\text{CMO}}^p(\widetilde{M})$  with  $H^p(\widetilde{M})$  and the fact that  $\|f_n\|_{\widetilde{\text{CMO}}^p(\widetilde{M})} \leq C\|f\|_{\widetilde{\text{CMO}}^p(\widetilde{M})}$ , we have

$$\begin{aligned} |\langle f - f_n, g \rangle| &\leq |\langle f - f_n, g - h \rangle| + |\langle f - f_n, h \rangle| \\ &\leq \|f - f_n\|_{\widetilde{\text{CMO}}^p(\widetilde{M})}\|g - h\|_{H^p(\widetilde{M})} + |\langle f - f_n, h \rangle| \\ &\leq C\varepsilon\|f\|_{\widetilde{\text{CMO}}^p(\widetilde{M})} + |\langle f - f_n, h \rangle|, \end{aligned}$$

which implies that  $\limsup_{n \rightarrow \infty} |\langle f - f_n, g \rangle| \leq C\varepsilon\|f\|_{\widetilde{\text{CMO}}^p(\widetilde{M})}$  and hence  $\lim_{n \rightarrow \infty} |\langle f - f_n, g \rangle| = 0$ . The proof of Lemma 3.1 is completed.  $\square$

*Proof of Theorem 1.2.* We first define  $T$  on  $\widetilde{\text{CMO}}^p(\widetilde{M})$  as follows. Given  $f \in \widetilde{\text{CMO}}^p(\widetilde{M})$ , by Lemma 3.1, there is a sequence  $\{f_n\} \subset L^2(\widetilde{M}) \cap \widetilde{\text{CMO}}^p(\widetilde{M})$  such that  $\|f_n\|_{\widetilde{\text{CMO}}^p(\widetilde{M})} \leq C\|f\|_{\widetilde{\text{CMO}}^p(\widetilde{M})}$ , and for each  $g \in L^2(\widetilde{M}) \cap H^p(\widetilde{M})$ ,  $\langle f_n, g \rangle \rightarrow \langle f, g \rangle$  as  $n \rightarrow \infty$ . Thus, for  $f \in \widetilde{\text{CMO}}^p(\widetilde{M})$ , we define  $\langle T(f), g \rangle := \lim_{n \rightarrow \infty} \langle T(f_n), g \rangle$  for each  $g \in L^2(\widetilde{M}) \cap H^p(\widetilde{M})$ .

To see that this limit exists, we note that  $\langle T(f_j - f_k), g \rangle = \langle f_j - f_k, T^*(g) \rangle$  since both  $f_j - f_k, g$  belong to  $L^2$  and  $T$  is bounded on  $L^2$ . Since  $T^*$  satisfies the same conditions as  $T$ , we have that  $T^*(g) \in L^2(\widetilde{M})$  and also  $T^*(g) \in H^p(\widetilde{M})$  by the note after Theorem C. Therefore, by Lemma 3.1,  $\langle f_j - f_k, T(g) \rangle$  tends to zero as  $j, k \rightarrow \infty$ . It is also easy to see that the definition of  $T(f)$  is independent of the choice of the sequence  $f_n$  that satisfies the conditions in Lemma 3.1.

By Lemma 3.1, to prove Theorem 1.2 it suffices to show that for each  $f \in L^2(\widetilde{M}) \cap \text{CMO}^p(\widetilde{M})$ ,

$$(3.6) \quad \|T(f)\|_{\text{CMO}^p(\widetilde{M})} \leq C\|f\|_{\text{CMO}^p(\widetilde{M})}$$

with a positive constant  $C$  independent of  $f$ .

To see this, let  $f \in L^2(\widetilde{M}) \cap \text{CMO}^p(\widetilde{M})$ . It suffices to show that for each open set  $\Omega \subset \widetilde{M}$  with finite measure,

$$(3.7) \quad \frac{1}{|\Omega|^{\frac{2}{p}-1}} \int_{\Omega} \sum_{k_1, k_2} \sum_{I \times J \subseteq \Omega} |D_{k_1} D_{k_2}(Tf)(x, y)|^2 \chi_I(x) \chi_J(y) \, dx \, dy \leq C\|f\|_{\text{CMO}^p(\widetilde{M})}^2.$$

To verify the above estimate, using the discrete Calderón reproducing identity (3.1) for  $f$ , we have

$$\begin{aligned} & \frac{1}{|\Omega|^{\frac{2}{p}-1}} \int_{\Omega} \sum_{k_1, k_2} \sum_{I \times J \subseteq \Omega} |D_{k_1} D_{k_2}(Tf)(x, y)|^2 \chi_I(x) \chi_J(y) \, dx \, dy \\ &= \frac{1}{|\Omega|^{\frac{2}{p}-1}} \int_{\Omega} \sum_{k_1, k_2} \sum_{I \times J \subseteq \Omega} \\ & \quad \times \left| D_{k_1} D_{k_2} T \left( \sum_{k'_1, k'_2} \sum_{I', J'} |I'| |J'| |D_{k'_1}(\cdot, x_{I'}) D_{k'_2}(\cdot, y_{J'}) \right. \right. \\ & \quad \quad \left. \left. \times \widetilde{D}_{k'_1} \widetilde{D}_{k'_2}(f)(x_{I'}, y_{J'}) \right) (x, y) \right|^2 \chi_I(x) \chi_J(y) \, dx \, dy \\ & \leq \frac{1}{|\Omega|^{\frac{2}{p}-1}} \int_{\Omega} \sum_{k_1, k_2} \sum_{I \times J \subseteq \Omega} \left| \sum_{k'_1, k'_2} \sum_{I', J'} |I'| |J'| \right. \\ & \quad \left. \times (D_{k_1} D_{k_2} T D_{k'_1} D_{k'_2})(x, x_{I'}, y, y_{J'}) \widetilde{D}_{k'_1} \widetilde{D}_{k'_2}(f)(x_{I'}, y_{J'}) \right|^2 \\ & \quad \times \chi_I(x) \chi_J(y) \, dx \, dy. \end{aligned}$$

Next we recall an almost-orthogonality argument related to singular integral operators, which appears as Proposition 4.2 in [HLL2].

**Proposition 3.3** ([HLL2]). *For each singular integral  $T$  satisfying (II-1)' through (II-6)', we have*

$$(3.8) \quad \begin{aligned} & |D_{j_1} D_{j_2} T D_{k_1} D_{k_2}(x_1, y_1, x_2, y_2)| \\ & \lesssim 2^{-|k_1-j_1|\varepsilon'_1} \frac{1}{V(x, y) + V_{2^{-(k_1 \wedge j_1)}}(x_1) + V_{2^{-(k_1 \wedge j_1)}}(y_1)} \left( \frac{2^{-(k_1 \wedge j_1)}}{2^{-(k_1 \wedge j_1)} + d(x_1, y_1)} \right)^{\varepsilon'_1} \\ & \quad \times 2^{-|k_2-j_2|\varepsilon'_2} \frac{1}{V(x, y) + V_{2^{-(k_2 \wedge j_2)}}(x_2) + V_{2^{-(k_2 \wedge j_2)}}(y_2)} \left( \frac{2^{-(k_2 \wedge j_2)}}{2^{-(k_2 \wedge j_2)} + d(x_2, y_2)} \right)^{\varepsilon'_2}, \end{aligned}$$

where the  $D_{k_i}$  are the same as in equation (2.6) and  $\varepsilon'_1, \varepsilon'_2 < \varepsilon$ .

From Proposition 3.3, we see that  $D_{k_1} D_{k_2} T D_{k'_1} D_{k'_2}(x, x', y, y')$  satisfies the same almost-orthogonality estimate as  $D_{k_1} D_{k_2} D_{k'_1} D_{k'_2}(x, x', y, y')$ ; see Lemma 3.2. Hence we have the same estimate as in the term (3.5). Therefore, applying the same proof

as that of Lemma 3.1, we obtain that inequality (3.7) holds, which implies that inequality (3.6) holds.

Finally, from the definition of  $T$  on  $\text{CMO}^p(\widetilde{M})$ , Lemma 3.1 and (3.6), we have for each  $g \in L^2(\widetilde{M}) \cap H^p(\widetilde{M})$ ,

$$\begin{aligned} |\langle T(f), g \rangle| &= \lim_{n \rightarrow \infty} |\langle T(f_n), g \rangle| \leq C \limsup_{n \rightarrow \infty} \|T(f_n)\|_{\text{CMO}^p(\widetilde{M})} \|g\|_{H^p(\widetilde{M})} \\ &\leq \limsup_{n \rightarrow \infty} C \|f_n\|_{\text{CMO}^p(\widetilde{M})} \|g\|_{H^p(\widetilde{M})} \\ &\leq C \|f\|_{\text{CMO}^p(\widetilde{M})} \|g\|_{H^p(\widetilde{M})}, \end{aligned}$$

which implies that  $\|T(f)\|_{\text{CMO}^p(\widetilde{M})} \leq C \|f\|_{\text{CMO}^p(\widetilde{M})}$ . The proof of Theorem 1.2 is complete.  $\square$

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