PROCEEDINGS OF THE
AMERICAN MATHEMATICAL SOCIETY
Volume 141, Number 8, August 2013, Pages 2699–2702
S 0002-9939(2013)11627-8
Article electronically published on April 3, 2013

LENGTH INEQUALITIES FOR RIEMANN SURFACES

A. F. BEARDON

(Communicated by Mario Bonk)

Abstract. We establish inequalities between the lengths of certain closed loops in the triply punctured sphere and in the twice-punctured disc.

1. Introduction

Let $\mathbb{H}$ be the hyperbolic plane $\{x + iy : y > 0\}$ with the hyperbolic metric $|dz|/y$ and hyperbolic distance ρ. The Uniformisation Theorem states that each hyperbolic Riemann surface $\mathcal{R}$ is conformally equivalent to a quotient space $\mathbb{H}/G$, where G is a discrete group of Möbius transformations (that is, a Fuchsian group) that acts on $\mathbb{H}$. The group G is isomorphic to the fundamental group of $\mathcal{R}$, no element of G (except the identity map) has a fixed point in $\mathbb{H}$, and the hyperbolic metric in $\mathbb{H}$ projects to give a unique metric (the hyperbolic metric) of curvature -1 on $\mathcal{R}$.

Let $\mathcal{R}$ be a hyperbolic Riemann surface which we write as $\mathbb{H}/G$. Let ζ be a point on $\mathcal{R}$, and let α be a closed loop on $\mathcal{R}$ that starts and ends at ζ. Then α lifts to a curve in $\mathbb{H}$ with endpoints, say, w and $g(w)$, where $g \in G$, and the hyperbolic length $\ell(\alpha)$ of α satisfies $\ell(\alpha) \geqslant \rho\big(w, g(w)\big)$, with equality when α is the projection of the geodesic segment from w to $g(w)$. Moreover, α is homotopically trivial on $\mathcal{R}$ if and only if g is the identity map in G. If β is another loop on $\mathcal{R}$, also starting and ending at ζ, then, in a similar way, $\ell(\beta) \geqslant \rho\big(w, h(w)\big)$ for some h in G. The author has proved [2, p. 198] that if g and h generate a non-cyclic group $\langle g, h\rangle$, then, for all z in $\mathbb{H}$,

$$\sinh \tfrac{1}{2}\rho\big(z, g(z)\big) \sinh \tfrac{1}{2}\rho\big(z, h(z)\big) \geqslant 1.$$

This is valid for every hyperbolic $\mathcal{R}$ and every z in $\mathbb{H}$, and it is stronger than what is customarily known as the Collar Lemma. It shows, for example, that providing $\langle g, h\rangle$ is not cyclic,

$$\sinh(\tfrac{1}{2}\ell(\alpha)) \sinh(\tfrac{1}{2}\ell(\beta)) \geqslant 1. \tag{1}$$

In this paper we establish stronger inequalities of a similar nature when $\mathcal{R}$ is a triply punctured sphere and a twice-punctured disc.

First, we consider $\mathcal{R}$ to be topologically a sphere punctured at the three points p_1, p_2 and p_3. Now suppose that $\zeta \in \mathcal{R}$ and, for $i = 1, 2, 3$, let γ_i be a simple closed curve that starts and ends at ζ and that separates p_i from the other two punctures

Received by the editors October 26, 2011.
2010 *Mathematics Subject Classification.* Primary 30F45; Secondary 30F35, 20H05, 20H10.
Key words and phrases. Riemann surfaces, hyperbolic metric, punctures.

©2013 American Mathematical Society
Reverts to public domain 28 years from publication

p_j and p_k. Now (1), applied to each pair γ_s and γ_t, yields

$$\sinh(\tfrac{1}{2}\ell_1)\sinh(\tfrac{1}{2}\ell_2)\sinh(\tfrac{1}{2}\ell_3) \geqslant 1,$$

where ℓ_s is the length of γ_s. Our first result strengthens this inequality.

Theorem 1.1. *Let $\mathcal{R}$ be a sphere punctured at the three points p_1, p_2 and p_3. Let ζ be any point of $\mathcal{R}$ and, for $j = 1, 2, 3$, let ℓ_j be the length of the shortest closed curve that starts and ends at ζ and that separates p_j from the other two punctures. Then*

$$\sinh(\tfrac{1}{2}\ell_1)\sinh(\tfrac{1}{2}\ell_2)\sinh(\tfrac{1}{2}\ell_3) \geqslant \left(2/\sqrt{3}\right)^3 = 1.5396\cdots. \tag{2}$$

This bound is attained if and only if ζ is at the common intersection of the three geodesics σ_j, where σ_j leaves p_j and is orthogonal to the geodesic joining the other two punctures.

Next, we assume that $\mathcal{R}$ is a twice-punctured disc. Here we may assume that $\mathcal{R} = \mathbb{D}\backslash\{p_1, p_2\}$, where $\mathbb{D}$ is the open unit disc in $\mathbb{C}$ and p_1 and p_2 are in $\mathbb{D}$. Select any point ζ in $\mathcal{R}$, and let ℓ_j, $j = 1, 2$, be the length of the shortest closed curve in $\mathcal{R}$ that starts and ends at ζ, and that separates the puncture p_j from the other puncture, and from $\partial\mathbb{D}$. Also, let ℓ be the length of the shortest closed geodesic that separates both punctures from $\partial\mathbb{D}$.

Theorem 1.2. *For the twice-punctured disc described above,*

$$\sinh(\tfrac{1}{2}\ell_1)\sinh(\tfrac{1}{2}\ell_2) \geqslant \cosh^2(\tfrac{1}{4}\ell)\cosh\rho(\zeta,\sigma), \tag{3}$$

where σ is the geodesic joining the two punctures.

2. Some hyperbolic geometry

The conformal isometries of $\mathbb{H}$ are the maps $f(z) = (az+b)/(cz+d)$, where $a, b, c, d \in \mathbb{R}$ and $ad - bc = 1$. Any such f may be represented by the pair of matrices

$$\pm\begin{pmatrix} a & b \\ c & d \end{pmatrix}$$

in $\mathrm{SL}(2,\mathbb{R})$, and $tr^2(f)$ is uniquely defined as $(a+d)^2$. Next, the *displacement function* for f is $\rho\big(z, f(z)\big)$, and this is given by

$$\sinh\tfrac{1}{2}\rho\big(z, f(z)\big) = \frac{|z - f(z)|}{2\sqrt{\mathrm{Im}[z]}\,\sqrt{\mathrm{Im}[f(z)]}}, \tag{4}$$

where $z = x + iy$ (see [2, p. 130]).

Now suppose that f has two distinct fixed points on $\mathbb{R}\cup\{\infty\}$. Then f is conjugate to a map $F(z) = kz$, where $k > 1$. Then it is easy to see (and well-known) that the displacement function of F attains its minimum value T_F on the positive imaginary axis and that T_F, which is the *translation length* of F, is given by $tr^2(F) = 4\cosh^2(\frac{1}{2}T_F)$. Since all terms here are invariant under conjugation, we see that

$$tr^2(f) = 4\cosh^2(\tfrac{1}{2}T_f). \tag{5}$$

For more details on these and related facts, see [2].

3. The proof of Theorem 1.1

As any triply punctured sphere is conformally equivalent to $\mathbb{C}_\infty\backslash\{0,1,\infty\}$, we may assume that $\mathcal{R} = \mathbb{C}_\infty\backslash\{0,1,\infty\}$. Then (see [1, pp. 277–282]) $\mathcal{R} = \mathbb{H}/\Gamma(2)$, where $\Gamma(2)$, which is a congruence subgroup of the Modular group, is generated by $z \mapsto z+2$ and $z \mapsto z/(2z+1)$. Further, the hyperbolic quadrilateral in $\mathbb{H}$ with vertices -1, 0, 1 and ∞ is a fundamental polygon for G. The quotient map $\lambda : \mathbb{H} \to \mathcal{R}$ is the elliptic modular function, and, using the construction of λ given in [1], we may take (with the appropriate interpretation) $p_1 = 0 = \lambda(\infty)$, $p_2 = 1 = \lambda(0)$ and $p_3 = \infty = \lambda(1)$. Finally, the stabilizers of ∞, 0 and 1 in $\Gamma(2)$ are generated by g_1, g_2 and g_3, respectively, where

$$g_1(z) = z+2, \quad g_2(z) = \frac{z}{2z+1}, \quad g_3(z) = g_1 g_2^{-1}(z) = \frac{3z-2}{2z-1}.$$

Now suppose that w in $\mathbb{H}$ projects to ζ in $\mathcal{R}$. Then, for $j = 1,2,3$, $\ell_j = \sigma\big(w, g_j(w)\big)$, so that (2) is equivalent to the assertion that

$$\inf_{z\in\mathbb{H}} \Big(\sinh \tfrac{1}{2}\rho(z, g_1 z) \sinh \tfrac{1}{2}\rho(z, g_2 z) \sinh \tfrac{1}{2}\rho(z, g_3 z)\Big) = \big(2/\sqrt{3}\big)^3. \tag{6}$$

We shall now establish (6). First, (4) shows that

$$\sinh \tfrac{1}{2}\rho\big(z, g_1(z)\big) \sinh \tfrac{1}{2}\rho\big(z, g_2(z)\big) \sinh \tfrac{1}{2}\rho\big(z, g_3(z)\big) = \frac{1}{y}.\frac{|z|^2}{y}.\frac{|z-1|^2}{y},$$

so that (6) is equivalent to

$$\inf_{z\in\mathbb{H}} \frac{|z(z-1)|^2}{y^3} = \left(\frac{2}{\sqrt{3}}\right)^3.$$

If we change the variable to $u + iv = z - \frac{1}{2}$ and write

$$\Phi(u,v) = \frac{(u^2+v^2-\frac{1}{4})^2 + v^2}{v^3},$$

we see that (6) is equivalent to

$$\Phi(u,v) \geqslant \Phi(0, \tfrac{1}{2}\sqrt{3}) = \left(\frac{2}{\sqrt{3}}\right)^3. \tag{7}$$

We now establish the inequality in (7) (the equality is true). If $v \leqslant \frac{1}{2}$, then $\Phi(u,v) \geqslant v^2/v^3 \geqslant 2 > (2/\sqrt{3})^3$. If not, then $v > \frac{1}{2}$ so that

$$\Phi(u,v) \geqslant \Phi(0,v) = \frac{(v^2-\frac{1}{4})^2 + v^2}{v^3} = \frac{(v^2+\frac{1}{4})^2}{v^3} = \varphi(v),$$

say, with equality if and only if $u = 0$. By considering $\varphi'(v)$ we see that $\varphi(v)$ attains its minimum when $4v^2 = 3$, and (6) is proved. This argument shows that equality occurs in (2) if and only if $u = 0$ and $v = \sqrt{3}/2$ or, equivalently, when $z = (1+i\sqrt{3})/2$, which is at the intersection of the altitudes of the hyperbolic triangle in $\mathbb{H}$ with vertices 0, 1 and ∞.

4. The proof of Theorem 1.2

We may assume that $\mathcal{R} = \mathbb{D}\backslash\{a, -a\}$ for some a in $(0, 1)$, and we may take G to be generated by

$$g(z) = z + t, \quad h(z) = \frac{z}{z+1},$$

where $t > 4$. The region Σ bounded by the two lines $x = \pm t/2$ and the two circles $|z \pm 1| = 1$ is a fundamental region for G and the sides of Σ are paired by g and h. An examination of the geometry of the actions of g and h shows that ℓ is the translation length of $h^{-1}g$, and if we compute $h^{-1}g$ and use (5) we find that $t = 4\cosh^2(\ell/4)$.

Now choose any z in $\mathbb{H}$ with $z = x + iy$. Then, from (4) and [2, p. 162],

$$\begin{aligned} \sinh \tfrac{1}{2}\rho\big(z, g(z)\big) \sinh \tfrac{1}{2}\rho\big(z, h(z)\big) &= \left(\frac{t}{2y}\right)\left(\frac{|z|^2}{2y}\right) \\ &= \tfrac{1}{4}t \cosh \rho(z, \mathcal{I}), \end{aligned}$$

where $\mathcal{I}$ is the positive imaginary axis, and this gives (3).

References

[1] Ahlfors, L. V., *Complex Analysis*, McGraw-Hill (third edition), 1978. MR510197 (80c:30001)

[2] Beardon, A. F., *The Geometry of Discrete Groups*, Graduate Texts in Mathematics, No. 91, Springer-Verlag, 1983. MR698777 (85d:22026)

CMS, University of Cambridge, Wilberforce Road, Cambridge CB3 OWB, United Kingdom