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A NOTE ON SHEAVES WITHOUT SELF-EXTENSIONS ON THE PROJECTIVE n-SPACE

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Abstract. Let $\mathbf{P}^n$ be the projective n-space over the complex numbers. In this note we show that an indecomposable rigid coherent sheaf on $\mathbf{P}^n$ has a trivial endomorphism algebra. This generalizes a result of Drézet for $n = 2$.

Let $\mathbf{P}^n$ be the projective n-space over the complex numbers. Recall that an *exceptional sheaf* is a coherent sheaf E such that $\operatorname{Ext}^i(E, E) = 0$ for all $i > 0$ and $\operatorname{End} E \cong \mathbb{C}$. Drézet proved in [4] that if E is an indecomposable sheaf over $\mathbf{P}^2$ such that $\operatorname{Ext}^1(E, E) = 0$ then its endomorphism ring is trivial, and also that $\operatorname{Ext}^2(E, E) = 0$. Moreover, the sheaf is locally free. Partly motivated by this result, we prove in this short note that if E is an indecomposable coherent sheaf over the projective n-space such that $\operatorname{Ext}^1(E, E) = 0$, then we automatically get that $\operatorname{End} E \cong \mathbb{C}$. The proof involves reducing the problem to indecomposable linear modules without self-extensions over the polynomial algebra. In the second part of this article, we look at the Auslander-Reiten quiver of the derived category of coherent sheaves over the projective n-space. It is known ([11]) that if $n > 1$, then all the components are of type $\mathbb{Z}A_\infty$. Then, using the Bernstein-Gelfand-Gelfand correspondence ([3]) we prove that each connected component contains at most one sheaf. We also show that in this case the sheaf lies on the boundary of the component.

Throughout this article, S will denote the polynomial ring in $n + 1$ variables with coefficients in $\mathbb{C}$. Its Koszul dual is the exterior algebra in $n + 1$ variables $x_0, x_1, \ldots, x_n$ which we denote by R. It is well known that R is a graded, local, finite dimensional algebra over $\mathbb{C}$. Moreover, R is a self-injective algebra; that is, the notions of free and of injective R-modules coincide. Denote by $\operatorname{Lin} R$ and $\operatorname{Lin} S$ the categories of linear R-modules (linear S-modules respectively), that is, of the modules having a linear free resolution. It is well known that both $\operatorname{Lin} R$ and $\operatorname{Lin} S$ are closed under extensions and cokernels of monomorphisms. We have mutual inverse Koszul dualities:

$$\begin{tikzcd} \operatorname{Lin} R \arrow[r, shift left, "\mathcal{E}"] & \operatorname{Lin} S \arrow[l, shift left, "\mathcal{F}"] \end{tikzcd}$$

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given by $\mathcal{E}(M) = \bigoplus_{n\geq 0} \operatorname{Ext}^n_R(M, \mathbb{C})$ with the obvious action of S on $\mathcal{E}(M)$ and with the inverse duality $\mathcal{F}$ defined in a similar way.

Let $\operatorname{mod}^Z R$ be the category of finitely generated graded R-modules and graded (degree 0) homomorphisms, where we denote by $\operatorname{Hom}_R(M,N)_0$ the degree zero homomorphisms from M to N. Denote by $\underline{\operatorname{mod}}^Z R$ the stable category of finitely generated graded R-modules. The objects of $\underline{\operatorname{mod}}^Z R$ are the finitely generated graded R-modules, and the morphisms are given by

$$\underline{\operatorname{Hom}}_R(M,N)_0 = \operatorname{Hom}_R(M,N)_0/\mathcal{P}(M,N)_0$$

where $\mathcal{P}(M,N)_0$ is the space of graded homomorphisms from M to N factoring through a free R-module. The stable graded category $\underline{\operatorname{mod}}^Z R$ is a triangulated category where the shift functor is given by the first cosyzygy; that is, every distinguished triangle in the stable category is of the form $A \to B \to C \to \Omega^{-1}A$ where $\Omega\colon \underline{\operatorname{mod}}^Z R \to \underline{\operatorname{mod}}^Z R$ denotes the syzygy functor. Note also that the stable category $\underline{\operatorname{mod}}^Z R$ has a Serre duality:

$$D\underline{\operatorname{Hom}}_R(X, \Omega^{-m}Y)_0 \cong \underline{\operatorname{Hom}}_R(Y, \Omega^{m+1}X(n+1))_0$$

for each integer m and all R-modules X and Y, where $X(n+1)$ is the graded shift of X. It is immediate to see that we can derive the well-known Auslander-Reiten formula from the above in the case when $m = 1$.

Let $M = \bigoplus M_i$ be a finitely generated graded module over the exterior algebra R and let ξ be a homogeneous element of degree 1 in R. Following [3], we can view the left multiplication by ξ on M as inducing a complex of K-vector spaces

$$L_\xi M\colon \quad \cdots M_{i-1} \xrightarrow{\cdot\xi} M_i \xrightarrow{\cdot\xi} M_{i+1} \to \cdots .$$

We call the graded module M *nice* or *proper* if the homology $H_i(L_\xi M) = 0$ for each $i \neq 0$ and for each homogeneous element $\xi \neq 0$ of degree 1. It is well known that a graded module M is free if and only if $H_i(L_\xi M) = 0$ for each $i \in \mathbb{Z}$ and for each nonzero homogeneous element $\xi \in R_1$.

The derived category $\mathcal{D}^b(\operatorname{coh}\mathbf{P}^n)$ of coherent sheaves over the projective space is also, triangulated, and the BGG correspondence (see [3])

$$\Phi\colon \underline{\operatorname{mod}}^Z R \to \mathcal{D}^b(\operatorname{coh}\mathbf{P}^n)$$

is an exact equivalence of triangulated categories that assigns to each finitely generated graded R-module the equivalence class of a bounded complex of vector bundles on $\mathbf{P}^n$. If $M = \bigoplus M_i$, we use also M_i to denote the trivial bundle with finite dimensional fiber M_i and we set

$$\Phi(M)\colon \quad \ldots \longrightarrow \mathcal{O}(i)\otimes M_i \xrightarrow{\delta^i} \mathcal{O}(i+1)\otimes M_{i+1} \longrightarrow \ldots$$

where the differentials δ^i are defined in the following way. Let $x_0, x_1, \ldots, x_n$ be a basis of V, and let $\xi_0, \xi_1, \ldots, \xi_n$ denote the dual basis of V^*. Then $\delta^i = \sum_{j=0}^n \xi_j \otimes x_j$. Moreover for each nice R-module M, the complex $\Phi(M)$ is quasi-isomorphic to the stalk complex of a vector bundle over the projective n-space, and all the vector bundles can be obtained in this way from the nice R-modules.

Throughout this article if M is a graded module, then its graded shift, denoted by $M(i)$, is the graded module given by $M(i)_n = M_{i+n}$ for all integers n. If M is a graded module, we denote its truncation at the k-th level by $M_{\geq k} = M_k \oplus M_{k+1} \oplus \cdots$.

1. The endomorphism ring of a rigid sheaf

Let F be an indecomposable coherent sheaf over the projective n-space. Assume also that F is *rigid*, that is, $\mathrm{Ext}^1(F,F) = 0$. We prove in this section that the endomorphism ring of F is trivial. This is done by reducing the problem to indecomposable rigid linear modules over the polynomial algebra S in $n+1$ variables.

Let M be a finitely generated, graded, indecomposable nonprojective S-module. We say that M has no graded self-extensions if every short exact sequence of graded S-modules $0 \to M \to X \to M \to 0$ splits, that is, $\mathrm{Ext}^1_S(M,M)_0 = 0$.

In order to prove our next result we recall the following definition and facts from [6]. We say that a module $M \in \mathrm{mod}^Z S$ has a *linear presentation* if it has a graded free presentation $P_1 \to P_0 \to M \to 0$ where P_0 is generated in degree 0 and P_1 is generated in degree 1. Obviously such a linear presentation is always minimal. A stronger notion is that of a *linear* module. We say that a graded module M is linear if it has a minimal graded free resolution:

$$\cdots P_k \to \cdots P_1 \to P_0 \to M \to 0$$

where for each $i \geq 0$, the free module P_i is generated in degree i. If we denote by $\mathcal{L}_S$ the subcategory of $\mathrm{mod}^Z S$ consisting of the modules with a linear presentation, we then have an exact equivalence of categories $\mathcal{L}_S \cong \mathcal{L}_{S/J^2}$ where J denotes the maximal homogeneous ideal of S ([6]). This equivalence is given by assigning to each module M having a linear presentation over S the module M/J^2M. We start with the following background result:

Lemma 1.1. *An exact sequence* $0 \to M \to E \to N \to 0$ *in* $\mathcal{L}_S$ *splits if and only if the induced sequence* $0 \to M/J^2M \to E/J^2E \to N/J^2N \to 0$ *splits too.*

Proof. To prove this, it is enough to show that a monomorphism $M \to E$ in $\mathcal{L}_S$ splits if and only if the induced map $M/J^2M \to E/J^2E$ is also a splittable monomorphism and one direction is trivial. For the other direction, recall that the category $\mathcal{L}_S$ is equivalent to the category $\mathcal{C}$ whose objects are graded morphisms $f\colon P_1 \to P_0$ between free S-modules, where P_0 is generated in degree zero, P_1 is generated in degree one, and the degree one component of f is a monomorphism from the degree one component of P_1 to the degree one component of P_0. A morphism in $\mathcal{C}$ from $f\colon P_1 \to P_0$ to $g\colon Q_1 \to Q_0$ is a pair (u_1, u_0) of graded homomorphisms such that the diagram

$$\begin{tikzcd} P_1 \arrow[r, "f"] \arrow[d, "u_1"] & P_0 \arrow[d, "u_0"] \\ Q_1 \arrow[r, "g"] & Q_0 \end{tikzcd}$$

is commutative. Let M and E be two modules with linear presentations $P_1 \xrightarrow{f} P_0$ and $Q_1 \xrightarrow{g} Q_0$, and let $h\colon M \to E$ be a homomorphismm such that the induced map $\tilde{h}\colon\; M/J^2M \to E/J^2E$ is a splittable monomorphism. We have an exact commutative diagram:

$$\begin{tikzcd}
P_1/J^2P_1 \arrow[r, "\tilde{f}"] \arrow[d, "\tilde{h}_1"] & P_0/J^2P_0 \arrow[r, "\tilde{\pi}_M"] \arrow[d, "\tilde{h}_0"] & M/J^2M \arrow[r] \arrow[d, "\tilde{h}"] & 0 \\
Q_1/J^2Q_1 \arrow[r, "\tilde{g}"] \arrow[d, dotted, "\tilde{q}_1"] & Q_0/J^2Q_0 \arrow[r, "\tilde{\pi}_E"] \arrow[d, dotted, "\tilde{q}_0"] & E/J^2E \arrow[r] \arrow[d, "\tilde{q}"] & 0 \\
P_1/J^2P_1 \arrow[r, "\tilde{f}"] & P_0/J^2P_0 \arrow[r, "\tilde{\pi}_M"] & M/J^2M \arrow[r] & 0
\end{tikzcd}$$

where $\tilde{q}\tilde{h} = 1_{M/J^2M}$. Observe that the liftings $\tilde{q}_1$ and $\tilde{q}_0$ also have the property that $\tilde{q}_1\tilde{h}_1 = 1_{P_1/J^2P_1}$ and $\tilde{q}_0\tilde{h}_0 = 1_{P_0/J^2P_0}$. By lifting to $\mathcal{L}_S$ we obtain the following exact commutative diagram:

$$\begin{tikzcd}
P_1 \arrow[r, "f"] \arrow[d, "h_1"] & P_0 \arrow[r, "\pi_M"] \arrow[d, "h_0"] & M \arrow[r] \arrow[d, "h"] & 0 \\
Q_1 \arrow[r, "g"] \arrow[d, "q_1"] & Q_0 \arrow[r, "\pi_E"] \arrow[d, "q_0"] & E \arrow[r] & 0 \\
P_1 \arrow[r, "f"] & P_0 \arrow[r, "\pi_M"] & M \arrow[r] & 0
\end{tikzcd}$$

and we have that $q_1h_1 = 1_{P_1}$ and $q_0h_0 = 1_{P_0}$. There exists a unique homomorphism $l\colon E \to M$ such that $l\pi_E = \pi_M q_0$, and it follows immediately that $lh = 1_M$. □

We will also need the following results about graded modules over the polynomial algebra $S = \mathbb{C}[x_0, \dots x_n]$:

Lemma 1.2. *Let $0 \to A \overset{f}{\to} B \overset{g}{\to} C \to 0$ be an exact sequence in $\operatorname{mod}^{\mathbb{Z}} S$ where A is a linear module and C is semisimple generated in a single degree $m \geq 0$. Then the sequence splits.*

Proof. It suffices to show that the sequence $0 \to A_{\geq m} \overset{f_{\geq m}}{\longrightarrow} B_{\geq m} \overset{g_{\geq m}}{\longrightarrow} C \to 0$ splits in the category of graded S-modules since we can easily construct a right inverse to g from a right inverse to $g_{\geq m}$. Each of the modules $A_{\geq m}, B_{\geq m}, C$ is an m-shift of a linear S-module, so by Koszul duality we obtain an exact sequence over the exterior algebra R:

$$0 \to \mathcal{E}(C) \to \mathcal{E}(B_{\geq m}) \to \mathcal{E}(A_{\geq m}) \to 0.$$

But R is self-injective and $\mathcal{E}(C)$ is free, so this sequence splits. The lemma now follows immediately. □

Lemma 1.3. *Let C be a linear S-module. Let i be a positive integer and assume that $\operatorname{Ext}^1_S(C_{\geq i}, C)_0 = 0$. Then $\operatorname{Ext}^1_S(C_{\geq i}, C_{\geq i})_0 = 0$.*

Proof. Assume to the contrary that $\mathrm{Ext}^1_S(C_{\geq i}, C_{\geq i})_0 \neq 0$. We have the following pushout diagram of graded modules:

$$\begin{tikzcd}
0 \arrow[r] & C_{\geq i} \arrow[r, "\alpha"] \arrow[d, "j"] & X \arrow[r, "\beta"] \arrow[d, "h"] & C_{\geq i} \arrow[r] \arrow[d, equal] & 0 \\
0 \arrow[r] & C \arrow[r, "\gamma"] & Y \arrow[r, "\delta"] & C_{\geq i} \arrow[r] & 0
\end{tikzcd}$$

where j is the inclusion map. The bottom sequence splits by assumption so we have homomorphisms $q\colon C_{\geq i} \to Y$ and $p\colon Y \to C$ such that $\delta q = 1$ and $p\gamma = 1$. But then each graded component of γ, δ splits so we obtain an induced split exact sequence $0 \to C_{\geq i} \to Y_{\geq i} \to C_{\geq i} \to 0$ and an induced commutative diagram:

$$\begin{tikzcd}
0 \arrow[r] & C_{\geq i} \arrow[r, "\alpha"] \arrow[d, equal] & X \arrow[r, "\beta"] \arrow[d, "h"] & C_{\geq i} \arrow[r] \arrow[d, equal] & 0 \\
0 \arrow[r] & C_{\geq i} \arrow[r, "\gamma_{\geq i}"] & Y_{\geq i} \arrow[r, "\delta_{\geq i}"] & C_{\geq i} \arrow[r] & 0
\end{tikzcd}$$

and since the two extensions represent the same element in Ext^1 they both split. $\square$

The following result holds for an arbitrary Koszul algebra over the complex numbers.

Lemma 1.4. *Let Γ be a Koszul algebra and let $M \in \mathcal{L}_\Gamma$ be an indecomposable nonprojective Γ-module of Loewy length two having no graded self-extensions. Then $\mathrm{End}_\Gamma(M)_0 \cong \mathbb{C}$.*

Proof. The proof follows the same idea as the one given in [8], but we present it here for the reader's convenience. Assume that there exists a nonzero graded homomorphism $f\colon M \to M$ that is not an isomorphism, and let N denote its image and L its kernel. Furthermore, let T denote the cokernel of the inclusion map $N \to M$. Note that T is also generated in degree zero and that L lives in degree 1 and possibly also in degree zero. Applying $\mathrm{Hom}_\Gamma(T, -)$ to the exact sequence $0 \to L \to M \to N \to 0$ and taking only the degree zero parts, we obtain an exact sequence

$$\to \mathrm{Ext}^1_\Gamma(T, M)_0 \to \mathrm{Ext}^1_\Gamma(T, N)_0 \to \mathrm{Ext}^2_\Gamma(T, L)_0.$$

But if

$$\cdots \to P_2 \to P_1 \to P_0 \to T \to 0$$

is the beginning of a graded minimal projective resolution of T, the second term P_2 is generated in degrees 2 or higher; hence $\mathrm{Hom}_\Gamma(P_2, L)_0 = 0$ and therefore $\mathrm{Ext}^2_\Gamma(T, L)_0$ vanishes too. We obtain a surjection $\mathrm{Ext}^1_\Gamma(T, M)_0 \to \mathrm{Ext}^1_\Gamma(T, N)_0$ and hence a commutative exact diagram of graded Γ-modules of Loewy length 2 and degree zero maps:

$$\begin{tikzcd}
0 \arrow[r] & M \arrow[r] \arrow[d, "f"] & X \arrow[r] \arrow[d] & T \arrow[r] \arrow[d, equal] & 0 \\
0 \arrow[r] & N \arrow[r] & M \arrow[r] & T \arrow[r] & 0
\end{tikzcd}$$

This pushout yields an exact sequence of graded Γ-modules:

$$0 \to M \to N \oplus X \to M \to 0.$$

Our assumption on M implies that this sequence must split. The Krull-Remak-Schmidt theorem yields a contradiction, since $N \neq 0$, M is indecomposable and $\ell(N) < \ell(M)$. ☐

We can proceed now with the proof of our main result:

Theorem 1.5. *Let F be an indecomposable coherent sheaf over the projective space $\mathbf{P}^n$ such that* $\operatorname{Ext}^1(F,F) = 0$. *Then,* $\operatorname{End} F \cong \mathbb{C}$.

Proof. By Serre's theorem, we may identify the category of coherent sheaves over $\mathbf{P}^n$ with the quotient category $Q \operatorname{mod}^Z S$ of $\operatorname{mod}^Z S$. Recall that the objects of $Q \operatorname{mod}^Z S$ are the objects of $\operatorname{mod}^Z S$ modulo the graded modules of finite dimension. Let $\widetilde{X}$ denote the sheafification of X, that is, the image of X in the quotient category. If X and Y are two graded S-modules, then $\widetilde{X} \cong \widetilde{Y}$ in $Q \operatorname{mod}^Z S$ if and only if for some integer k, their truncations $X_{\geq k}$ and $Y_{\geq k}$ are isomorphic as graded S-modules. Now, let F be an indecomposable sheaf with $\operatorname{Ext}^1(F,F) = 0$. We may assume that $F = \widetilde{X}$ where X is a linear S-module up to some shift ([1]) and that X has no finite dimensional submodules. Then (see [10] for instance), $\operatorname{End}(F) \cong \varinjlim \operatorname{Hom}_S(X_{\geq k}, X)_0$ and $\operatorname{Ext}^1(F,F) \cong \varinjlim \operatorname{Ext}^1_S(X_{\geq k}, X)_0$. For each k, we have an exact sequence of graded S-modules

$$0 \to X_{\geq k+1} \to X_{\geq k} \to S_k \to 0$$

where S_k denotes the semisimple module $X_{\geq k}/X_{\geq k+1}$ concentrated in degree k. Applying $\operatorname{Hom}_S(-, X)$ and taking degrees, we obtain a long exact sequence

$$0 \to \operatorname{Hom}_S(X_{\geq k}, X)_0 \to \operatorname{Hom}_S(X_{\geq k+1}, X)_0 \to \operatorname{Ext}^1_S(S_k, X)_0$$

$$\to \operatorname{Ext}^1_S(X_{\geq k}, X)_0 \to \operatorname{Ext}^1_S(X_{\geq k+1}, X)_0$$

since $\operatorname{Hom}_S(S_k, X)_0 = 0$ by our assumption on X. $\operatorname{Ext}^1_S(S_k, X)_0$ also vanishes by Lemma 1.2, since having a nontrivial extension $0 \to X \to Y \to S_k \to 0$ yields a nonsplit exact sequence of degree k shifts of linear modules $0 \to X_{\geq k} \to Y_{\geq k} \to S_k \to 0$. Since for each k, $\operatorname{Hom}_S(X_{\geq k}, X)_0 = \operatorname{End}_S(X_{\geq k})$, we see that we have $\operatorname{End}(F) \cong \operatorname{End}_S(X)_0 \cong \operatorname{End}_S(X_{\geq k})_0$ for each $k \geq 0$. At the same time we have a sequence of embeddings

$$\operatorname{Ext}^1_S(X, X)_0 \hookrightarrow \cdots \hookrightarrow \operatorname{Ext}^1_S(X_{\geq k}, X)_0 \hookrightarrow \operatorname{Ext}^1_S(X_{\geq k+1}, X)_0 \hookrightarrow \cdots .$$

Thus, the assumption that $\operatorname{Ext}^1(F,F) = 0$ implies that for each k we must have $\operatorname{Ext}^1_S(X_{\geq k}, X)_0 = 0$, in particular, $\operatorname{Ext}^1_S(X, X)_0 = 0$. X is a shift of a linear module so we may apply Lemma 1.1. We now apply Lemma 1.4 to conclude that the sheaf F has trivial endomorphism ring. ☐

2. Auslander-Reiten components containing sheaves

We start with the following general observation:

Lemma 2.1. *Let* $A \xrightarrow{\left[\begin{smallmatrix} u_1 \\ u_2 \end{smallmatrix}\right]} B_1 \oplus B_2 \xrightarrow{[\,v_1\ v_2\,]} C \to A[1]$ *be a triangle in a Krull-Schmidt triangulated category $\mathcal{T}$, and assume that C is indecomposable and that both maps v_1, v_2 are nonzero. Then* $\operatorname{Hom}_{\mathcal{T}}(A, C) \neq 0$.

Proof. We prove that $v_1u_1 \neq 0$. Assume that $v_1u_1 = 0$. Then, since the composition $\begin{bmatrix} v_1 & 0 \end{bmatrix}\begin{bmatrix} u_1 \\ u_2 \end{bmatrix}: A \to C$ is equal to 0, we have an induced homomorphism $\psi\colon C \to C$ such that $\psi\begin{bmatrix} v_1 & v_2 \end{bmatrix} = \begin{bmatrix} v_1 & 0 \end{bmatrix}$. Thus $\psi v_1 = v_1$ and $\psi v_2 = 0$. Since C is indecomposable, ψ is either an isomorphism or it is nilpotent. It cannot be nilpotent since $v_1 \neq 0$, and it cannot be an isomorphism since $v_2 \neq 0$; hence a contradiction. $\square$

The BGG correspondence Φ is an exact equivalence of triangulated categories; hence for every graded R-module M, we have that $\Phi(\Omega M) = \Phi(M)[-1]$. An easy computation also shows that for each finitely generated graded module M and integer i, we have that $\Phi(M(i)) \otimes \mathcal{O}(i) = \Phi(M)[i]$. It turns out that the modules corresponding to sheaves are located on the boundary of the Auslander-Reiten component containing them. This and more follows immediately from the next result by playing with the BGG correspondence.

Proposition 2.2. *Let B and C be two R-modules corresponding to sheaves under the BGG correspondence. Then, for each $i > 0$ $\underline{\mathrm{Hom}}_R(\tau^i B, C)_0 = 0$.*

Proof. Since taking syzygies and graded shifts are self-equivalences of $\underline{\mathrm{mod}}^Z R$, we have by [11] that C is a weakly Koszul module, and so we can use the formula $\tau B = \Omega^2 B(n+1)$, and so $\tau^i B = \Omega^{2i} B(ni+i)$ for all $i \geq 1$. We then have

$$\Phi(\tau^i B) = \Phi(\Omega^{2i} B(ni+i)) = \Phi(\Omega^{2i} B)[ni+i] \otimes \mathcal{O}(-ni-i) = \Phi(B)[ni-i] \otimes \mathcal{O}(-ni-i).$$

Applying the BGG correspondence, we obtain

$$\begin{aligned}\underline{\mathrm{Hom}}_R(\tau^i B, C)_0 &\cong \mathrm{Hom}_{\mathcal{D}^b(\mathrm{coh}(\mathbf{P}^n))}(\Phi(\tau^i B), \Phi(C)) \\ &= \mathrm{Hom}_{\mathcal{D}^b(\mathrm{coh}(\mathbf{P}^n))}(\Phi(B)[ni-i] \otimes \mathcal{O}(-ni-i), \Phi(C)) \\ &\cong \mathrm{Hom}_{\mathcal{D}^b(\mathrm{coh}(\mathbf{P}^n))}(\Phi(B)(-ni-i)[ni-i], \Phi(C)) = 0\end{aligned}$$

where the last equality holds since both $\Phi(C)$ and $\Phi(B)(-ni-i)$ are sheaves, $n \geq 2$, so $ni - i \geq 1$. $\square$

We have the following immediate consequence by letting $B = C$ and $i = 1$ in the previous proposition and applying Lemma 2.1:

Corollary 2.3. *Let C be an indecomposable R-module corresponding to a sheaf under the BGG correspondence, where $n \geq 2$. Then the Auslander-Reiten sequence ending at C has indecomposable middle term.* $\square$

We end with the following observation:

Proposition 2.4. *Let $\mathcal{C}$ be a connected component of the Auslander-Reiten quiver of $\mathcal{D}^b(\mathrm{coh}\,\mathbf{P}^n)$ for $n > 1$. Then $\mathcal{C}$ contains at most one indecomposable coherent sheaf.*

Proof. By the previous results, if $\mathcal{C}$ contains a sheaf, then this sheaf must lie on the boundary. Under the BGG correspondence, two sheaves in $\mathcal{C}$ must correspond to two modules of the form C and $\tau^i C$ for some $i \geq 1$. But by applying the BGG correspondence we get stalk complexes concentrated in different degrees; hence we obtain a contradiction. $\square$

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