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LIFTING TO MAXIMAL RIGID OBJECTS IN 2-CALABI-YAU TRIANGULATED CATEGORIES

YUNLI XIE AND PIN LIU

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ABSTRACT. We show that a tilting module over the endomorphism algebra of a maximal rigid object in a 2-Calabi-Yau triangulated category lifts to a maximal rigid object in this 2-Calabi-Yau triangulated category. This strengthens recent work of Fu and Liu for cluster-tilting objects.

INTRODUCTION

Tilting theory in cluster categories and, more generally, Hom-finite Calabi-Yau triangulated categories have recently been widely investigated. The study of such categories was originally motivated by their links to cluster algebras, and indeed there has been a considerable amount of activity and a lot of results in this direction. We refer the reader to the recent survey [K].

However, the study of such categories has also contributed to new developments in the theory of finite-dimensional (and, more generally, non-commutative) algebras. The endomorphism algebras of cluster-tilting objects in the Calabi-Yau triangulated categories have been further developed, and it has been revealed that they have very nice properties (see [KR] for instance). Cluster-tilting objects are always maximal rigid objects, while the converse is not true in general. There exist 2-Calabi-Yau triangulated categories in which maximal rigid objects are not cluster tilting. The first examples of such categories were given by Burban-Iyama-Keller-Reiten in [BIKR]. Cluster tubes, introduced by Barot-Kussin-Lenzing in [BKL], are another family of 2-Calabi-Yau triangulated categories without cluster-tilting objects. In [BMV] Buan-Marsh-Vatne classified maximal rigid objects of cluster tubes, none of which is cluster tilting. The endomorphism algebras of maximal rigid objects in the 2-Calabi-Yau triangulated categories are studied in [V, Y, ZZ].

It is an interesting idea to investigate algebras derived equivalent to the endomorphism algebras of cluster-tilting objects and maximal rigid objects. Understanding their tilting modules is a step in this direction ([FL, HJ, S]). The aim of this note is to give an answer as to how to identify tilting modules over the endomorphism algebras of maximal rigid objects in 2-Calabi-Yau triangulated categories.

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The second author is the corresponding author.

We show that tilting modules over such algebras lift to maximal rigid objects in the corresponding 2-Calabi-Yau triangulated category. Namely, we prove the following theorem.

Theorem. *Let R be a maximal rigid object in the 2-Calabi-Yau triangulated category $\mathcal{C}$, Γ be the endomorphism algebra of R and T be a tilting module over Γ. Then T lifts to a maximal rigid object in $\mathcal{C}$.*

1. Preliminaries

In this section we review some useful notation and results.

1.1. **Tilting modules.** Let k be an algebraically closed field and A be a finite-dimensional algebra. Let $\operatorname{mod} A$ be the category of finite-dimensional right A-modules. For an A-module T, let $\operatorname{add} T$ denote the full subcategory of $\operatorname{mod} A$ with objects all direct summands of direct sums of copies of T. Then T is called a tilting module in $\operatorname{mod} A$ if

- $\operatorname{pd}_A T \leq 1$,
- $\operatorname{Ext}^1_A(T, T) = 0$,
- there is an exact sequence $0 \to A \to T^0 \to T^1 \to 0$, with T^0, T^1 in $\operatorname{add} T$.

This is the original definition of tilting modules from [HR], and it was proved in [B] that the third axiom can be replaced by the following:

- the number of indecomposable direct summands of T (up to isomorphism) is the same as the number of simple A-modules.

1.2. **Approximation.** Let $\mathcal{A}$ be an additive category and $\mathcal{B}$ be an additive subcategory of $\mathcal{A}$. For an object A in $\mathcal{A}$, a map $f : B \to A$ with $B \in \mathcal{B}$ is called a right $\mathcal{B}$-approximation if for all objects B' in $\mathcal{B}$, the sequence $\operatorname{Hom}_{\mathcal{C}}(B', B) \to \operatorname{Hom}_{\mathcal{C}}(B', A) \to 0$ is exact. A minimal right $\mathcal{B}$-approximation is a right $\mathcal{B}$-approximation $f : B \to A$ such that for every $g : B \to B$ such that $fg = f$, the map g is an isomorphism. Dually, we have the concepts of left $\mathcal{B}$-approximations and minimal left $\mathcal{B}$-approximations.

The full subcategory $\mathcal{B}$ is called functorially finite if for every object A in $\mathcal{A}$, there exist a right $\mathcal{B}$-approximation ending in A and a left $\mathcal{B}$-approximation starting in A. This notation was introduced in [AS].

The following is well-known and straightforward to check.

Lemma 1.1. *Let $\mathcal{B}$ be a full additive subcategory of $\mathcal{A}$ and A an object of $\mathcal{A}$. If there is a right $\mathcal{B}$-approximation of A, then there is a minimal right $\mathcal{B}$-approximation of A, unique up to isomorphism.*

1.3. **Maximal rigid objects in 2-Calabi-Yau triangulated categories.** Let k be an algebraically closed field and $\mathcal{C}$ be a Krull-Schmidt triangulated k-linear category with split idempotents and suspension functor S. We suppose that all Hom-spaces of $\mathcal{C}$ are finite-dimensional and that $\mathcal{C}$ admits a Serre functor Σ; i.e., for any X, Y in $\mathcal{C}$, we have the following bifunctorial isomorphisms:

$$\operatorname{Hom}_{\mathcal{C}}(X, Y) \simeq D \operatorname{Hom}_{\mathcal{C}}(Y, \Sigma X),$$

where $D = \operatorname{Hom}_k(-, k)$ is the usual duality. We suppose that $\mathcal{C}$ is Calabi-Yau of CY-dimension 2; i.e. there is an isomorphism of triangle functors

$$S^2 \xrightarrow{\sim} \Sigma.$$

For $X, Y \in \mathcal{C}$ and $n \in \mathbb{Z}$, we put as usual

$$\mathrm{Ext}^n_{\mathcal{C}}(X, Y) = \mathrm{Hom}_{\mathcal{C}}(X, S^n Y).$$

Thus the Calabi-Yau property can be written as the following bifunctorial isomorphisms:

$$D\,\mathrm{Ext}^1(X, Y) \simeq \mathrm{Ext}^1(Y, X), \text{ for any } X, Y.$$

An object R of $\mathcal{C}$ is rigid if $\mathrm{Ext}^1_{\mathcal{C}}(R, R) = 0$. It is maximal rigid if it is rigid and $\mathrm{Ext}^1_{\mathcal{C}}(X \oplus R, X \oplus R) = 0$ implies that $X \in \mathrm{add}\, R$, where $\mathrm{add}\, R$ denotes the additive closure of R. An object M of $\mathcal{C}$ is finitely presented by R if there is a triangle in $\mathcal{C}$

$$R_1 \to R_0 \xrightarrow{f} M \to SR_1$$

with R_0, R_1 in $\mathrm{add}\, R$. The morphism f is necessarily a right $\mathrm{add}\, R$-approximation of M, and, conversely, the cone of any $\mathrm{add}\, R$-approximation of an object M finitely presented by R belongs to $\mathrm{add}\, SR$ (see [Y]). Let $\mathrm{pr}(R)$ denote the (full) subcategory of $\mathcal{C}$ of objects finitely presented by R.

Let Γ be the endomorphism algebra of R. Let $\mathrm{mod}\,\Gamma$ denote the category of finite-dimensional right modules over Γ. The following result describes the relationship between a 2-CY triangulated category and the corresponding endomorphism algebra of a maximal rigid object (see [Y]).

Lemma 1.2. *The functor $F = \mathrm{Hom}_{\mathcal{C}}(R, -) : \mathcal{C} \to \mathrm{mod}\,\Gamma$ induces an equivalence*

$$\mathrm{pr}(R)/\,\mathrm{add}\, SR \xrightarrow{\sim} \mathrm{mod}\,\Gamma,$$

where the category on the left has the same objects as $\mathrm{pr}(R)$, with morphisms given by morphisms in $\mathcal{C}$ modulo maps factoring through $\mathrm{add}\, SR$.

Thus the functor F induces an equivalence from $\mathrm{add}\, R$ to the category of projective modules in $\mathrm{mod}\,\Gamma$.

Moreover, it is shown in [ZZ] that all maximal rigid objects have the same number of pairwise non-isomorphic indecomposable direct summands. That is,

Lemma 1.3. *All maximal rigid objects in the 2-Calabi-Yau triangulated category have the same number of indecomposable direct summands (up to isomorphism).*

2. Main result

As before, let R be a maximal rigid object in the 2-Calabi-Yau triangulated category $\mathcal{C}$, let Γ be the endomorphism algebra of R in $\mathcal{C}$ and let F be the functor $\mathrm{Hom}_{\mathcal{C}}(R, -) : \mathcal{C} \to \mathrm{mod}\,\Gamma$. Notice that F is dense, so for each Γ-module M^+ there is some object $M \in \mathrm{pr}(R)$ with $FM = M^+$. We write $\mathcal{C}(X, Y)$ for the set of morphisms from X to Y in the category $\mathcal{C}$. In this section, we prove that the tilting modules over Γ lift to maximal rigid objects in $\mathcal{C}$.

We need the following crucial proposition.

Proposition 2.1. *Let M^+, N^+ be two Γ-modules of projective dimension at most one and let M, N be two objects in $\mathcal{C}$ corresponding to M^+, N^+ respectively. If*

$$\mathrm{Ext}^1_{\Gamma}(M^+, N^+) = 0 \text{ and } \mathrm{Ext}^1_{\Gamma}(N^+, M^+) = 0,$$

then $\mathrm{Ext}^1_{\mathcal{C}}(M, N) = 0$ and $\mathrm{Ext}^1_{\mathcal{C}}(N, M) = 0$.

Proof. Let $\tilde{S}(N, SM)$ be the class of maps $\phi : N \to SM$ which factor through an object from $\operatorname{add} SR$.

First we claim that $\tilde{S}(N, SM) = 0$. In fact, combining this with Lemma 1.1, we have the following "minimal $\operatorname{add} R$-approximation" triangle:

$$R_1^N \xrightarrow{g} R_0^N \xrightarrow{f} N \to SR_1^N,$$

where f is the minimal right $\operatorname{add} R$-approximation of N. Since R is rigid, it is not hard to check that $\tilde{S}(N, SM)$ coincides with the kernel of

$$\mathcal{C}(N, SM) \xrightarrow{\mathcal{C}(f,SM)} \mathcal{C}(R_0^N, SM).$$

Applying the functor $\operatorname{Hom}_{\mathcal{C}}(-, M)$ to the triangle

$$S^- R_0^N \xrightarrow{-S^- f} S^- N \to R_1^N \xrightarrow{g} R_0^N,$$

one gets the following exact sequence:

$$\mathcal{C}(R_0^N, M) \xrightarrow{\mathcal{C}(g,M)} \mathcal{C}(R_1^N, M) \to \mathcal{C}(S^- N, M) \xrightarrow{\mathcal{C}(S^- f,M)} \mathcal{C}(S^- R_0^N, M).$$

Thus we have

$$(*) \qquad \tilde{S}(N, SM) = \operatorname{Ker} \mathcal{C}(f, SM) \simeq \operatorname{Ker} \mathcal{C}(S^- f, M) \simeq \operatorname{Coker} \mathcal{C}(g, M).$$

Applying the functor F to the triangle

$$R_1^N \xrightarrow{g} R_0^N \xrightarrow{f} N \to SR_1^N,$$

we get the following projective resolution of N^+ in $\operatorname{mod} \Gamma$:

$$0 \to FR_1^N \xrightarrow{Fg} FR_0^N \to N^+ \to 0,$$

since we consider the minimal $\operatorname{add} R$-approximation triangle and N^+ is of projective dimension at most 1. Thus

$$\operatorname{Ext}^1_\Gamma(N^+, M^+) = \operatorname{Coker} \operatorname{Hom}_\Gamma(Fg, M^+).$$

Note that R is rigid, which implies that

$$\operatorname{Coker} \operatorname{Hom}_\Gamma(Fg, M^+) = \operatorname{Coker} \mathcal{C}(g, M)$$

by the definition of quotient category. Combining this with $(*)$, one gets

$$\tilde{S}(N, SM) \simeq \operatorname{Ext}^1_\Gamma(N^+, M^+) = 0.$$

Alternatively, if we denote by $\tilde{S}(M, SN)$ the class of maps $\phi : M \to SN$ which factor through an object from $\operatorname{add} SR$, then $\tilde{S}(M, SN) = 0$.

As before, we have the following minimal $\operatorname{add} R$-approximation triangle for M:

$$R_1^M \to R_0^M \to M \xrightarrow{\beta} SR_1^M.$$

Applying the functor $\operatorname{Hom}_{\mathcal{C}}(N, -)$ to the triangle

$$SR_1^M \to SR_0^M \xrightarrow{\alpha} SM \to S^2 R_1^M,$$

we get the exact sequence

$$\mathcal{C}(N, SR_0^M) \xrightarrow{\mathcal{C}(N,\alpha)} \mathcal{C}(N, SM) \to \mathcal{C}(N, S^2 R_1^M).$$

Since

$$\operatorname{Im} \mathcal{C}(N, \alpha) \subset \tilde{S}(N, SM) = 0,$$

we get the injective map

$$\mathcal{C}(N, SM) \hookrightarrow \mathcal{C}(N, S^2 R_1^M),$$

which can be written as follows by the definition of Serre functor and the 2-Calabi-Yau property:

$$D\mathcal{C}(M, SN) \hookrightarrow D\mathcal{C}(SR_1^M, SN).$$

Thus the dual $\mathcal{C}(SR_1^M, SN) \to \mathcal{C}(M, SN)$ of the injective map is surjective, which means that

$$\mathcal{C}(M, SN) \simeq \operatorname{Im} \mathcal{C}(\beta, SN) \subset \tilde{S}(M, SN).$$

That is, $\operatorname{Ext}^1_{\mathcal{C}}(M, N) = 0$. Thanks to the 2-Calabi-Yau property, we get $\operatorname{Ext}^1_{\mathcal{C}}(N, M)$ $= 0$, too. $\square$

By the definition of quotient category, we can use the same notation for a Γ-module and its preimage in $\mathcal{C}$ under the projection $\operatorname{pr}(R) \to \operatorname{pr}(R)/\operatorname{add} SR \xrightarrow{\sim} \operatorname{mod} \Gamma$. We now prove our main result.

Theorem 2.2. *Let R be a maximal rigid object in the 2-Calabi-Yau triangulated category $\mathcal{C}$ and let Γ be the endomorphism algebra of R. Let T be a tilting module over Γ; then T lifts to a maximal rigid object in $\mathcal{C}$.*

Proof. Note that T is a tilting module over Γ, so T has no self-extension in $\mathcal{C}$ by Proposition 2.1. That is, T is rigid in $\mathcal{C}$.

Suppose that T is not maximal, which means that there exists X not from $\operatorname{add} T$ such that $\operatorname{Ext}^1_{\mathcal{C}}(X \oplus T, X \oplus T) = 0$. On the other hand, because T is a tilting module, the number of indecomposable direct summands of T (up to isomorphism) is the same as the number of simple Γ-modules which is the number of indecomposable direct summands of R. Thus the number of pairwise non-isomorphic indecomposable direct summands of the rigid object $X \oplus T$ is greater than that of the maximal rigid object R, which is a contradiction to Lemma 1.3. $\square$

3. The cluster-tilting objects case

Cluster-tilting objects in 2-Calabi-Yau triangulated categories and the corresponding endomorphism algebras were originally defined and studied in [KR]. Let $\mathcal{C}$ be a 2-Calabi-Yau triangulated category. An object T in $\mathcal{C}$ is called cluster tilting if it satisfies that

- $\operatorname{Ext}^1_{\mathcal{C}}(T, T) = 0$ and
- for any $X \in \mathcal{C}$, if $\operatorname{Hom}_{\mathcal{C}}(X, ST) = 0$, then X belongs to $\operatorname{add} T$.

Any 2-Calabi-Yau triangulated category admits rigid objects (0 is viewed as a trivial rigid object) and also admits maximal rigid objects if $\mathcal{C}$ is skeletally small. As described in the introduction, there are 2-Calabi-Yau triangulated categories which contain no cluster-tilting objects. Cluster-tilting objects are obviously maximal rigid objects. But the converse is not true in general (see more in [BIKR, BMV, KZ]). Zhou-Zhu [ZZ] proved that the converse is true if the 2-Calabi-Yau triangulated category admits a cluster-tilting object.

Lemma 3.1 (Zhou-Zhu). *If the 2-Calabi-Yau triangulated category admits a cluster-tilting object, then every maximal rigid object is cluster tilting.*

Thus if the 2-Calabi-Yau triangulated category $\mathcal{C}$ admits a cluster-tilting object, our result indicates that the tilting modules over the endomorphism algebra of a cluster-tilting object in $\mathcal{C}$ lift to cluster-tilting objects in this 2-Calabi-Yau triangulated category. This coincides with Fu-Liu's result in [FL].

Theorem 3.2. *Let $\mathcal{C}$ be a 2-Calabi-Yau triangulated category with a cluster-tilting object T and let Γ be the endomorphism algebra of T. Let L be a tilting module over Γ; then L lifts to a cluster-tilting object in $\mathcal{C}$.*

Proof. This follows directly from Theorem 2.2 in combination with Lemma 3.1. □

Notice that Fu-Liu's proof is heavily based on the fact that the Auslander-Reiten translation of the 2-CY-tilted algebra (the endomorphism algebra of a cluster-tilting object in the 2-Calabi-Yau triangulated category) is induced by the Auslander-Reiten translation of the 2-Calabi-Yau triangulated category. But this is not true for the endomorphism algebra of a maximal rigid object, since $\tau_{\mathcal{C}}$, the Auslander-Reiten translation of 2-Calabi-Yau triangulated category, is not closed in the subcategory $\mathrm{pr}(R)$ generally. The techniques used in this note are totally different from Fu-Liu's work, and our result strengthens theirs.

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Department of Mathematics, Sichuan University, 610064 Chengdu, People's Republic of China–and–Department of Mathematics, Southwest Jiaotong University, 610031 Chengdu, People's Republic of China

E-mail address: xieyunli@home.swjtu.edu.cn

Department of Mathematics, Southwest Jiaotong University, 610031 Chengdu, People's Republic of China

E-mail address: liupin@home.swjtu.edu.cn