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LOCAL AND GLOBAL WELL-POSEDNESS FOR THE CRITICAL SCHRÖDINGER-DEBYE SYSTEM

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Abstract. We establish local well-posedness results for the Initial Value Problem associated to the Schrödinger-Debye system in dimensions $N = 2, 3$ for data in $H^s \times H^\ell$, with s and ℓ satisfying $\max\{0, s-1\} \leq \ell \leq \min\{2s, s+1\}$. In particular, these include the energy space $H^1 \times L^2$. Our results improve the previous ones obtained by B. Bidégaray, and by A. J. Corcho and F. Linares. Moreover, in the critical case ($N = 2$) and for initial data in $H^1 \times L^2$, we prove that solutions exist for all times, thus providing a negative answer to the open problem mentioned by G. Fibich and G. C. Papanicolau concerning the formation of singularities for these solutions.

1. Introduction

We consider the Initial Value Problem (IVP) for the Schrödinger-Debye system

$$\begin{cases} iu_t + \frac{1}{2}\Delta u = uv, & t \geq 0,\ x \in \mathbb{R}^N\ (N = 1, 2, 3), \\ \mu v_t + v = \lambda |u|^2, & \mu > 0,\ \lambda = \pm 1, \\ u(x,0) = u_0(x), \quad v(x,0) = v_0(x), \end{cases} \tag{1.1}$$

where $u = u(x,t)$ is a complex-valued function, $v = v(x,t)$ is a real-valued function and Δ is the Laplacian operator in the spacial variable. This model describes the propagation of an electromagnetic wave through a nonresonant medium whose material response time is relevant. See Newell and Moloney [16] for a more complete discussion of this model.

In the absence of delay ($\mu = 0$), the system (1.1) reduces to the cubic Nonlinear Schrödinger equation (**NLS**)

$$iu_t + \tfrac{1}{2}\Delta u = \lambda u\,|u|^2\,, \tag{1.2}$$

which is focusing or defocusing for $\lambda = -1$ and $\lambda = 1$, respectively. Similarly, the sign of the parameter λ provides an analogous classification of (1.1).

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For sufficiently regular data, the *mass* of the solution u of the system (1.1) is invariant. More precisely,

$$\int_{\mathbb{R}^N} |u(x,t)|^2 dx = \int_{\mathbb{R}^N} |u_0(x)|^2 dx. \tag{1.3}$$

Other conservation laws for this system are not known, but the following pseudo-Hamiltonian structure holds:

$$\frac{d}{dt}E(t) = 2\lambda\mu \int_{\mathbb{R}^N} (v_t)^2 dx, \tag{1.4}$$

where

$$E(t) = \int_{\mathbb{R}^N} \left\{|\nabla u|^2 + \lambda|u|^4 - \lambda\mu^2(v_t)^2\right\} dx = \int_{\mathbb{R}^N} \left\{|\nabla u|^2 + 2v|u|^2 - \lambda v^2\right\} dx. \tag{1.5}$$

The system (1.1) can be decoupled by solving the second equation with respect to v,

$$v(t) = e^{-t/\mu} v_0(x) + \tfrac{\lambda}{\mu} \int_0^t e^{-(t-t')/\mu} |u(t')|^2 \, dt', \tag{1.6}$$

to obtain the integro-differential equation

$$\begin{cases} iu_t + \frac{1}{2}\Delta u = e^{-t/\mu} u v_0(x) + \frac{\lambda}{\mu} u \int_0^t e^{-(t-t')/\mu} |u(t')|^2 dt', \quad x \in \mathbb{R}^N, \ t \geq 0, \\ u(x,0) = u_0(x). \end{cases} \tag{1.7}$$

The rest of this introduction is organized as follows: in section 1.1 we review the previous existing results regarding the local and global theory for (1.1). In section 1.2 we describe our new results in dimensions $N = 2, 3$.

1.1. **Overview of former results in dimensions $N = 1, 2, 3$.** We begin with a review of the local and global theory for the Cauchy problem (1.1) with initial data (u_0, v_0) in Sobolev spaces $H^s(\mathbb{R}^N) \times H^\ell(\mathbb{R}^N)$, $N = 1, 2, 3$.

Bidégaray ([2] and [3]) established the following local results:

Theorem 1.1 ([3]). *Let $N = 1, 2, 3$ and $(u_0, v_0) \in H^s(\mathbb{R}^N) \times H^s(\mathbb{R}^N)$. The IVP (1.7) has a unique solution:*

(a) $u \in L^\infty\left([0,T]; H^s(\mathbb{R}^N)\right)$ *if* $s > N/2$,
(b) $u \in L^\infty\left([0,T]; H^1(\mathbb{R}^N)\right)$ *if* $s = 1$,
(c) $u \in C\left([0,T]; L^2(\mathbb{R}^N)\right) \cap L^{8/N}\left([0,T]; L^4(\mathbb{R}^N)\right)$ *if* $s = 0$,

where $T = T(\|u_0\|_{H^s}, \|v_0\|_{H^s}) > 0$. Moreover, the solution u depends continuously on the initial data (u_0, v_0).

These results were obtained by a fixed-point procedure applied to the Duhamel formulation for the integro-differential equation (1.7), using the Strichartz estimates for the unitary Schrödinger group

$$S(t)f(x) = \left(\mathrm{e}^{-it|\xi|^2/2} \widehat{f}(\xi)\right)^\vee (x). \tag{1.8}$$

Following the same approach, Corcho and Linares ([9]) improved the results stated in Theorem 1.1 in the one-dimensional case. More precisely, they established the following assertions:

Theorem 1.2 ([9]). *Let $0 < s \leq 1$, $q \in [2, \infty]$, $2/r = 1/2 - 1/q$ and $(u_0, v_0) \in H^s(\mathbb{R}) \times H^\ell(\mathbb{R})$. The IVP (1.7) has a unique solution:*

(a) $u \in \mathcal{X}_T^{s,r,q} := C([0,T];\ H^s(\mathbb{R})) \cap L^r([0,T]; L^q(\mathbb{R}))$ *if* $0 < s < 1/2$ *and* $\ell = s$,

(b) $u \in \mathcal{X}_T^{s,r,q}$ *and* $u_x \in L^\infty\big(\mathbb{R}; L^2([0,T])\big)$ *if* $s = 1/2$ *and* $0 \leq \ell \leq 1/2$,

(c) $u \in \mathcal{X}_T^{s,r,q}$ *and* $u_x \in L^\infty\big(\mathbb{R}; L^2([0,T])\big)$ *if* $1/2 < s \leq 1$ *and* $s - 1/2 < \ell \leq s$,

where $T = T(\mu, \|u_0\|_{H^s}, \|v_0\|_{H^\ell}) > 0$. Moreover, the map $(u_0, v_0) \longmapsto u(t)$ is locally Lipschitz and $v \in C\big([0,T];\ H^\ell(\mathbb{R})\big)$.

The new ingredients used in the proof of Theorem 1.2 are commutator estimates for fractional Sobolev spaces and the smoothing effect for the Schrödinger group

$$\|D^{1/2} S(t) u_0\|_{L_x^\infty L_T^2} \leq C \|u_0\|_{L^2},$$

deduced by Kenig, Ponce and Vega (see [14, 15]). Furthermore, the authors also showed that although the fixed-point procedure is performed only on the function u, equation (1.6) can be used to obtain the persistence property of the solution v in $H^s(\mathbb{R}^N)$ in the cases described in Theorem 1.1.

Concerning global existence, it was also proved in [9] that the local-in-time results for the solution u of the integro-differential equation (1.7), given in Theorem 1.1(c) and Theorem 1.2(b) and (c), can be extended to all positive times. However, the method used does not provide control of the evolution in time of the H^s-norm of the corresponding solution v. Indeed, contrary to the NLS equation, (1.1) does not possess a Hamiltonian structure; hence the extension to any positive times of the local-in-time solutions (u, v) is not straightforward.

Recently, however, Corcho and Matheus (see [10]) studied the case $N = 1$ in the framework of Bourgain spaces and obtained the following local and global well-posedness results for the system:

Theorem 1.3 ([10]). *For any $(u_0, v_0) \in H^s(\mathbb{R}) \times H^\ell(\mathbb{R})$, where*

$$|s| - 1/2 \leq \ell < \min\{s + 1/2,\ 2s + 1/2\} \quad \text{and} \quad s > -1/4, \tag{1.9}$$

there exists a time $T = T(\|u_0\|_{H^s}, \|v_0\|_{H^\ell}) > 0$ and a unique solution $(u(t), v(t))$ of the initial value problem (1.1) in the time interval $[0,T]$, satisfying

$$(u, v) \in C\left([0,T]; H^s(\mathbb{R}) \times H^\ell(\mathbb{R})\right).$$

Moreover, the map $(u_0, v_0) \longmapsto (u(t), v(t))$ is locally Lipschitz. In addition, in the case $\ell = s$ with $-3/14 < s \leq 0$, the local solutions can be extended to any time interval $[0,T]$.

The global results in Theorem 1.3 are based on a good control of the L^2-norm of the solution v, which provides global well-posedness in $L^2 \times L^2$. Global well-posedness below L^2 regularity is then obtained via the *I-method* introduced by Colliander, Keel, Staffilani, Takaoka and Tao in [6].

Regarding the formation of singularities in the critical case ($N = 2$), Fibich and Papanicolaou ([11]) studied this system in the focusing case using the lens transformations, but did not derive any result as to the blow-up of the solutions. On the other hand, from a numerical point of view, Besse and Bidégaray ([1]) used two different methods suggesting that blow-up occurs for initial data $u_0(x, y) = \mathrm{e}^{-(x^2+y^2)}$ and $v_0 = \lambda |u_0|^2$. However, prior to the present paper, the blow-up problem remained open.

1.2. **Main results in dimensions $N = 2, 3$.** In this paper we give a negative answer to the question of the existence of blow-up solutions for initial data in $H^1(\mathbb{R}^2) \times L^2(\mathbb{R}^2)$ (see Theorem 3.1). Note that this result is not in contradiction with the numerical simulations in [1]. Indeed, in the latter, the suggested blow-up occurs for the norm $\|u(\cdot,t)\|_{L^\infty}$, which, in two dimensions, is not controlled by $\|u(\cdot,t)\|_{H^1}$. Also, contrary to the NLS case, we prove that the blow-up occurs neither in the defocusing nor in the focusing case. This is due to the delay induced by the term μv_t in the left-hand side of the second equation of (1.1), which prevents the solution from concentrating critically. As expected, this behavior does not depend on the size of μ as long as this parameter stays positive. This was already remarked in [1]: if (u, v) is a solution to (1.1) for a value of $\mu > 0$, then $\Big(\tilde{u}(x,t), \tilde{v}(x,t)\Big) = \big(\mu^{1/2} u(\mu^{1/2}x, \mu t),\, \mu v(\mu^{1/2}x, \mu t)\big)$ yields a solution to (1.1) for $\mu = 1$.

In order to prove our global result and overcome the difficulty caused by the absence of conservation of the energy of (1.1), we use careful control of its derivative (1.4) for solutions in $H^1(\mathbb{R}^2) \times L^2(\mathbb{R}^2)$. This method requires the availability of a local theory in this space, a case which is not covered in the previous literature and had to be derived here as well. More precisely, we prove local well-posedness in dimensions $N = 2, 3$ for initial data in $H^s \times H^\ell$ with s and ℓ satisfying $\max\{0, s-1\} \le \ell \le \min\{2s, s+1\}$ (see Theorem 2.1 and Figure 1).

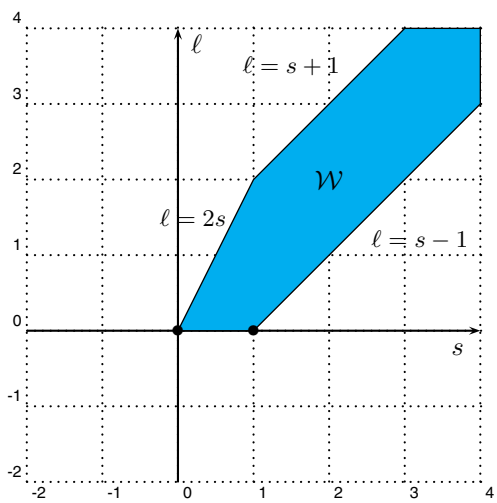

FIGURE 1. The region $\mathcal{W}$, bounded by the lines $\ell = 0$, $\ell = s - 1$, $\ell = 2s$ and $\ell = s + 1$, corresponds to the set of indices (s, ℓ) of our local well-posedness results for system (1.1).

2. Local well-posedness in dimensions $N = 2, 3$

In this section we obtain local well-posedness for the system (1.1) following the approach used by Ginibre, Tsutsumi and Velo in [13] for the Zakharov system. As in [13] we measure the solutions in appropriate Bourgain space, or Fourier restriction, norms. More specifically, we consider the solution (u, v) of the system (1.1) in the space $\mathcal{Y}_{s,\ell} = X^{s,b} \times H^{\ell,c}$, the completion of the product of Schwartz spaces

$\mathcal{S}(R^{N+1}) \times \mathcal{S}(\mathbb{R}^{N+1})$ with respect to the norm

$$\|(u,v)\|_{\mathcal{Y}_{s,\ell}} = \|u\|_{X^{s,b}} + \|v\|_{H^{\ell,c}},$$

where

$$\|v\|_{H^{\ell,c}} := \left\|(1+|\xi|)^{\ell}(1+|\tau|)^{c}\widehat{v}(\xi,\tau)\right\|_{L^2_{\xi,\tau}}, \tag{2.10}$$

and

$$\|u\|_{X^{s,b}} := \left\|(1+|\xi|)^{s}(1+|\tau+\tfrac{1}{2}|\xi|^2|)^{b}\widehat{u}(\xi,\tau)\right\|_{L^2_{\xi,\tau}} \tag{2.11}$$

is the Fourier restriction norm associated to the Schrödinger group $S(t)$ defined in (1.8). In these definitions $\widehat{f}(\xi,\tau)$ denotes the space-time Fourier transform of $f(x,t)$ and $|\xi|$ is the Euclidean norm of the frequency vector $\xi \in \mathbb{R}^N$.

We recall that $X^{s,b} \hookrightarrow C\left(\mathbb{R}; H^s(\mathbb{R}^N)\right)$ and $H^{\ell,c} \hookrightarrow C\left(\mathbb{R}; H^{\ell}(\mathbb{R}^N)\right)$ for all $s, \ell \in \mathbb{R}$ if b, $c > 1/2$.

Now we state the main local well-posedness result.

Theorem 2.1. *Let $N = 2, 3$. For any $(u_0, v_0) \in H^s(\mathbb{R}^N) \times H^{\ell}(\mathbb{R}^N)$, with s and ℓ satisfying the conditions*

$$\max\{0, s-1\} \le \ell \le \min\{2s, s+1\}, \tag{2.12}$$

there exists a positive time $T = T(\|u_0\|_{H^s}, \|v_0\|_{H^{\ell}})$ and a unique solution $(u(t), v(t))$ of the initial value problem (1.1) on the time interval $[0,T]$ such that

(i) $(\psi_T\, u, \psi_T\, v) \in X^{s,b} \times H^{\ell,c}$,

(ii) $(u,v) \in C\left([0,T]; H^s(\mathbb{R}^N) \times H^{\ell}(\mathbb{R}^N)\right)$

for suitable b and c close to $\frac{1}{2}+$ (ψ_T denotes, as usual, a cutoff function for the time interval $[0,T]$). Moreover, the map $(u_0, v_0) \longmapsto (u(t), v(t))$ is locally Lipschitz from $H^s(\mathbb{R}^N) \times H^{\ell}(\mathbb{R}^N)$ into $C\left([0,T];\, H^s(\mathbb{R}^N) \times H^{\ell}(\mathbb{R}^N)\right)$.

2.1. **Preliminary estimates.** In the sequel, we use the following notation. For $\lambda \in \mathbb{R}$,

$$[\lambda]_+ = \begin{cases} \lambda & \text{if} \quad \lambda > 0, \\ \varepsilon\ (0 < \varepsilon \ll 1) & \text{if} \quad \lambda = 0, \\ 0 & \text{if} \quad \lambda < 0, \end{cases}$$

and we denote by $\lambda\pm$ a number slightly larger, respectively smaller, than λ. The bracket $\langle\cdot\rangle$ is defined as $\langle\cdot\rangle = 1 + |\cdot|$.

We introduce the variables

$$\sigma_i = \tau_i + \tfrac{1}{2}|\xi_i|^2,\ \xi_i \in \mathbb{R}^N,\ \tau_i \in \mathbb{R}\ (i = 1,2) \quad \text{and} \quad \sigma = \tau \in \mathbb{R} \tag{2.13}$$

with the convolution structure

$$\xi = \xi_1 - \xi_2 \quad \text{and} \quad \tau = \tau_1 - \tau_2. \tag{2.14}$$

In terms of these variables, the resonance relation for system (1.1) is the following:

$$\sigma_1 - \sigma_2 - \sigma = \tfrac{1}{2}(|\xi_1|^2 - |\xi_2|^2). \tag{2.15}$$

Lemma 2.2. *Let $N = 2, 3$, $b_0 = \frac{1}{2}+$, $0 \le \gamma \le 1$ and a, a_1 and a_2 be nonnegative numbers satisfying the conditions*

$$(1-\gamma)\max\{a, a_1, a_2\} \le b_0 \le (1-\gamma)(a + a_1 + a_2), \tag{2.16}$$

$$(1-\gamma)a < b_0. \tag{2.17}$$

Let m be such that

$$m \geq N/2 + 1 - (1-\gamma)\frac{(a + a_1 + a_2)}{b_0} \geq 0 \tag{2.18}$$

with strict inequality on the left of (2.18) *if equality holds on the right of* (2.16) *or if* $a_1 = 0$. *In addition, let* $a' \geq \gamma a$, $a_1' \geq \gamma a_1$, $a_2' \geq \gamma a_2$ *and let* $h, h_1, h_2 \in L^2(\mathbb{R}^{N+1})$ *be such that* $\mathcal{F}^{-1}(\langle\sigma\rangle^{-a'}\widehat{h})$, $\mathcal{F}^{-1}(\langle\sigma_i\rangle^{-a_i'}\widehat{h_i})$ $(i = 1, 2)$ *have support in* $|t| \leq CT$. *Then, for*

$$\theta = \gamma \sum_{j=a,a_1,a_2} j(1 - [j' - 1/2]_+/j'), \tag{2.19}$$

the inequalities

$$\int \frac{|\widehat{h}(\xi,\tau)\widehat{h}_1(\xi_1,\tau_1)\widehat{h}_2(\xi_2,\tau_2)|}{\langle\sigma\rangle^a\langle\sigma_1\rangle^{a_1}\langle\sigma_2\rangle^{a_2}\langle\xi\rangle^m} \lesssim T^\theta \|h\|_{L^2}\|h_1\|_{L^2}\|h_2\|_{L^2} \tag{2.20}$$

and

$$\int \frac{|\widehat{h}(\xi,\tau)\widehat{h}_1(\xi_1,\tau_1)\widehat{h}_2(\xi_2,\tau_2)|}{\langle\sigma\rangle^a\langle\sigma_1\rangle^{a_1}\langle\sigma_2\rangle^{a_2}\langle\xi_2\rangle^m} \lesssim T^\theta \|h\|_{L^2}\|h_1\|_{L^2}\|h_2\|_{L^2} \tag{2.21}$$

hold.

Proof. These estimates follow from Lemma 3.2 in [13] for the case $\sigma_i = \tau_i + |\xi_i|^2$ and $\sigma = \tau \pm |\xi|$, changing the terms σ_i $(i = 1, 2)$ and σ by $\sigma_i = \tau_i + \frac{1}{2}|\xi_i|^2$ $(i = 1, 2)$ and $\sigma = \tau$, respectively. □

2.2. **Bilinear estimates.** It is well known that in the framework of Bourgain spaces local well-posedness results can usually be reduced to the proof of adequate k-linear estimates. In the present case, to prove Theorem 2.1 it suffices to establish the following two bilinear estimates:

Proposition 2.3. *Let* $s \geq 0$, $\ell \geq \max\{0, s-1\}$ *and the functions* u *and* v *be supported in time in the region* $|t| \leq CT$. *Then, the bilinear estimate*

$$\|uv\|_{X^{s,-b_1}} \lesssim T^\theta \|u\|_{X^{s,b_2}}\|v\|_{H^{\ell,c}}$$

holds provided $c = \frac{1}{2} + \varepsilon$, $b_1 = \frac{1}{2} - \varepsilon_1$ *and* $b_2 = \frac{1}{2} + \varepsilon_2$ *for an adequate selection of the parameters* $0 \leq \varepsilon, \varepsilon_1, \varepsilon_2 \ll 1$.

Proposition 2.4. *Let* $s \geq 0$, $\ell \leq \min\{2s, s+1\}$ *and the functions* u *and* w *be supported in time in the region* $|t| \leq CT$. *Then, the bilinear estimate*

$$\|(u \cdot \overline{w})\|_{H^{\ell,-b}} \lesssim T^\theta \|w\|_{X^{s,b_3}}\|u\|_{X^{s,b_3}}$$

holds provided $b = \frac{1}{2} - \varepsilon$, $b_3 = \frac{1}{2} + \varepsilon_3$ *for an adequate selection of the parameters* $0 \leq \varepsilon, \varepsilon_3 \ll 1$.

The proofs of Propositions 2.3 and 2.4 follow similar arguments as the ones used in [13] to prove Lemmas 3.4 and 3.5 for the Zakharov system in all dimensions. Thus, here we present only the proof of Proposition 2.3, corresponding to Lemma 3.4 of [13] in our context (dimensions $N = 2, 3$), followed by a brief sketch of the proof of Proposition 2.4.

2.3. **Proof of Proposition 2.3.** We define

$$\widehat{h}(\xi,\tau)=\langle\xi\rangle^{\ell}\langle\sigma\rangle^{c}\widehat{v}(\xi,\tau)\quad\text{and}\quad\widehat{h}_2(\xi_2,\tau_2)=\langle\xi_2\rangle^{s}\langle\sigma_2\rangle^{b_2}\widehat{u}(\xi_2,\tau_2).$$

In order to estimate $\|uv\|_{X^{s,-b_1}}$ by duality arguments, we take the scalar product with a generic function in X^{-s,b_1} with Fourier transform $\langle\xi_1\rangle^{s}\langle\sigma_1\rangle^{-b_1}\widehat{h_1}(\xi_1,\tau_1)$ and $h_1\in L^2(\mathbb{R}^{N+1})$. Then, the bilinear estimate in Proposition 2.3 takes the form

$$|S(h,h_1,h_2)|\lesssim T^{\theta}\|h\|_{L^2}\|h_1\|_{L^2}\|h_2\|_{L^2}, \tag{2.22}$$

where

$$S(h,h_1,h_2)=\int\frac{\widehat{h}\widehat{h}_1\widehat{h}_2\langle\xi_1\rangle^{s}}{\langle\sigma\rangle^{c}\langle\sigma_1\rangle^{b_1}\langle\sigma_2\rangle^{b_2}\langle\xi_2\rangle^{s}\langle\xi\rangle^{\ell}}. \tag{2.23}$$

First, we note that if $0\le s\le\ell$, then we have

$$\frac{\langle\xi_1\rangle^{s}}{\langle\xi_2\rangle^{s}\langle\xi\rangle^{\ell}}\lesssim 1.$$

Then, taking $(\varepsilon,\varepsilon_1,\varepsilon_2)=(0,\varepsilon,\varepsilon)$ and applying (2.20) in Lemma 2.2 with

$$(a',a_1',a_2')=(a,a_1,a_2)=(c,b_1,b_2)=(\tfrac12,\ \tfrac12-\varepsilon,\ \tfrac12+\varepsilon), \tag{2.24}$$

$$(1-\gamma)=\tfrac{N+2}{3}b_0, \tag{2.25}$$

and

$$m=0\ge N/2+1-(1-\gamma)\frac{(c+b_1+b_2)}{b_0}=0, \tag{2.26}$$

we obtain

$$|S|\lesssim\int\frac{|\widehat{h}\widehat{h}_1\widehat{h}_2|}{\langle\sigma\rangle^{c}\langle\sigma_1\rangle^{b_1}\langle\sigma_2\rangle^{b_2}}\lesssim T^{\theta}\|h\|_{L^2}\|h_1\|_{L^2}\|h_2\|_{L^2}. \tag{2.27}$$

To estimate the functional S in the case $s\ge\ell$ we divide the analysis into two cases by considering two integration subregions:

Case 1: $|\xi_1|\le 2|\xi_2|$. Here, $s\ge 0$ implies $\langle\xi_1\rangle^{s}\lesssim\langle\xi_2\rangle^{s}$, so the contribution S_1 of this subregion to S is given by

$$|S_1|\lesssim\int\frac{|\widehat{h}\widehat{h}_1\widehat{h}_2|}{\langle\sigma\rangle^{c}\langle\sigma_1\rangle^{b_1}\langle\sigma_2\rangle^{b_2}\langle\xi\rangle^{\ell}}\lesssim T^{\theta}\|h\|_{L^2}\|h_1\|_{L^2}\|h_2\|_{L^2}, \tag{2.28}$$

where we have used (2.20) in Lemma 2.2 with conditions (2.24), (2.25) and

$$m=\ell\ge N/2+1-(1-\gamma)\frac{(c+b_1+b_2)}{b_0}=0. \tag{2.29}$$

Case 2: $|\xi_1|\ge 2|\xi_2|$. In this situation we have that $|\xi|\sim|\xi_1|$. Also, from the resonance relation (2.15) it follows that

$$|\sigma_1-\sigma_2-\sigma|=\tfrac12\Big||\xi_1|^2-|\xi_2|^2\Big|\ge\tfrac38|\xi_1|^2\Longrightarrow|\xi_1|^2\le\tfrac98\max\{|\sigma|,|\sigma_1|,|\sigma_2|\}. \tag{2.30}$$

Then, in view of (2.30) and using the fact that $s\ge\ell$, we estimate the contribution S_2 of this subregion to S by

$$|S_2|\lesssim\int\frac{|\widehat{h}\widehat{h}_1\widehat{h}_2|\langle\xi_1\rangle^{s-\ell}}{\langle\sigma\rangle^{c}\langle\sigma_1\rangle^{b_1}\langle\sigma_2\rangle^{b_2}\langle\xi_2\rangle^{s}}\lesssim\int\frac{|\widehat{h}\widehat{h}_1\widehat{h}_2|\langle\sigma^*\rangle^{\rho(s,\ell)}}{\langle\sigma\rangle^{c}\langle\sigma_1\rangle^{b_1}\langle\sigma_2\rangle^{b_2}\langle\xi_2\rangle^{s}}, \tag{2.31}$$

where $\rho(s,\ell)=\frac{s-\ell}{2}$ and $\sigma^*=\max\big\{\langle\sigma\rangle,\langle\sigma_1\rangle,\langle\sigma_2\rangle\big\}$.

Now we consider the condition

$$\rho(s,\ell) = \frac{s-\ell}{2} \leq \min\{c, b_1, b_2\} = \frac{1}{2} - \varepsilon_1 \Longleftrightarrow s - 1 + 2\varepsilon_1 \leq \ell, \tag{2.32}$$

which guarantees that $b-\rho$, $b_1-\rho$ and $b_2-\rho$ are nonnegative. Next, we establish conditions that allow us to apply (2.20) in Lemma 2.2, with

$$(a', a_1', a_2') = (a, a_1, a_2) = \begin{cases} (c-\rho,\, b_1,\, b_2) & \text{if} \quad \sigma^* = \sigma, \\ (b,\, b_1-\rho,\, b_2) & \text{if} \quad \sigma^* = \sigma_1, \\ (b,\, b_1,\, b_2-\rho) & \text{if} \quad \sigma^* = \sigma_2, \end{cases} \tag{2.33}$$

and $m = s$, to obtain the desired estimate

$$|S_2| \lesssim \int \frac{|\widehat{h}\widehat{h}_1\widehat{h}_2|\langle\sigma^*\rangle^\rho}{\langle\sigma\rangle^c\langle\sigma_1\rangle^{b_1}\langle\sigma_2\rangle^{b_2}\langle\xi_2\rangle^s} \lesssim T^\theta \|h\|_{L^2}\|h_1\|_{L^2}\|h_2\|_{L^2}. \tag{2.34}$$

For this purpose, it suffices to take $0 < \gamma < 1$ such that

$$b_0 \leq (1-\gamma)(c + b_1 + b_2 - \rho), \tag{2.35}$$

$$s \geq N/2 + 1 - (1-\gamma)\frac{(c + b_1 + b_2 - \rho)}{b_0} \geq 0. \tag{2.36}$$

Choosing $\gamma' = \gamma\frac{c+b_1+b_2-\rho}{c+b_1+b_2} \in (0,1)$ we find that conditions (2.35)-(2.36) are equivalent to taking γ' such that

$$b_0 \leq (1-\gamma')(c + b_1 + b_2) - \rho(s,\ell), \tag{2.37}$$

$$s \geq N/2 + 1 - (1-\gamma')\frac{(c + b_1 + b_2)}{b_0} + \frac{\rho(s,\ell)}{b_0} \geq 0. \tag{2.38}$$

Now, we take $(\varepsilon, \varepsilon_1, \varepsilon_2) = (0, \varepsilon, \varepsilon)$ and γ' satisfying $(1-\gamma') = \frac{N+2}{3}b_0$, and then from (2.29) we have

$$s > \ell + \frac{\rho(s,\ell)}{b_0} \geq \underbrace{N/2 + 1 - (1-\gamma')\frac{(c + b_1 + b_2)}{b_0}}_{=0} + \frac{\rho(s,\ell)}{b_0} \geq 0,$$

as desired. This completes the proof.

2.4. **Proof of Proposition 2.4.** Following the same ideas as in the proof of Proposition 2.3, in this case we need to estimate the following functional:

$$W(h, h_1, h_2) = \int \frac{\widehat{h}\widehat{h}_1\widehat{h}_2\langle\xi\rangle^\ell}{\langle\sigma\rangle^b\langle\sigma_1\rangle^{b_3}\langle\sigma_2\rangle^{b_3}\langle\xi_1\rangle^s\langle\xi_2\rangle^s}. \tag{2.39}$$

Unlike the Zakharov system, here we do not have the presence of the derivative term ($|\xi|$) in the numerator of W. Thus, we estimate (2.39) in the same way as in the proof of Lemma 3.5 in [13] ($N = 2, 3$), replacing $|\xi|\langle\xi\rangle^\ell$ by $\langle\xi\rangle^\ell$, which is equivalent to changing $\ell + 1$ by ℓ in the internal computations.

Remark 2.5. In the case $|\ell - s| < 1$ the bilinear estimates in Propositions 2.3 and 2.4 hold for small positive numbers $\varepsilon, \varepsilon_1, \varepsilon_2$ and ε_3, which allows taking $b = \frac{1}{2}+$ and $c = \frac{1}{2}+$ in Theorem 2.1 so that the corresponding immersions $X^{s,b} \hookrightarrow C\left(\mathbb{R}; H^s(\mathbb{R}^N)\right)$ and $H^{\ell,b} \hookrightarrow C\left(\mathbb{R}; H^\ell(\mathbb{R}^N)\right)$ are guaranteed. On the other hand, if $|\ell - s| = 1$, one must take $b = c = 1/2$ in Theorem 2.1. Then, to guarantee the required immersions, we need to establish extra bilinear estimates in the norms

$$\|u\|_{\tilde{X}^s} := \left\|\langle\xi\rangle^s\langle\tau + \tfrac{1}{2}|\xi|^2\rangle^{-1}\widehat{u}(\xi,\tau)\right\|_{L^2_\xi L^1_\tau}$$

and

$$\|v\|_{\tilde{H}^\ell} := \left\|\langle\xi\rangle^\ell\langle\tau\rangle^{-1}\widehat{v}(\xi,\tau)\right\|_{L^2_\xi L^1_\tau},$$

which follow in the same manner as in the proof of Lemmas 3.6 and 3.7 in [13].

2.5. **Proof of Theorem 2.1.** The proof follows the now standard contraction method applied to a localized in time cut-off integral formulation associated to the system (1.1) in the Bourgain spaces defined by the norms (2.10) and (2.11) (see [13], for example, for complete details of a similar proof). As is well known, the success of this method relies almost exclusively on the availability of certain multilinear estimates in these norms for the nonlinear terms of the equations. In our case these estimates are the ones obtained in Propositions 2.3 and 2.4.

We start with the following integral system:[1]

(2.40)

$$u(\cdot,t) = \Phi_1(u,v) := \psi_1(t)S(t)u_0 - i\psi_T(t)\int_0^t S(t-t')\psi_{2T}^2(t')u(\cdot,t')v(\cdot,t')dt',$$

(2.41)

$$v(\cdot,t) = \Phi_2(u,v) := \psi_1(t)v_0 + \psi_T(t)\int_0^t \left[\frac{\lambda}{\mu}|\psi_{2T}(t')u(\cdot,t')|^2 - \psi_{2T}(t')v(\cdot,t')\right]dt'.$$

Here $S(t)$ is given by (1.8), $\psi_1 \in C^\infty(\mathbb{R};\ \mathbb{R}^+)$ is an even function, such that $0 \le \psi_1 \le 1$ and

$$\psi_1(t) = \begin{cases} 1, & |t| \le 1, \\ 0, & |t| \ge 2, \end{cases} \tag{2.42}$$

and $\psi_T(t) = \psi_1(t/T)$ for $0 \le T \le 1$.

Now, choosing T sufficiently small, we solve (2.40)-(2.41) by contraction in the complete metric space

$$\Sigma = \left\{(u,v) \in X^{s,b} \times H^{\ell,c};\ \|u\|_{X^{s,b}} \le N_1 \text{ and } \|v\|_{H^{\ell,c}} \le N_2\right\},$$

with $b, c = \frac{1}{2}+$ and the induced norm $\|(u,v)\|_\Sigma := \|u\|_{X^{s,b}} + \|v\|_{H^{\ell,c}}$, where the positive constants $N_j, j = 1,2$, will be chosen below.

Using the properties of spaces $X^{s,b}$ and $H^{\ell,c}$ (see (2.19) and Lemma 2.1 in [13]) and the bilinear estimates obtained in Propositions 2.3 and 2.4, we get

$$\begin{aligned} \|\Phi_1(u,v)\|_{X^{s,b}} &\le c_0\|u_0\|_{H^s} + c_1T^{1-b-a_1}\|\psi_{2T}^2uv\|_{X^{s,-a_1}} \\ &\le c_0\|u_0\|_{H^s} + c_1T^{\theta_1}\|u\|_{X^{s,b}}\|v\|_{H^{\ell,c}} \\ &\le c_0\|u_0\|_{H^s} + c_1T^{\theta_1}N_1N_2 \end{aligned} \tag{2.43}$$

and

$$\begin{aligned} \|\Phi_2(u,v)\|_{H^{\ell,c}} &\le c_0\|v_0\|_{H^\ell} + c_2T^{1-c-a_2}\left(\|\psi_{2T}^2|u|^2\|_{H^{\ell,-a_2}}\|\psi_{2T}^2v\|_{H^{\ell,-a_2}}\right) \\ &\le c_0\|v_0\|_{H^\ell} + c_2T^{\theta_2}\left(\|u\|^2_{X^{s,b}} + \|v\|_{H^{\ell,c}}\right) \\ &\le c_0\|v_0\|_{H^\ell} + c_2T^{\theta_2}(N_1^2 + N_2), \end{aligned} \tag{2.44}$$

where $-\frac{1}{2} < -a_1 \le 0 \le b \le 1-a_1,\ -\frac{1}{2} < -a_2 \le 0 \le c \le 1-a_2$ and $\theta_i,\ i = 1,2,$ are positive.

[1]Observe that for the v equation (2.41), we are not using the standard Duhamel, or variation of parameters, form as in (1.6). Both forms could be used here, but this choice makes computations slightly simpler.

Taking $N_1 = 2c_0\|u_0\|_{H^s}$ and $N_2 = 2c_0\|v_0\|_{H^\ell}$ from (2.43) and (2.44), it follows that $\Phi(u,v) = (\Phi_1(u,v), \Phi_2(u,v)) \in \Sigma$ for $(u,v) \in \Sigma$ and for small enough T satisfying

$$T^{\theta_1} \leq \frac{1}{2c_1 N_2} \quad \text{and} \quad T^{\theta_2} \leq \frac{N_2}{2c_2(N_1^2 + N_2)}.$$

The contraction condition can be obtained in a similar way, and the proof is finished.

Remark 2.6. In the case $b = c = 1/2$ the proof of Theorem 2.1 follows by similar arguments, but by using the modified norms

$$|||u|||_s := \|u\|_{X^{s,1/2}} + \left\|\langle\xi\rangle^s \langle\tau + \tfrac{1}{2}|\xi|^2\rangle^{-1}\widehat{u}(\xi,\tau)\right\|_{L^2_\xi L^1_\tau}$$

and

$$|||v|||_\ell := \|u\|_{H^{\ell,1/2}} + \left\|\langle\xi\rangle^\ell \langle\tau\rangle^{-1}\widehat{v}(\xi,\tau)\right\|_{L^2_\xi L^1_\tau}$$

in order to obtain the immersions in the spaces $C\left(\mathbb{R}; H^s(\mathbb{R}^N)\right)$ and $C\left(\mathbb{R}; H^\ell(\mathbb{R}^N)\right)$.

3. Global well-posedness for the critical model

As mentioned in the introduction, in dimension $N = 2$, the system (1.1) is a perturbation of the scaling-critical cubic NLS equation (1.2). In this section we derive *a priori* estimates in the energy space $H^1(\mathbb{R}^2) \times L^2(\mathbb{R}^2)$ for the focusing and defocusing cases of (1.1), which allow us to extend the local solutions obtained in the previous section to all positive times.

Theorem 3.1. *Let $(u_0, v_0) \in H^1(\mathbb{R}^2) \times L^2(\mathbb{R}^2)$. Then, for all $T > 0$, there exists a unique solution*

$$(u,v) \in C\left([0,T];\ H^1(\mathbb{R}^2) \times L^2(\mathbb{R}^2)\right)$$

to the Initial Value Problem (1.1).

Proof. In view of the local well-posedness result detailed in the last section and the conservation of the L^2-norm of u, we only need to obtain an *a priori* bound for the function

$$f(t) := \|\nabla u(\cdot,t)\|^2_{L^2} + \|v(\cdot,t)\|^2_{L^2}.$$

We begin by estimating $\|v(\cdot,t)\|^2_{L^2}$. Using the explicit representation for v, given by

$$v(x,t) = e^{-t/\mu} v_0(x) + \tfrac{\lambda}{\mu}\int_0^t e^{-(t-t')/\mu}|u(x,t')|^2\,dt', \tag{3.45}$$

and applying the Minkowski and Gagliardo-Nirenberg inequalities, we get

$$\begin{aligned} \|v(\cdot,t)\|_{L^2} &\leq \|v_0\|_{L^2} + \tfrac{1}{\mu}\int_0^t e^{-(t-t')/\mu}\|u(\cdot,t')\|^2_{L^4}\,dt' \\ &\leq \|v_0\|_{L^2} + \tfrac{\beta^2}{\mu}\int_0^t \|u(\cdot,t')\|_{L^2}\|\nabla u(\cdot,t')\|_{L^2}\,dt' \\ &= \|v_0\|_{L^2} + \tfrac{\beta^2\|u_0\|_{L^2}}{\mu}\int_0^t \|\nabla u(\cdot,t')\|_{L^2}\,dt', \end{aligned} \tag{3.46}$$

where β is the constant from the Gagliardo-Nirenberg inequality. Now, we use Hölder's inequality to obtain, from (3.46),

$$\begin{aligned}\|v(\cdot,t)\|_{L^2}^2 &\le 2\|v_0\|_{L^2}^2 + 2\left(\tfrac{\beta^2\|u_0\|_{L^2}}{\mu}\int_0^t \|\nabla u(\cdot,t')\|_{L^2}\,dt'\right)^2 \\ &\le 2\|v_0\|_{L^2}^2 + \tfrac{2\beta^4\|u_0\|_{L^2}^2}{\mu^2}\,t\int_0^t \|\nabla u(\cdot,t')\|_{L^2}^2\,dt' \\ &\le 2\|v_0\|_{L^2}^2 + \tfrac{2\beta^4\|u_0\|_{L^2}^2}{\mu^2}\,t\int_0^t f(t')\,dt'.\end{aligned} \tag{3.47}$$

On the other hand, from (1.5), we have

$$\begin{aligned}\|\nabla u(\cdot,t)\|_{L^2}^2 &= E(t) - 2\int_{\mathbb{R}^2} v(\cdot,t)|u(\cdot,t)|^2dx + \lambda\|v(\cdot,t)\|_{L^2}^2 \\ &\le |E(t)| + \left|2\int_{\mathbb{R}^2} v(\cdot,t)|u(\cdot,t)|^2dx\right| + \|v(\cdot,t)\|_{L^2}^2.\end{aligned} \tag{3.48}$$

We begin by treating the term $\int 2v|u|^2dx$, which, in view of (3.45), can be rewritten in the form

$$2\int_{\mathbb{R}^2} v(\cdot,t)|u(\cdot,t)|^2dx = A(t) + B(t), \tag{3.49}$$

where

$$A(t) = 2e^{-t/\mu}\int_{\mathbb{R}^2} v_0|u|^2dx$$

and

$$B(t) = \tfrac{2\lambda}{\mu}\int_{\mathbb{R}^2} |u(x,t)|^2\int_0^t e^{-(t-t')/\mu}|u(x,t')|^2dt'dx.$$

We now proceed with the estimates of the terms $A(t)$ and $B(t)$. To estimate $A(t)$ we use the Hölder and Gagliardo-Nirenberg inequalities to obtain

$$\begin{aligned}|A(t)| &\le 2e^{-t/\mu}\|v_0\|_{L^2}\|u(\cdot,t)\|_{L^4}^2 \\ &\le 2\|v_0\|_{L^2}\beta^2\|u(\cdot,t)\|_{L^2}\|\nabla u(\cdot,t)\|_{L^2} \\ &= 2\|v_0\|_{L^2}\beta^2\|u_0\|_{L^2}\|\nabla u(\cdot,t)\|_{L^2} \\ &\le 4\|v_0\|_{L^2}^2\beta^4\|u_0\|_{L^2}^2 + \tfrac14\|\nabla u(\cdot,t)\|_{L^2}^2.\end{aligned} \tag{3.50}$$

Now we estimate $B(t)$. By the Hölder, Minkowski and Gagliardo-Nirenberg inequalities we get

$$\begin{aligned}|B(t)| &\le \tfrac{2}{\mu}\|u(\cdot,t)\|_{L^4}^2\int_0^t e^{-\frac{(t-t')}{\mu}}\|u(\cdot,t')\|_{L^4}^2dt' \\ &\le \tfrac{1}{\mu}\int_0^t 2\|u(\cdot,t)\|_{L^4}^2\|u(\cdot,t')\|_{L^4}^2dt' \\ &\le \tfrac{1}{\mu}\int_0^t \|u(\cdot,t)\|_{L^4}^4dt' + \tfrac{1}{\mu}\int_0^t \|u(\cdot,t')\|_{L^4}^4dt' \\ &\le \tfrac{\beta^4}{\mu}\,t\,\|u_0\|_{L^2}^2\|\nabla u(\cdot,t)\|_{L^2}^2 + \tfrac{\beta^4}{\mu}\|u_0\|_{L^2}^2\int_0^t \|\nabla u(\cdot,t')\|_{L^2}^2dt' \\ &\le \tfrac14\|\nabla u(\cdot,t)\|_{L^2}^2 + \tfrac{\beta^4}{\mu}\|u_0\|_{L^2}^2\int_0^t f(t')dt',\end{aligned} \tag{3.51}$$

for all $t \in [0, T_\mu]$, where $T_\mu = \frac{\mu}{4\beta^4\|u_0\|_{L^2}^2}$. Then, (3.50) and (3.51) yield

$$\begin{aligned}\left|2\int_{\mathbb{R}^2} v(\cdot,t)|u(\cdot,t)|^2dx\right| &\leq 4\|v_0\|_{L^2}^2\beta^4\|u_0\|_{L^2}^2 + \tfrac{1}{2}\|\nabla u(\cdot,t)\|_{L^2}^2 \\ &\quad + \tfrac{\beta^4}{\mu}\|u_0\|_{L^2}^2\int_0^t f(t')dt',\end{aligned} \tag{3.52}$$

for all $t \in [0, T_\mu]$.

Next, we estimate the growth of $E(t)$. Using (1.4), we have

$$\begin{aligned}|E(t)| &= \left|E_0 + 2\lambda\mu\int_0^t\left(\int_{\mathbb{R}^2} v_t^2(x,t')dx\right)dt'\right| \\ &\leq |E_0| + 2\mu\int_0^t\left(\int_{\mathbb{R}^2} v_t^2(x,t')dx\right)dt' \\ &\leq |E_0| + \frac{2}{\mu}\int_0^t\left(\int_{\mathbb{R}^2}\big(\lambda|u(x,t')|^2 - v(x,t')\big)^2dx\right)dt' \quad \text{(by (1.1))} \\ &\leq |E_0| + \frac{4}{\mu}\int_0^t\left(\|u(\cdot,t')\|_{L^4}^4 + \|v(\cdot,t')\|_{L^2}^2\right)dt' \\ &\leq |E_0| + \frac{4}{\mu}\int_0^t\left(\beta^4\|u_0\|_{L^2}^2\|\nabla u(t')\|_{L^2}^2 + \|v(t')\|_{L^2}^2\right)dt' \\ &\leq |E_0| + \frac{4}{\mu}\left(\beta^4\|u_0\|_{L^2}^2 + 1\right)\int_0^t f(t')dt'.\end{aligned} \tag{3.53}$$

Collecting the information in (3.48), (3.52) and (3.53) we have

$$\begin{aligned}\|\nabla u(\cdot,t)\|_{L^2}^2 \leq 2|E_0| + 8\|v_0\|_{L^2}^2\beta^4\|u_0\|_{L^2}^2 + 2\|v(\cdot,t)\|_{L^2}^2 \\ + \tfrac{2}{\mu}\left(5\beta^4\|u_0\|_{L^2}^2 + 4\right)\int_0^t f(t')dt',\end{aligned}$$

for all $t \in \big[0, T_\mu\big]$, from which follows

$$\begin{aligned}f(t) \leq 2|E_0| + 8\|v_0\|_{L^2}^2\beta^4\|u_0\|_{L^2}^2 + 3\|v(\cdot,t)\|_{L^2}^2 \\ + \tfrac{2}{\mu}\left(5\beta^4\|u_0\|_{L^2}^2 + 4\right)\int_0^t f(t')dt',\end{aligned} \tag{3.54}$$

for all $t \in \big[0, T_\mu\big]$.

Finally, we combine (3.47) with (3.54) to obtain

$$f(t) \leq \alpha_0 + \alpha_1\int_0^t f(t')dt', \quad \text{for all} \quad t \in \big[0, T_\mu\big], \tag{3.55}$$

with

$$\begin{aligned}\alpha_0 &= 2|E_0| + 4\|v_0\|_{L^2}^2\left(2\beta^4\|u_0\|_{L^2}^2 + \tfrac{3}{2}\right), \\ \alpha_1 &= \tfrac{2}{\mu}\left(5\beta^4\|u_0\|_{L^2}^2 + \tfrac{19}{4}\right).\end{aligned}$$

Hence, by Gronwall's lemma,

$$f(t) \leq \alpha_0 e^{\alpha_1 t}, \quad t \in \big[0, T_\mu\big].$$

Since the time $T_\mu = \frac{\mu}{4\beta^4\|u_0\|_{L^2}^2}$ depends only on the conserved quantity $\|u_0\|_{L^2}$, we can iterate this procedure in order to extend this solution to all positive times. Note, however, that the solution can blow up at infinity. □

4. Concluding remarks

4.1. **Global well-posedness in $L^2 \times L^2$ $(N = 2, 3)$.** We observe that following the same ideas outlined in Remark 5.5 in [10] we can extend our local results in $L^2 \times L^2$ to any positive time T.

4.2. **Global well-posedness in $H^1 \times H^1$ $(N = 1)$.** The results in [10] concerning local well-posedness in one dimension do not include the case $(u_0, v_0) \in H^1 \times L^2$. However, they include $(u_0, v_0) \in H^1 \times H^1$. Our proof in Section 3 can be adapted in order to obtain global well-posedness in this situation, as well. Indeed, putting

$$g(t) = f(t) + \|v_x(\cdot, t)\|_{L^2} = \|v(\cdot, t)\|_{L^2}^2 + \|u_x(\cdot, t)\|_{L^2}^2 + \|v_x(\cdot, t)\|_{L^2}$$

and using the inequality

$$\|u\|_{L^4}^2 \leq \|u\|_{L^\infty} \|u\|_{L^2} \leq \|u\|_{H^1} \|u\|_{L^2}$$

instead of the one-dimensional Gagliardo-Nirenberg, we obtain as in (3.55) that

$$f(t) \leq \tilde{\alpha}_0 + \tilde{\alpha}_1 \int_0^t f(t')dt', \quad \text{for all} \quad t \in \left[0, \tilde{T}\right],$$

for some $\tilde{T} = \tilde{T}(\|u_0\|_{L^2}, \mu, \beta)$ and where $\tilde{\alpha}_i$, $i = 0, 1$, depend exclusively on the initial data.

Furthermore, differentiating (3.45) with respect to x and taking the L^2-norm yield

$$\begin{aligned} \|v_x(\cdot, t)\|_{L^2} &\leq \|v_{0x}\|_{L^2} + \tfrac{2}{\mu} \int_0^t \|u(t')\|_{L^\infty} \|u_x(t')\|_{L^2} dt' \\ &\leq \|v_{0x}\|_{L^2} + \tfrac{2C}{\mu} \int_0^t \|u(t')\|_{H^1} \|u_x(t')\|_{L^2} dt' \\ &\leq \|v_{0x}\|_{L^2} + \tfrac{2C}{\mu} \left(\tfrac{\tilde{T}}{2} \|u_0\|_{L^2}^2 + \tfrac{3}{2} \int_0^t f(t')dt' \right), \end{aligned} \tag{4.56}$$

where we have used the Sobolev embedding $H^1(\mathbb{R}) \hookrightarrow L^\infty(\mathbb{R})$. Finally, we obtain

$$g(t) \leq \tilde{\alpha}_0 + \|v_{0x}\|_{L^2} + \tfrac{C\tilde{T}}{\mu} \|u_0\|_{L^2}^2 + \left(\tfrac{3C}{\mu} + \tilde{\alpha}_1 \right) \int_0^t g(t')dt', \quad \text{for all} \quad t \in \left[0, \tilde{T}\right].$$

We conclude as in Theorem 3.1.

4.3. **On the blow-up in $H^1 \times H^1$ $(N = 2)$.** Since, in two dimensions, $\|u\|_{H^1}$ does not control $\|u\|_{L^\infty}$, the considerations in the previous remark do not apply. However, our global result for initial data $(u_0, v_0) \in H^1 \times L^2$ shows that a possible blow-up in $H^1 \times H^1$ can only occur for $\|\nabla v\|_{L^2}$.

4.4. **Comparison between the cubic NLS and the Schrödinger-Debye equations.** In the next table, we summarize all known results concerning the local and global well-posedness for these equations. It illustrates the regularization induced by the delay μ in the Schrödinger-Debye system.

N	**Cubic NLS (H^s)**	**Schrödinger-Debye ($H^s \times H^\ell$)**
1	l.w.p: $s \geq 0$ ([12], [17], [5])	l.w.p: $\lvert s\rvert - \frac{1}{2} \leq \ell < \min\{s + \frac{1}{2}; 2s + \frac{1}{2}\}$ ([10])
	g.w.p: $s \geq 0$ (Conservation of L^2 norm)	g.w.p: $-\frac{3}{14} \leq \ell = s \leq 0$ ([10]) or $(s, \ell) = (1, 1)$
2	l.w.p: $s \geq 0$ ([12], [4], [5])	l.w.p: $\max\{0, s-1\} \leq \ell \leq \min\{2s, s+1\}$
	g.w.p ($\lambda = 1$): $s > \frac{1}{3}$ ([7]) g.w.p ($\lambda = \pm 1$): $s \geq 0$ for small L^2 norm ([4])	g.w.p ($\lambda = \pm 1$): $(s, \ell) = (1, 0)$ and $(s, \ell) = (0, 0)$
3	l.w.p: $s \geq \frac{1}{2}$ ([12], [5])	l.w.p: $\max\{0, s-1\} \leq \ell \leq \min\{2s, s+1\}$
	g.w.p ($\lambda = 1$): $s > \frac{4}{5}$ ([8]) g.w.p ($\lambda = \pm 1$): $s \geq \frac{1}{2}$ for small $H^{1/2}$ norm ([4])	g.w.p ($\lambda = \pm 1$): $(s, \ell) = (0, 0)$

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