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ON THE EMBEDDING OF THE ATTRACTOR GENERATED BY NAVIER-STOKES EQUATIONS INTO FINITE DIMENSIONAL SPACES

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ABSTRACT. For 2-D Navier-Stokes equations on a C^2 bounded domain Ω, a class of nonlinear homeomorphisms is constructed from the attractor of Navier-Stokes to curves in $\mathbb{R}^N$ for sufficiently large N. The construction uses an ε-net on Ω (so does not use the information "near" the boundary) and is more physically perceivable compared to abstract common embeddings.

1. Introduction

The problem of existence of an attractor for 2-D Navier-Stokes (N-S) equations, initiated by Hopf [12], was first addressed by the works of Foiaş and Prodi [5] and Ladyzhenskaya [15]. In its most general case this attractor exists in H (see Section 2 for definitions of function spaces) and has finite box-counting dimensions. As the attractor hosts the ultimate dynamics of an N-S system, there is a natural interest in mappings from the attractor into $\mathbb{R}^N$ since such an embedding might yield to a dynamically equivalent finite dimensional system of ODE's.

When a dynamical system possesses an inertial manifold (so that the mapping of the attractor into $\mathbb{R}^N$ is through a Lipschitz manifold on which the attractor lies), constructing such an equivalent finite dimensional system of ODE's is relatively easy [3]. To obtain finite dimensional dynamics in systems like 2-D Navier-Stokes, for which existence of an inertial manifold is not known, an alternate method should be devised. One such method with partial success is that of Mañé's projections. This method is a result of studying the more general problem of embedding a compact finite dimensional subset, K, of a Banach space, X, into $\mathbb{R}^N$. The first results in this direction, when X is finite dimensional, are due to Mañé [16], Ben-Artzi et al. [1] and Eden et al. [3]. They considered projections into finite dimensional subspaces of X. Foiaş and Olson [4] extend this result to the case that X is not necessarily finite dimensional. Projecting the original PDE using these maps, a finite system of ordinary differential equations is obtained. This system of ODE's, however, lacks uniqueness of solutions, in general, and hence does not yield to a dynamical system [3]. This suggests a "relaxed" definition of dynamical systems as a "generalized" dynamical system with equivalent "generalized" dynamics compared to the underlying PDE. To obtain the original "generalized" dynamics from the

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projected ODE's, the projection mappings should have Hölder inverses. This fact is shown to be true in a very general setting by Hunt and Kaloshin [13], although most of the above cited references also obtain such results for their mappings.

One problem with Mañé's projections is that the mappings and the corresponding system of ODE's do not have a physically meaningful interpretation [17, Section 16.1.1]. In this paper we construct nonlinear homeomorphisms between the attractor generated by N-S equations, $\mathcal{A}$, and curves in $\mathbb{R}^N$. Being nonlinear and having the range as curves in $\mathbb{R}^N$, these homeomorphisms do not fall into the category of the mappings considered in the works above. This class of mappings possesses a physically tangible explanation and gives a different characterization of the attractor. Finding equivalent finite dimensional dynamics through such new descriptions might be easier as they are occurring in $\mathbb{R}^N$ rather than in the Hilbert spaces in which $\mathcal{A}$ lies. The class considered here has two important properties: First, the construction does not use the fact that $\mathcal{A}$ has finite dimensions and depends mainly on the invariance of $\mathcal{A}$ with respect to the solution operator. Second, the mapping is obtained without using the information of the flow "near" boundaries, suggesting that the ultimate dynamics occur inside the domain of the flow. This result is in itself appealing, as it continues to hold in the case of nonhomogeneous (time-independent) boundary conditions (see [21] and Section 5).

The mapping is constructed using the idea of "determining nodes" introduced by Foiaş and Temam [6]. They showed that if two solutions of 2-D Navier-Stokes equations converge to each other on a (suitable) set of finite points in the domain, then the two solutions will converge. They also conjectured that these nodal values might uniquely determine the elements of $\mathcal{A}$. For the case of a periodic domain with analytic force, Friz and Robinson [8] showed that each point on the attractor can be identified by its values at N distinct points (N sufficiently large) in the domain, from which Foiaş and Temam's conjecture follows (see also [7] and [19] for related results). Our main theorem (Theorem 3.2) shows that under mild assumptions on the forcing term, a similar result to this conjecture holds with the condition that the "trajectories" passing through two elements of $\mathcal{A}$ coincide on (one of) these sets of finite points. As the values of a given trajectory at these finite number of points and at different times can be viewed as a curve in $\mathbb{R}^N$ (with N determined by the number of points), we obtain a mapping between a point of $\mathcal{A}$ and a curve in $\mathbb{R}^N$.

2. Preliminaries

The usual Lebesgue and Sobolev spaces on a domain Ω are denoted by $L^q(\Omega)$ and $W^{m,q}(\Omega)$, $1 \le q \le \infty$, $m = 0, 1, 2, \ldots$, respectively.[1] The norm associated with a normed space X is indicated by $\|\cdot\|_X$; for example, $\|\cdot\|_{W^{m,q}(\Omega)}$ is the norm in $W^{m,q}(\Omega)$. For the L^2-norm we simply use $\|\cdot\|$. $L^q(a,b;X)$, $1 \le q < \infty$, is used to indicate space of functions $f : (a,b) \to X$ such that

$$\|f\|_{L^q(a,b;X)} := \begin{cases} \left(\displaystyle\int_a^b \|f(t)\|_X^q dt \right)^{\frac{1}{q}} & 1 \le q < \infty, \\ \operatorname*{ess\,sup}_{t\in(a,b)} \|f(t)\|_X & q = \infty \end{cases}$$

is finite.

[1] We may occasionally allow m to be a noninteger when referring to fractional Sobolev spaces.

$C^m(\Omega)$ refers to the space of functions m-times differentiable with bounded derivatives in Ω. $C_0^\infty(\Omega)$ is defined to be the space of all smooth functions with compact support in Ω, and $W_0^{m,q}(\Omega)$ is the completion of $C_0^\infty(\Omega)$ in the norm of $W^{m,q}(\Omega)$. For a vector $\boldsymbol{u}$ and a space of scalar functions, X, $\boldsymbol{u} \in X$ means each component of $\boldsymbol{u}$ is in X. For $\boldsymbol{u}$, $\boldsymbol{v} \in L^2(\Omega)$, the inner product of $\boldsymbol{u}$ and $\boldsymbol{v}$ is

$$(\boldsymbol{u}, \boldsymbol{v}) = \int_\Omega u_i v_i d\boldsymbol{x},$$

where u_i is the i-th component of vector $\boldsymbol{u}$ and a summation is implied on repeated indices over all components.

Let $\mathcal{D}(\Omega) = \{\boldsymbol{u} \in C_0^\infty : \operatorname{div} \boldsymbol{u} = 0 \text{ in } \Omega\}$. Then H^m is the completion of $\mathcal{D}(\Omega)$ in the $W^{m,2}(\Omega)$ norm, with the simplified notation $H := H^0$.

2.1. **The global attractor of the Navier-Stokes equations.** Let Ω be a bounded domain in $\mathbb{R}^2$ with C^2 boundary. For homogeneous boundary conditions the N-S equations can be written in functional form (in H) as

$$\frac{\mathrm{d}\boldsymbol{u}}{\mathrm{d}t} + \nu A\boldsymbol{u} + P(\boldsymbol{u}.\nabla\boldsymbol{u}) = \boldsymbol{f}. \tag{2.1}$$

Here, $A = -P\Delta$ is the Stokes operator and P is the orthogonal projection operator from $\left(L^2(\Omega)\right)^2$ into H. The above, of course, needs to be augmented with an initial condition, $\boldsymbol{u}(\boldsymbol{x}, 0) = \boldsymbol{u}_0 \ \in H$. It is well known (see e.g. [20, 10]) that for any $\boldsymbol{u}_0 \in H$, (2.1) has a unique weak solution $\boldsymbol{u} \in L^2(0, T; H^1) \cap L^\infty(0, T; H)$ for any $T > 0$ and $\boldsymbol{f} \in H$. These solutions can be redefined on a set of times of measure zero such that $\boldsymbol{u}(t) \in H$ for all $t \in [0, T)$. We define the solution operator $S(t) : H \to H$, for all $t \geq 0$, as $S(t)\boldsymbol{v} = \boldsymbol{u}(t)$, where $\boldsymbol{u}(t)$ is the solution to (2.1) with $\boldsymbol{u}_0 = \boldsymbol{v}$. For $X \subset H$, $S(t)X := \bigcup_{\boldsymbol{v} \in X} S(t)\boldsymbol{v}$. Note that by uniqueness of the solutions, $S(t)S(s) = S(s)S(t) = S(s + t)$.

Definition 2.1. A set $A \subset H$ is called *absorbing* for the N-S equations if for any bounded set $X \in H$, there is $\tau = \tau(A, X)$ such that for any $t > \tau$, $S(t)X \subset A$.

Definition 2.2. The *global attractor* $\mathcal{A}$ (of the N-S equations) is "the" set $\mathcal{A} \in H$ with the following properties:

(1) $\mathcal{A}$ is compact.
(2) $\mathcal{A}$ is *invariant*; that is, $S(t)\mathcal{A} = \mathcal{A}, \forall t \geq 0$.
(3) For all bounded $X \subset H$,

$$\operatorname{dist}(S(t)X, \mathcal{A}) := \sup_{x \in S(t)X} \inf_{a \in \mathcal{A}} \|x - a\|_H \to 0 \qquad \text{as } t \to \infty.$$

(4) $\mathcal{A}$ is *maximal*, in the sense that there is no proper subset of $\mathcal{A}$ with the above properties.

For a bounded Ω of class C^2 as considered above, it has been shown by Foiaş and Prodi [5] (cf. Ladyzhenskaya [15]) that

$$\|S(t)\boldsymbol{u}_0\|_H \leq c_0(\|\boldsymbol{f}\|_H, \nu, \Omega), \tag{2.2}$$

$$\|S(t)\boldsymbol{u}_0\|_{H^1} \leq c_1(\varepsilon, \|\boldsymbol{f}\|_H, \nu, \Omega), \tag{2.3}$$

for all $t > \epsilon$ and $\boldsymbol{u}_0 \in H$. So $K = \{\boldsymbol{u} \in H : \|\boldsymbol{u}\|_H \leq c_0 \text{ and } \|\boldsymbol{u}\|_{H^1} \leq c_1\}$ is a compact absorbing set in H (considering a fixed ε), and hence the global attractor for the Navier-Stokes equations exists; see e.g. [18, Theorem 3.2]. The attractor generated by Navier-Stokes equations for a given ν and $\boldsymbol{f}$ on a domain Ω is referred to as $\mathcal{A}(\Omega, \nu, \boldsymbol{f})$.

2.2. **Properties of the global attractor.** As will be demonstrated, if a property holds for the solutions of the Navier-Stokes equations independent of the initial condition, it also holds for all points of the attractor, $\mathcal{A}$. We now use this fact to arrive at an attractor smooth enough to prove our following theorems. Since Ω is of class C^2, then by [10, Theorem 5.6] and the Sobolev embedding theorem (cf. [15]),

$$\sup_{t \geq \varepsilon} \|S(t)\boldsymbol{u}\|_{H^2} \leq c_2' \left(\|\boldsymbol{f}\|_H, \nu, \Omega, \varepsilon\right),$$

$$\sup_{\boldsymbol{x} \in \Omega,\ t \geq \varepsilon} \left|\frac{\partial S(t)\boldsymbol{u}}{\partial t}\right| \leq c_3' \left(\|\boldsymbol{f}\|_H, \nu, \Omega, \varepsilon\right),$$

for all $\varepsilon > 0$ and all $\boldsymbol{u} \in H$. Since the attractor is invariant, for any $t > 0$ and any point $\boldsymbol{u} \in \mathcal{A}$, we may find at least one $\boldsymbol{u}_{-t} \in \mathcal{A}$ (as will be clear in the next paragraph, $\boldsymbol{u}_{-t}$ is indeed unique) such that $\boldsymbol{u} = S(t)\boldsymbol{u}_{-t}$. So if we fix ε in the above inequalities and choose a proper t, the above inequalities hold for any $\boldsymbol{u} \in \mathcal{A}$ for that fixed ε. This removes the dependence of c_2' and c_3' on ε, and we get

$$\|S(t)\boldsymbol{u}\|_{H^2} \leq c_2 \left(\|\boldsymbol{f}\|_H, \nu, \Omega\right), \tag{2.4}$$

$$\sup_{\boldsymbol{x} \in \Omega,\ t \in \mathbb{R}} \left|\frac{\partial S(t)\boldsymbol{u}}{\partial t}\right| \leq c_3 \left(\|\boldsymbol{f}\|_H, \nu, \Omega\right), \tag{2.5}$$

for all $\boldsymbol{u} \in \mathcal{A}$. Similarly, (2.2) and (2.3) hold for all $\boldsymbol{u}_0 \in \mathcal{A}$.

One of the central ideas in what follows is the notion of time on the attractor. In fact, for initial conditions $\boldsymbol{u}_0 \in \mathcal{A}$, it is possible to extend the definition of the solution operator, $S(t)$, to include negative values of time. One consequence of continuous dependence result of Knops and Payne [14] (see Lemma 4.2) is backward uniqueness for solutions of 2-D Navier-Stokes equations (at least for the ones that are as smooth as the points of the attractor constructed above), which concludes that $S(t)$ is injective: $\boldsymbol{u}_0 = \boldsymbol{v}_0$ if $S(t)\boldsymbol{u}_0 = S(t)\boldsymbol{v}_0$, for some $t > 0$. Since the attractor is invariant in time, for any $t > 0$ and $\boldsymbol{u} \in \mathcal{A}$, we set $S(-t)\boldsymbol{u} = \boldsymbol{u}_{-t}$ where $\boldsymbol{u}_{-t} \in \mathcal{A}$ satisfies $S(t)\boldsymbol{u}_{-t} = \boldsymbol{u}$ [18]. Note that this does not imply that the Navier-Stokes equations are solvable backwards in time.

3. Construction of the mappings

In this section we construct a mapping from the attractor into curves in $\mathbb{R}^N$ for N sufficiently large. For $\boldsymbol{u} \in \mathcal{A}$, the following lemma due to Foiaş and Temam provides us with an estimate for $\|\boldsymbol{u}\|_H$ in terms of values of $\boldsymbol{u}$ on a set of discrete points in Ω.

Lemma 3.1 (Foiaş and Temam [6]). *Let Ω be a bounded Lipschitz domain in $\mathbb{R}^2$, with $\mathcal{E}_\Omega$ an ε-net over Ω. Then for any $\boldsymbol{u} \in W^{m,2}$, $m > 1$,*

$$\|\boldsymbol{u}\|_{L^2(\Omega)} \leq c_4\, \eta^{(\mathcal{E}_\Omega)}(\boldsymbol{u}) + c_5\, \varepsilon^{\alpha}\, \|\boldsymbol{u}\|_{W^{m,2}(\Omega)}, \tag{3.1}$$

where the c_i's are positive constants, $0 < \alpha \leq m - 1$ *and*

$$\eta^{(\mathcal{E}_\Omega)}(\boldsymbol{u}) = \max_{\boldsymbol{x}_n \in \mathcal{E}_\Omega} |\boldsymbol{u}(\boldsymbol{x}_n)|. \tag{3.2}$$

In view of the above lemma, we show in the next theorem that we can find a finite number of points in Ω, away from the boundary, such that if the trajectories (for $t < 0$) passing through two points, $\boldsymbol{u}$ and $\boldsymbol{v}$, coincide on these points, then $\boldsymbol{u} = \boldsymbol{v}$.

Theorem 3.2. *Let $\mathcal{A}(\Omega, \nu, \boldsymbol{f})$ be the attractor generated by the Navier-Stokes equations under the assumptions of Section 2.2. Then there is an ε-net, $\mathcal{E}_\Omega$, over Ω such that if for $\boldsymbol{u}^*$, $\boldsymbol{v}^* \in \mathcal{A}$,*

$$\eta^{(\mathcal{E}_\Omega)} (S(-t)\boldsymbol{u}^* - S(-t)\boldsymbol{v}^*) = 0, \qquad \forall\, t > t_0 \geq 0,$$

then $\boldsymbol{u}^* = \boldsymbol{v}^*$.

Proof. The proof is in line with the proof of Theorem 3.1 of [6]. For $\tau > t_0$, consider the following Navier-Stokes equations on the attractor for $0 \leq t \leq \tau - t_0$:

$$\begin{aligned} &\frac{\mathrm{d}\boldsymbol{u}}{\mathrm{d}t} + \nu A\boldsymbol{u} + P(\boldsymbol{u}.\nabla\boldsymbol{u}) = \boldsymbol{f}, && \frac{\mathrm{d}\boldsymbol{v}}{\mathrm{d}t} + \nu A\boldsymbol{v} + P(\boldsymbol{v}.\nabla\boldsymbol{v}) = \boldsymbol{f}, \\ &\boldsymbol{u}(0) = S(-\tau)\boldsymbol{u}^* = \boldsymbol{u}_0, && \boldsymbol{v}(0) = S(-\tau)\boldsymbol{v}^* = \boldsymbol{v}_0. \end{aligned}$$

Subtracting the two equations, with $\boldsymbol{w} = \boldsymbol{u} - \boldsymbol{v}$ and $\boldsymbol{w}_0 = \boldsymbol{u}_0 - \boldsymbol{v}_0$, we get

$$\begin{aligned} &\frac{\mathrm{d}\boldsymbol{w}}{\mathrm{d}t} + \nu A\boldsymbol{w} + P(\boldsymbol{u}.\nabla\boldsymbol{w}) + P(\boldsymbol{w}.\nabla\boldsymbol{v}) = 0, \\ &\boldsymbol{w}(0) = \boldsymbol{w}_0. \end{aligned} \tag{3.3}$$

Taking the inner product of the above equation with $A\boldsymbol{w}$ over Ω, we find

$$(\frac{\mathrm{d}\boldsymbol{w}}{\mathrm{d}t}, A\boldsymbol{w}) + \nu(A\boldsymbol{w}, A\boldsymbol{w}) + (P(\boldsymbol{u}.\nabla\boldsymbol{w}), A\boldsymbol{w}) + (P(\boldsymbol{w}.\nabla\boldsymbol{v}), A\boldsymbol{w}) = 0.$$

Since for smooth solutions under consideration

$$(\frac{\mathrm{d}\boldsymbol{w}}{\mathrm{d}t}, A\boldsymbol{w}) = \frac{1}{2}\frac{d}{dt}\|\nabla\boldsymbol{w}\|^2,$$

for all t, we have

$$\begin{aligned} \frac{1}{2}\frac{d}{dt}\|\nabla\boldsymbol{w}\|^2_{L^2(\Omega)} + \nu\|A\boldsymbol{w}\|^2_{L^2(\Omega)} &\leq |(\boldsymbol{u}.\nabla\boldsymbol{w}, A\boldsymbol{w})| + |(\boldsymbol{w}.\nabla\boldsymbol{v}, A\boldsymbol{w})| \\ &\leq c_1'\|\boldsymbol{u}\|_{W^{2,2}(\Omega)}\,\|\boldsymbol{w}\|_{W^{1,2}(\Omega)}\,\|A\boldsymbol{w}\|_{L^2(\Omega)} \\ &\quad + c_1'\|\boldsymbol{w}\|_{W^{1,2}(\Omega)}\,\|\boldsymbol{v}\|_{W^{2,2}(\Omega)}\,\|A\boldsymbol{w}\|_{L^2(\Omega)}, \end{aligned}$$

where we have used the inequality (see e.g. [2])

$$|(\boldsymbol{u}_1.\nabla\boldsymbol{u}_2, \boldsymbol{u}_3)| \leq c'\|\boldsymbol{u}_1\|_{W^{m_1,2}}\,\|\boldsymbol{u}_2\|_{W^{m_2+1,2}}\,\|\boldsymbol{u}_3\|_{W^{m_3,2}}, \tag{3.4}$$

for $m_1 + m_2 + m_3 > 1$. Note that $\|A\boldsymbol{w}\|_{L^2(\Omega)}$ is a norm equivalent to $\|\boldsymbol{w}\|_{W^{2,2}(\Omega)}$ [20], and by (2.4) we obtain

$$\begin{aligned} \frac{1}{2}\frac{d}{dt}\|\nabla\boldsymbol{w}\|^2_{L^2(\Omega)} + \nu\|A\boldsymbol{w}\|^2_{L^2(\Omega)} &\leq c_2'\|\boldsymbol{w}\|_{W^{1,2}(\Omega)}\,\|A\boldsymbol{w}\|_{L^2(\Omega)} \\ &\leq \frac{(c_2')^2}{2\varepsilon_1}\|\boldsymbol{w}\|^2_{W^{1,2}(\Omega)} + \frac{\varepsilon_1}{2}\|A\boldsymbol{w}\|^2_{L^2(\Omega)}, \end{aligned}$$

where we have used Young's inequality in the last step. Now choosing ε_1 such that $c'_4 = \nu - \varepsilon_1/2 > 0$ and using the interpolation inequality

$$\|\boldsymbol{w}\|^2_{W^{1,2}(\Omega)} \leq c'_5 \|\boldsymbol{w}\|_{L^2(\Omega)} \|\boldsymbol{w}\|_{W^{2,2}(\Omega)}$$

and Lemma 3.1, we get

$$\begin{aligned}
\frac{1}{2}\frac{d}{dt}\|\nabla \boldsymbol{w}\|^2_{L^2(\Omega)} &+ c'_4\|A\boldsymbol{w}\|^2_{L^2(\Omega)} \\
&\leq \frac{(c'_2)^2 c'_5}{2\varepsilon_1}\|\boldsymbol{w}\|_{L^2(\Omega)}\|\boldsymbol{w}\|_{W^{2,2}(\Omega)} \\
&\leq \frac{(c'_2)^2 c'_5}{2\varepsilon_1}\left(c_4\eta^{(\mathcal{E}_\Omega)}(\boldsymbol{w}) + c_5\varepsilon\|\boldsymbol{w}\|_{W^{2,2}(\Omega)}\right)\|\boldsymbol{w}\|_{W^{2,2}(\Omega)} \\
&\leq \frac{(c'_2)^2 c'_5 c_4}{2\varepsilon_1}\eta^{(\mathcal{E}_\Omega)}(\boldsymbol{w})\,\|\boldsymbol{w}\|_{W^{2,2}(\Omega)} + c'_6\varepsilon\|A\boldsymbol{w}\|^2_{L^2(\Omega)}. \qquad (3.5)
\end{aligned}$$

If we pick ε such that $c'_7 = c'_4 - c'_6\varepsilon > 0$ and if on such a ε-net the assumption of the theorem is satisfied (that is, (3.2) holds), then it follows that $\eta^{(\mathcal{E}_\Omega)}(\boldsymbol{w}) = 0$ for all $0 \leq t < \tau - t_0$. So the above inequality reads

$$\frac{1}{2}\frac{d}{dt}\|\nabla \boldsymbol{w}\|^2_{L^2(\Omega)} + c'_7\|A\boldsymbol{w}\|^2_{L^2(\Omega)} \leq 0, \qquad 0 \leq t < \tau - t_0,$$

or

$$\frac{d}{dt}\|\nabla \boldsymbol{w}\|^2_{L^2(\Omega)} + c'_8\|\nabla \boldsymbol{w}\|^2_{L^2(\Omega)} \leq 0, \qquad 0 \leq t < \tau - t_0,$$

which upon integration over $[0, \tau - t_0)$ gives

$$e^{c'_8(\tau - t_0)}\,\|\nabla \boldsymbol{w}(\tau - t_0)\|^2 - \|\nabla \boldsymbol{w}(0)\|^2 \leq 0.$$

Since $\boldsymbol{w}(\tau - t_0) = S(-t_0)\boldsymbol{u}^* - S(-t_0)\boldsymbol{v}^*$ and $\boldsymbol{w}(0) = S(-\tau)\boldsymbol{u}^* - S(-\tau)\boldsymbol{v}^*$, we get

$$e^{c'_8(\tau - t_0)}\,\|\nabla\left(S(-t_0)\boldsymbol{u}^* - S(-t_0)\boldsymbol{v}^*\right)\|^2 \leq \|\nabla\left(S(-\tau)\boldsymbol{u}^* - S(-\tau)\boldsymbol{v}^*\right)\|^2.$$

By (2.3) it follows that

$$\|\nabla\left(S(-t_0)\boldsymbol{u}^* - S(-t_0)\boldsymbol{v}^*\right)\|^2 \leq c_2\, e^{-c'_8(\tau - t_0)},$$

which in the limit $\tau \to \infty$ gives $\|\nabla\left(S(-t_0)\boldsymbol{u}^* - S(-t_0)\boldsymbol{v}^*\right)\| = 0$ and by Poincaré inequality yields $S(-t_0)\boldsymbol{u}^* = S(-t_0)\boldsymbol{v}^*$, but then

$$\boldsymbol{u}^* = S(t_0)S(-t_0)\boldsymbol{u}^* = S(t_0)S(-t_0)\boldsymbol{v}^* = \boldsymbol{v}^*.$$

□

The proof of Theorem 3.2 gives conditions for ε such that if (3.2) holds (on $\mathcal{E}_\Omega$), then the statement of the theorem follows and hence leads to the following definition:

Definition 3.3. An ε-net on Ω is called a *qualified net* if $\varepsilon < (\nu/c'_6)$, where c'_6 is given in the proof of Theorem 3.2.

Consider the attractor $\mathcal{A}(\Omega, \nu, \boldsymbol{f})$ generated by the Navier-Stokes equations and a qualified net $\mathcal{E}_\Omega = \{\boldsymbol{x}_n : 1 \leq n \leq N\}$. In the following, we construct a mapping $\gamma : \boldsymbol{u} \longrightarrow \gamma(\boldsymbol{u})$, where $\gamma(\boldsymbol{u})$ is a curve in $(\mathbb{R}^2)^{N(\mathcal{E}_\Omega)}$, parametrized naturally by time t. At each fixed time, the point of $(\mathbb{R}^2)^{N(\mathcal{E}_\Omega)}$ that is on the curve has coordinates that specify the values of the solution $\boldsymbol{u}$ at the points of $\mathcal{E}_\Omega$. Also, note that it follows immediately from the continuity of the solution of 2-D N-S equations that $\gamma(\boldsymbol{u})$ is continuous with respect to its parameter t.

For $\boldsymbol{u} \in \mathcal{A}$ and $t \in \mathbb{R}$ let $\gamma^{(\mathcal{E}_\Omega)}(\boldsymbol{u})(t) \in (\mathbb{R}^2)^{N(\mathcal{E}_\Omega)}$ be the point of $(\mathbb{R}^2)^{N(\mathcal{E}_\Omega)}$ whose $(2i-1)$-component is given by the first component of the vector $(S(t)\boldsymbol{u})(\boldsymbol{x}_n)$ and whose $(2i)$-component is the second component of $(S(t)\boldsymbol{u})(\boldsymbol{x}_n)$, $1 \leq i \leq N(\mathcal{E}_\Omega)$. This gives a map from $(\boldsymbol{u}, t) \in \mathcal{A} \times \mathbb{R}$ into $\gamma^{(\mathcal{E}_\Omega)}(\boldsymbol{u})(t) \in (\mathbb{R}^2)^{N(\mathcal{E}_\Omega)}$. Since

$$\begin{aligned} \eta^{(\mathcal{E}_\Omega)}(S(t)\boldsymbol{u}) &= \max_{\boldsymbol{x}_n \in \mathcal{E}_\Omega} |(S(t)\boldsymbol{u})(\boldsymbol{x}_n)| \leq \sqrt{\sum_{\boldsymbol{x}_n \in \mathcal{E}_\Omega} |(S(t)\boldsymbol{u})(\boldsymbol{x}_n)|^2} = |\gamma^{(\mathcal{E}_\Omega)}(\boldsymbol{u})(t)| \\ &\leq \sqrt{N(\mathcal{E}_\Omega)} \max_{\boldsymbol{x}_n \in \mathcal{E}_\Omega} |(S(t)\boldsymbol{u})(\boldsymbol{x}_n)| = \sqrt{N(\mathcal{E}_\Omega)}\ \eta^{(\mathcal{E}_\Omega)}(S(t)\boldsymbol{u}), \end{aligned} \tag{3.6}$$

it follows that on the set of points $\gamma^{(\mathcal{E}_\Omega)}(\boldsymbol{u})(t)$, for any $\boldsymbol{u}$ and t, $\eta^{(\mathcal{E}_\Omega)}(S(t)\boldsymbol{u})$ is a norm equivalent to the Euclidean norm.

Let $\gamma^{(\mathcal{E}_\Omega)}(\boldsymbol{u}) = \bigcup_{t<0} \gamma^{(\mathcal{E}_\Omega)}(\boldsymbol{u})(t)$, so $\gamma^{(\mathcal{E}_\Omega)}(\boldsymbol{u})$ is a curve in $(\mathbb{R}^2)^{N(\mathcal{E}_\Omega)}$ which is obtained by the projection on $\mathcal{E}_\Omega$ of the trajectory passing through $\boldsymbol{u}$ at $t = 0$. We write $\gamma^{(\mathcal{E}_\Omega)}(\boldsymbol{u}) = \gamma^{(\mathcal{E}_\Omega)}(\boldsymbol{v})$ if $\gamma^{(\mathcal{E}_\Omega)}(\boldsymbol{u})(t) = \gamma^{(\mathcal{E}_\Omega)}(\boldsymbol{v})(t)$ for all $t < t_0 \leq 0$. Let Γ be the set of all curves $\lambda : (-\infty, 0) \to (\mathbb{R}^2)^{N(\mathcal{E}_\Omega)}$ of class C^1. Then by (3.6) and Theorem 3.2, we observe that the mapping

$$\gamma^{(\mathcal{E}_\Omega)} : \boldsymbol{u} \in \mathcal{A} \longrightarrow \gamma^{(\mathcal{E}_\Omega)}(\boldsymbol{u}) \in \Gamma$$

is injective.

Since it is possible to have many qualified nets, $\mathcal{E}_\Omega$, and a corresponding $\gamma^{(\mathcal{E}_\Omega)}$, we obtain a class of mappings which then by the results of the next section are unique up to a homeomorphism. When there is no confusion about the underlying qualified net for a mapping, we use a simpler notation $\gamma(\boldsymbol{u})$ to refer to the image of $\boldsymbol{u}$.

Remark 3.4. It is always possible to choose a *qualified net*, $\mathcal{E}_\Omega$, such that

$$\operatorname{dist}(\boldsymbol{x}, \partial\Omega) > \varepsilon, \qquad \text{for all } \boldsymbol{x} \in \mathcal{E}_\Omega.$$

So to construct the mapping, no information is needed "near" the boundary.

4. Properties of mappings $\gamma^{(\mathcal{E}_\Omega)}$

Here we show continuity of $\gamma^{(\mathcal{E}_\Omega)}$ and continuity of its inverse. Continuity has the major consequence that the range of the mappings is a compact subset of $\mathbb{R}^N$, and together with the continuity of the inverse, it shows that the ranges of $\gamma^{(\mathcal{E}_\Omega)}$ and $\gamma^{(\mathcal{E}'_\Omega)}$ corresponding to two qualified nets $\mathcal{E}_\Omega$ and $\mathcal{E}'_\Omega$ are homeomorphic. To proceed with continuity results, we first introduce a topology on Γ. The following construction is along the ideas of Galdi and Rionero [11].

For any $-\tau \in (-\infty, 0)$,

$$d_{-\tau}(\gamma_1, \gamma_2) = \int_{-\tau}^{0} |\gamma_1(t) - \gamma_2(t)|\, dt, \qquad \gamma_1, \gamma_2 \in \Gamma$$

(as can be easily verified), defines a pseudo-metric in Γ (the set of all C^1 curves in $(\mathbb{R}^2)^{N(\mathcal{E}_\Omega)}$ defined on $(-\infty, 0)$). Here $|\cdot|$ is the Euclidean norm.

Let $h : (0,1) \to (-\infty, 0)$ be defined by $h(s) = 1 - 1/s^\alpha$ for some $0 < \alpha$.[2] For any $\gamma \in \Gamma$ and $0 < \varepsilon < 1$, let $B_\varepsilon^{(\gamma)} = \{\gamma' \in \Gamma : d_{h(\varepsilon)}(\gamma, \gamma') < \varepsilon\}$, and consider the

[2] h can be any nondecreasing function as long as the construction of a topology is considered. The specific form assumed here is merely for ease of algebraic operations in the theorems that follow.

following family of sets:

$$\mathcal{B} = \{B_\varepsilon^{(\gamma)} : \gamma \in \Gamma,\ 0 < \varepsilon < 1\}.$$

Next, observe that $\mathcal{B}$ can serve as a base for a topology on Γ [11]. Designating the empty set and all sets representable as a union of sets of $\mathcal{B}$ as "open", we arrive at a topology $\mathcal{T}$ on Γ.

The family of pseudo-metrics $d_{h(s)}$, $0 < s < 1$, can be used to define the following metric on Γ:

$$d(\gamma_1, \gamma_2) = \int_0^1 \frac{d_{h(s)}(\gamma_1, \gamma_2)}{1 + d_{h(s)}(\gamma_1, \gamma_2)} ds.$$

Let us denote by $\mathcal{T}_w$ the topology induced by the metric d on Γ; then we have [11].

Lemma 4.1. *$\mathcal{T}_w$ is weaker than $\mathcal{T}$.*

The following logarithmic convexity argument of Knops and Payne [14] plays a key role in proving continuity:

Lemma 4.2. *Let $\mathcal{A}(\Omega, \nu, \boldsymbol{f})$ be the attractor generated by Navier-Stokes equations (under the assumptions of Section 2.2). For $\boldsymbol{u}, \boldsymbol{v} \in \mathcal{A}$ and $t_1, t_2 \in \mathbb{R}$ with $t_1 < t_2$, $\boldsymbol{w}(t) = S(t)\boldsymbol{u} - S(t)\boldsymbol{v}$ satisfies*

$$\|\boldsymbol{w}(t)\|_{L^2(\Omega)} \leq \exp\left(\frac{c_7}{2c_6}\left(t - \lambda t_1 - (1-\lambda)t_2\right)\right) \|\boldsymbol{w}(t_1)\|^{\lambda}_{L^2(\Omega)} \|\boldsymbol{w}(t_2)\|^{(1-\lambda)}_{L^2(\Omega)},$$

for any $t_1 < t < t_2$ with $\lambda = \dfrac{e^{c_6 t} - e^{c_6 t_2}}{e^{c_6 t_1} - e^{c_6 t_2}}$.

In their proof, Knops and Payne require that one of $\nabla \boldsymbol{u} - (\nabla \boldsymbol{u})^T$ or $\nabla \boldsymbol{v} - (\nabla \boldsymbol{v})^T$ belongs to $L^\infty(\Omega \times (t_1, t_2))$, but the proof can be slightly changed to obtain the same result using (2.3) and (2.4) instead. In view of the above lemma we immediately have the following continuity result:

Theorem 4.3. *The mapping $\gamma^{(\mathcal{E}_\Omega)} : \mathcal{A} \to \Gamma$ is continuous if $\mathcal{A}$ is endowed with the topology of $L^2(\Omega)$ and Γ with the topology $\mathcal{T}$.*

Proof. Let $B \in \mathcal{T}$ be an open neighborhood of $\gamma(\boldsymbol{u})$. Then since $\mathcal{B}$ is a base for $\mathcal{T}$, it follows that there is $0 < \varepsilon$ (< 1) such that $B_\varepsilon^{(\gamma(\boldsymbol{u}))} \subset B$. So we need to show that there is $\delta = \delta(\varepsilon)$ such that

$$\boldsymbol{v} :\ \|\boldsymbol{u} - \boldsymbol{v}\| < \delta \Rightarrow d_{h(\varepsilon)}(\gamma(\boldsymbol{u}), \gamma(\boldsymbol{v})) < \varepsilon.$$

But by (3.6) and the Sobolev embedding theorem, we have

$$\begin{aligned} d_{h(\varepsilon)}(\gamma(\boldsymbol{u}), \gamma(\boldsymbol{v})) &= \int_{h(\varepsilon)}^0 |\gamma(\boldsymbol{u})(t) - \gamma(\boldsymbol{v})(t)| dt \\ &\leq \sqrt{N(\mathcal{E}_\Omega)} \int_{h(\varepsilon)}^0 \eta^{(\mathcal{E}_\Omega)}(S(t)\boldsymbol{u} - S(t)\boldsymbol{v}) dt \\ &\leq \sqrt{N(\mathcal{E}_\Omega)} \int_{h(\varepsilon)}^0 c_9' \|S(t)\boldsymbol{u} - S(t)\boldsymbol{v}\|_{W^{m,2}(\Omega)} dt, \end{aligned}$$

for some $1 < m < 2$. Using the more compact notation $\boldsymbol{w}(t) = S(t)\boldsymbol{u} - S(t)\boldsymbol{v}$ and the interpolation inequality

$$\|\boldsymbol{w}\|_{W^{m,2}(\Omega)} \leq c_{10}' \|\boldsymbol{w}\|^{\frac{m}{2}}_{W^{2,2}(\Omega)} \|\boldsymbol{w}\|^{1-\frac{m}{2}},$$

we obtain

$$d_{h(\varepsilon)}(\gamma(\boldsymbol{u}),\gamma(\boldsymbol{v})) \leq \sqrt{N(\mathcal{E}_\Omega)} \int_{h(\varepsilon)}^0 c_9' c_{10}' \|\boldsymbol{w}(t)\|_{W^{2,2}(\Omega)}^{\frac{m}{2}} \|\boldsymbol{w}(t)\|^{1-\frac{m}{2}} dt$$
$$\leq c_{11}' \int_{h(\varepsilon)}^0 \|\boldsymbol{w}(t)\|^{1-\frac{m}{2}} dt,$$

where we have used (2.4) in the last step.

For the given ε, let us use Lemma 4.2 with $t_1 = h(\varepsilon)$, $t_2 = 0$ and $\lambda = \dfrac{e^{c_6 t}-1}{e^{c_6 t_1}-1}$ to get

$$d_{h(\varepsilon)}(\gamma(\boldsymbol{u}),\gamma(\boldsymbol{v})) \leq c_{11}' \int_{h(\varepsilon)}^0 \left[\exp\left(\frac{c_7}{2c_6}(t-\lambda t)\right) \|\boldsymbol{w}(t_1)\|^{\lambda} \|\boldsymbol{w}(0)\|^{1-\lambda}\right]^{1-\frac{m}{2}} dt$$
$$\leq c_{12}'(t_1)\, \|\boldsymbol{w}(0)\| \int_{h(\varepsilon)}^0 e^{c_6 t} \|\boldsymbol{w}(0)\|^{(1-\frac{m}{2})\frac{1-e^{c_6 t}}{e^{c_6 t_1}-1}} dt$$
$$\leq c_{13}'(t_1) \frac{1}{|\ln \|\boldsymbol{w}(0)\||}$$

again, with the help of (2.2) and bounds for various powers of e in the interval $(t_1, 0)$. Noting that $\|\boldsymbol{w}(0)\| = \|\boldsymbol{u}-\boldsymbol{v}\|$, choosing

$$\delta < \exp\left(\min\{-1, -\frac{c_{13}'(t_1)}{\varepsilon}\}\right)$$

completes the proof. □

Remark 4.4. By Lemma 4.1, the mapping $\gamma^{(\mathcal{E}_\Omega)}$ is also continuous when Γ is furnished with the topology $\mathcal{T}_w$. Also, the topology on $\mathcal{A}$ can be replaced by a stronger topology, the most interesting of them being that of H^1.

Theorem 4.5. *The inverse of mapping $\gamma^{(\mathcal{E}_\Omega)}$ is continuous when Γ is furnished with the topology $\mathcal{T}_w$ and $\mathcal{A}$ with the topology of H^1.*

Proof. Let $\boldsymbol{u}, \boldsymbol{v} \in \mathcal{A}$. Then for $\boldsymbol{w}(t) = S(t)\boldsymbol{u} - S(t)\boldsymbol{v}$ by (3.5) and (2.4) we have

$$\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\|\nabla \boldsymbol{w}(t)\|^2 + c_7' \|A\boldsymbol{w}(t)\|^2 \leq c_{14}' \eta^{(\mathcal{E}_\Omega)}(\boldsymbol{w}(t))$$

or[3]

$$\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\|\nabla \boldsymbol{w}(t)\|^2 + c_8' \|\nabla \boldsymbol{w}(t)\|^2 \leq c_{14}' \eta(\boldsymbol{w}(t)).$$

Integrating the above for $h(s) \leq t < 0$ with the initial condition $\boldsymbol{w}(h(s)) = S(h(s))\boldsymbol{u} - S(h(s))\boldsymbol{v}$, we obtain

$$\|\nabla \boldsymbol{w}(0)\|^2 - e^{c_8' h(s)} \|\nabla \boldsymbol{w}(h(s))\|^2 \leq c_{14}' \int_{h(s)}^0 e^{c_8' t} \eta(\boldsymbol{w}(t)) dt \leq c_{14}' \int_{h(s)}^0 \eta(\boldsymbol{w}(t)) dt.$$

Noting that $\boldsymbol{w}(0) = \boldsymbol{u}-\boldsymbol{v}$ and using (2.3), the above inequality yields

$$\|\nabla(\boldsymbol{u}-\boldsymbol{v})\|^2 \leq c_{15}' e^{c_8' h(s)} + c_{14}' \int_{h(s)}^0 \eta(\boldsymbol{w}(t)) dt.$$

[3]We will use the simpler notation $\eta(\boldsymbol{w}(t))$ when the underlying ε-net on Ω is fixed without any confusion.

For any $0 < s' < s$, since $h(s') < h(s)$ we can increase the last term to obtain

$$\|\nabla(\boldsymbol{u}-\boldsymbol{v})\|^2 \leq c'_{15} e^{c'_8 h(s)} + c'_{14} \int_{h(s')}^{0} \eta(\boldsymbol{w}(t))dt,$$

and hence dividing by $1 + \int_{h(s')}^{0} \eta(\boldsymbol{w}(t))dt$ yields

$$\frac{\|\nabla(\boldsymbol{u}-\boldsymbol{v})\|^2}{1 + \int_{h(s')}^{0} \eta(\boldsymbol{w}(t))dt} \leq \frac{c'_{15} e^{c'_8 h(s)}}{1 + \int_{h(s')}^{0} \eta(\boldsymbol{w}(t))dt} + c'_{14} \frac{\int_{h(s')}^{0} \eta(\boldsymbol{w}(t))dt}{1 + \int_{h(s')}^{0} \eta(\boldsymbol{w}(t))dt}. \tag{4.1}$$

Using (2.4), (2.2) and the Sobolev embedding theorem, it follows that $\eta(\boldsymbol{w}(t)) \leq c'_{16}$, so for the term on the left hand side of the above we have

$$\frac{\|\nabla(\boldsymbol{u}-\boldsymbol{v})\|^2}{1 + \int_{h(s')}^{0} \eta(\boldsymbol{w}(t))dt} \geq \frac{\|\nabla(\boldsymbol{u}-\boldsymbol{v})\|^2}{1 - c'_{16} h(s')} \geq \frac{1}{c'_{17}} \frac{\|\nabla(\boldsymbol{u}-\boldsymbol{v})\|^2}{1 - h(s')} = \frac{(s')^\alpha}{c'_{17}} \|\nabla(\boldsymbol{u}-\boldsymbol{v})\|^2;$$

hence, (4.1) implies

$$\frac{(s')^\alpha}{c'_{17}} \|\nabla(\boldsymbol{u}-\boldsymbol{v})\|^2 \leq c'_{15} e^{c'_8 h(s)} + c'_{14} \frac{\int_{h(s')}^{0} \eta(\boldsymbol{w}(t))dt}{1 + \int_{h(s')}^{0} \eta(\boldsymbol{w}(t))dt}.$$

Taking the square root of both sides and integrating for $0 < s' < s$, we obtain

$$\begin{aligned}(s')^{1+\alpha/2} \|\nabla(\boldsymbol{u}-\boldsymbol{v})\| &\leq c'_{18}\, s'\, e^{c'_8 h(s)/2} + c'_{19} \int_0^s \left[\frac{\int_{h(s')}^{0} \eta(\boldsymbol{w}(t))dt}{1 + \int_{h(s')}^{0} \eta(\boldsymbol{w}(t))dt} \right]^{\frac{1}{2}} ds' \\ &\leq c'_{18}\, s'\, e^{c'_8 h(s)/2} + c'_{20} \left(\int_0^s ds' \right)^{\frac{1}{2}} [d(\gamma(\boldsymbol{u}), \gamma(\boldsymbol{v}))]^{\frac{1}{2}},\end{aligned}$$

where we have used Hölder inequality and (3.6) in the last step along with the fact that since the integrand in the last integral is positive, the limit of the integral can be increased from s to 1. So,

$$\|\nabla(\boldsymbol{u}-\boldsymbol{v})\| \leq c'_{18} \frac{e^{c'_8 h(s)/2}}{s^{(\alpha/2)}} + \frac{c'_{20}}{s^{(\alpha+1)/2}} [d(\gamma(\boldsymbol{u}), \gamma(\boldsymbol{v}))]^{\frac{1}{2}},$$

and continuity follows once for a given ε we choose s such that

$$c'_{18} \frac{e^{c'_8 h(s)/2}}{s^{(\alpha/2)}} < \frac{\varepsilon}{2}$$

and δ such that

$$\delta < \frac{\varepsilon^2\, s^{\alpha+1}}{(2c'_{20})^2}.$$

□

5. Non-homogeneous boundary conditions

For a Lipschitz domain $\Omega \subset \mathbb{R}^2$, let $B^{m-\frac{1}{2},2}(\partial\Omega)$ be the space of traces of functions in $W^{m,2}(\Omega)$, $1 \leq m < \infty$. Consider the nonhomogeneous Navier-Stokes equations (as in Section 2.2 on a bounded C^2 domain):

$$\begin{aligned} \frac{\partial \boldsymbol{u}}{\partial t} + \boldsymbol{u}.\nabla \boldsymbol{u} &= \nabla p + \nu \Delta \boldsymbol{u} + \boldsymbol{f}, \\ \operatorname{div} \boldsymbol{u} &= 0, \end{aligned} \tag{5.1}$$

with initial condition $\boldsymbol{u}(0) = \boldsymbol{u}_0 \in H$ and boundary condition

$$\boldsymbol{u}(\boldsymbol{x}) = \boldsymbol{u}^*(\boldsymbol{x}), \qquad \boldsymbol{x} \in \partial\Omega,\ t > 0.$$

For compatibility we, of course, require

$$\int_{\partial\Omega} \boldsymbol{u}^*.\boldsymbol{n} = 0,$$

and to avoid unnecessary complications, let us assume $\partial\Omega$ is connected [9, Section VIII.3]. To show the existence of weak solutions to (5.1), let us assume $\boldsymbol{f} \in H$ and, following Temam [21], a time independent boundary condition $\boldsymbol{u}^* \in B^{\frac{1}{2},2}(\partial\Omega)$. This ensures that for any $\alpha > 0$, $\boldsymbol{u}^*$ has an extension, $\boldsymbol{V} \in W^{1,2}(\Omega)$, such that [9]

$$-(\boldsymbol{w}.\nabla \boldsymbol{V}, \boldsymbol{w}) \leq \alpha \|w\|_{H^1}, \qquad \text{for all } \boldsymbol{w} \in H^1(\Omega).$$

If we write $\boldsymbol{u} = \boldsymbol{v} + \boldsymbol{V}$, then (5.1) has a weak solution if there is a weak solution, $\boldsymbol{v}$, to

$$\begin{aligned} \frac{\partial \boldsymbol{v}}{\partial t} - \nu\Delta \boldsymbol{v} + \boldsymbol{v}.\nabla \boldsymbol{v} &= \nabla p + \nu \Delta \boldsymbol{V} - \boldsymbol{v}.\nabla \boldsymbol{V} - \boldsymbol{V}.\nabla \boldsymbol{v} - \boldsymbol{V}.\nabla \boldsymbol{V} + \boldsymbol{f}, \\ \operatorname{div} \boldsymbol{v} &= 0, \end{aligned} \tag{5.2}$$

with initial condition $\boldsymbol{v}(0) = \boldsymbol{u}_0 - \boldsymbol{V}$ and homogeneous boundary conditions. A weak solution to (5.2) is defined (similar to Navier-Stokes equations) as a function $\boldsymbol{v} \in L^2(0, T; H^1) \cap L^\infty(0, T; H)$ such that it satisfies the above after taking its inner product with a test function and integrating by parts. The existence of such a solution, satisfying (2.2) and (2.3), and hence the existence of the attractor are shown by Temam [21].

To obtain the regularity results of (2.4) and (2.5), let us assume $\boldsymbol{u}^* \in B^{\frac{3}{2},2}(\partial\Omega)$, which guarantees that $\boldsymbol{V}$ can be chosen such that $\boldsymbol{V} \in W^{2,2}(\Omega)$. Then (2.4) and (2.5) follow by the same argument as the homogeneous case once we note that Lemmas 5.4 and 5.5 of [10] continue to hold when Navier-Stokes equations are replaced by (5.2) after obvious modifications of the proofs. Then all other theorems and lemmas in previous sections will be valid without any change, as they are either based on the equation for evolution of the difference between two solutions of N-S equations (and the boundary condition for such an equation is always homogeneous) and/or they use estimates (2.2), (2.3), (2.4) and (2.5).

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