HEISENBERG UNIQUENESS PAIRS IN THE PLANE.
THREE PARALLEL LINES

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ABSTRACT. A Heisenberg uniqueness pair is a pair (Γ, Λ), where Γ is a curve in the plane and Λ is a set in the plane, with the following property: any bounded Borel measure μ in the plane supported on Γ, which is absolutely continuous with respect to arc length and whose Fourier transform $\widehat{\mu}$ vanishes on Λ, must automatically be the zero measure. We characterize the Heisenberg uniqueness pairs for Γ as being three parallel lines $\Gamma = \mathbb{R} \times \{\alpha, \beta, \gamma\}$ with $\alpha < \beta < \gamma$, $(\gamma - \alpha)/(\beta - \alpha) \in \mathbb{N}$.

1. INTRODUCTION

The Heisenberg uncertainty principle states that both a function and its Fourier transform cannot be too localized at the same time (see [2] and [3]). M. Benedicks in [1] proved that given a nontrivial function $f \in L^1(\mathbb{R}^n)$, the Lebesgue measure of the set of points where $f \neq 0$ and the set of points where the Fourier transform $\widehat{f} \neq 0$ cannot be simultaneously finite. In this paper we consider a similar problem for measures supported on a subset of $\mathbb{R}^2$.

Let Γ be a smooth curve in the plane $\mathbb{R}^2$ and Λ a subset in $\mathbb{R}^2$. In [4], Hedenmalm and Montes-Rodríguez posed the problem of deciding when it is true that

$$\widehat{\mu}_{|\Lambda} = 0 \text{ implies } \mu = 0$$

for any Borel measure μ supported on Γ and absolutely continuous with respect to the arc length measure on Γ, where

$$\widehat{\mu}(\xi, \eta) = \int_{\mathbb{R}^2} e^{\pi i \langle (x,y),(\xi,\eta) \rangle} d\mu(x, y).$$

If this is the case, then (Γ, Λ) is called a *Heisenberg Uniqueness Pair* (HUP).

When Γ is the circle, Lev [7] and Sjölin [8] independently characterized the HUP for some "small" sets Λ.

In [4] Hedenmalm and Montes-Rodríguez characterized the HUP in the cases:

- Γ is the hyperbola $xy = 1$ and $\Lambda = (\alpha\mathbb{Z} \times \{0\}) \cup (\{0\} \times \beta\mathbb{Z})$, for $\alpha, \beta > 0$.
- Γ two parallel lines in $\mathbb{R}^2$.

In this note we present a result generalizing this last case. We characterize the HUP for Γ as being three parallel lines:

$$\Gamma = \mathbb{R} \times \{\alpha, \beta, \gamma\} \text{ with } \alpha < \beta < \gamma, \ (\gamma - \alpha)/(\beta - \alpha) \in \mathbb{N}.$$

2. Three parallel lines

Given a set $E \subset \mathbb{R}$ and a point $\xi \in E$, let us define:

- $\mathcal{A}_{loc}^{E,\xi} = \{$functions ψ defined on E such that there exist a small interval I_ξ around ξ and a function $\varphi \in L^1(\mathbb{R})$ such that $\psi(\zeta) = \widehat{\varphi}(\zeta)$, for $\zeta \in I_\xi \cap E\}$.
- $P^{1,p}[\mathcal{A}_{loc}^{E,\xi}] = \{$functions ψ defined on E such that there exist an interval I_ξ around ξ and functions $\varphi_0, \varphi_1 \in L^1(\mathbb{R})$ with $\psi^p(\zeta) + \widehat{\varphi}_1(\zeta)\psi(\zeta) + \widehat{\varphi}_0(\zeta) = 0$, for $\zeta \in I_\xi \cap E\}$.

Wiener's lemma [5, p. 57] states that if $\psi \in \mathcal{A}_{loc}^{E,\xi}$ and $\psi(\xi) \neq 0$, then $1/\psi \in \mathcal{A}_{loc}^{E,\xi}$. Observe also that if $\psi \in \mathcal{A}_{loc}^{E,\xi}$, then $\psi \in P^{1,p}[\mathcal{A}_{loc}^{E,\xi}]$. This is easy to see only if p is natural.

Due to invariance under translation and rescaling (see [4]) it will be sufficient to study the case when $\Gamma = \mathbb{R} \times \{0, 1, p\}$ for $p \in \mathbb{N}$, $p > 1$.

Given a set $\Lambda \subset \mathbb{R}^2$, we say that μ is an *admissible measure* if μ is a Borel measure in the plane absolutely continuous with respect to arc length with $supp\,\mu \subset \Gamma$ and $\widehat{\mu}_{|\Lambda} = 0$.

If μ is a measure absolutely continuous with respect to arc length on Γ, then there exist functions $f, g, h \in \mathrm{L}^1(\mathbb{R})$ such that

$$\widehat{\mu}(\xi, \eta) = \widehat{f}(\xi) + e^{\pi i \eta}\widehat{g}(\xi) + e^{p\pi i \eta}\widehat{h}(\xi), \quad \text{for any} \quad (\xi, \eta) \in \mathbb{R}^2.$$

In particular an admissible measure can be written in this form. Observe also that $\widehat{\mu}$ is 2-periodic with respect to the second variable. So, for any set $\Lambda \subset \mathbb{R}^2$, we may consider the periodized set

$$\mathcal{P}(\Lambda) = \{(\xi, \eta) \text{ such that } (\xi, \eta + 2k) \in \Lambda \text{ for some } k \in \mathbb{Z}\},$$

and it follows that (Γ, Λ) is a HUP if and only if $(\Gamma, \overline{\mathcal{P}(\Lambda)})$ is a HUP, where $\overline{\mathcal{P}(\Lambda)}$ stands for the closure of $\mathcal{P}(\Lambda)$ in $\mathbb{R}^2$.

We may think without loss of generality that Λ is a closed set in $\mathbb{R}^2$, 2-periodic with respect to the second coordinate.

We then have the following result.

Theorem 1. *Let $\Gamma = \mathbb{R} \times \{0, 1, p\}$, for some $p \in \mathbb{N}$, $p > 1$ and $\Lambda \subset \mathbb{R}^2$, closed and 2-periodic with respect to the second variable. Then (Γ, Λ) is a Heisenberg uniqueness pair if and only if*

$$(2.1) \qquad \mathfrak{F} := \Pi^3(\Lambda) \cup (\Pi^2(\Lambda) \setminus \Pi^{2^*}(\Lambda)) \cup (\Pi^1(\Lambda) \setminus \Pi^{1^*}(\Lambda))$$

is dense in $\mathbb{R}$.

$\Pi(\Lambda)$ means the projection of Λ on the axis $\mathbb{R} \times \{0\}$ and given a point $\xi \in \Pi(\Lambda)$, and $Img(\xi)$ corresponds to the set of points $\eta \in [0, 2)$ with $(\xi, \eta) \in \Lambda$. The sets in $\mathfrak{F}$ are defined as follows:

- $\Pi^1(\Lambda) = \{\xi \in \Pi(\Lambda)$ such that there is a unique $\eta_0 \in Img(\xi)\}$.
- $\Pi^2(\Lambda) = \{\xi \in \Pi(\Lambda)$ such that there are two different points $\eta_0, \eta_1 \in Img(\xi)$, and if there is another point $\eta_2 \in Img(\xi)$, then $\frac{e^{p\pi i \eta_1} - e^{p\pi i \eta_0}}{e^{\pi i \eta_1} - e^{\pi i \eta_0}} = \frac{e^{p\pi i \eta_2} - e^{p\pi i \eta_0}}{e^{\pi i \eta_2} - e^{\pi i \eta_0}}\}$.

- $\Pi^3(\Lambda) = \{\, \xi \in \Pi(\Lambda)$ such that there are at least three different points $\eta_0, \eta_1, \eta_2 \in Img(\xi)$ with $\frac{e^{p\pi i \eta_1} - e^{p\pi i \eta_0}}{e^{\pi i \eta_1} - e^{\pi i \eta_0}} \neq \frac{e^{p\pi i \eta_2} - e^{p\pi i \eta_0}}{e^{\pi i \eta_2} - e^{\pi i \eta_0}} \,\}$.

The following technical lemma is easy to prove and shows that the functions τ and Φ are well defined for $\xi \in \Pi^2(\Lambda)$.

Lemma 2. *Let $x, y, z \in \mathbb{C}$ be different with*

$$\tau = \frac{y^p - x^p}{y - x} = \frac{z^p - x^p}{z - x};$$

then

$$\frac{z^p - y^p}{z - y} = \tau \quad and \quad \Phi = x\tau - x^p = y\tau - y^p = z\tau - z^p.$$

Let χ be a function defined on $\Pi^1(\Lambda)$ as $\chi(\zeta) = e^{\pi i \eta}$, where $\eta \in Img(\zeta)$. We define the set $\Pi^{1^*}(\Lambda)$ as

- $\Pi^{1^*}(\Lambda) = \{\, \xi \in \Pi^1(\Lambda)$ such that $\chi \in P^{1,p}[\mathcal{A}_{loc}^{\Pi^1(\Lambda), \xi}] \,\}$.

Let τ, Φ be functions defined on $\Pi^2(\Lambda)$ as

$$\tau(\xi) = \frac{e^{p\pi i \eta_1} - e^{p\pi i \eta_0}}{e^{\pi i \eta_1} - e^{\pi i \eta_0}} \quad and \quad \Phi(\xi) = e^{\pi i \eta_0} \frac{e^{p\pi i \eta_1} - e^{p\pi i \eta_0}}{e^{\pi i \eta_1} - e^{\pi i \eta_0}} - e^{p\pi i \eta_0},$$

where $\eta_0, \ \eta_1 \in Img(\xi)$. We define the set $\Pi^{2^*}(\Lambda)$ as

- $\Pi^{2^*}(\Lambda) = \{\, \xi \in \Pi^2(\Lambda)$ such that $\tau, \Phi \in \mathcal{A}_{loc}^{\Pi^2(\Lambda), \xi} \,\}$.

The next lemma will be needed for the proof of the necessity of condition (2.1) in Theorem 1.

Lemma 3. *Let I be an interval in $\mathbb{R}$ with $\Pi^{2^*}(\Lambda)$ dense in I. Then there exists a subinterval $I' \subset I$ with $I' \subset \Pi^{2^*}(\Lambda) \cup \Pi^3(\Lambda)$.*

Proof. Pick an arbitrary point $\widetilde{\xi} \in I \cap \Pi^{2^*}(\Lambda)$. Since $\tau, \Phi \in \mathcal{A}_{loc}^{\Pi^2(\Lambda), \widetilde{\xi}}$ and $\Pi^{2^*}(\Lambda)$ is dense in I, we can extend the functions τ, Φ continuously on a neighborhood of $\widetilde{\xi}$. Let $\widetilde{\eta} \neq \widetilde{\varrho} \in Img(\widetilde{\xi})$. Then

$$|\tau(\widetilde{\xi})| = \left| \frac{e^{p\pi i \widetilde{\eta}} - e^{p\pi i \widetilde{\varrho}}}{e^{\pi i \widetilde{\eta}} - e^{\pi i \widetilde{\varrho}}} \right| < p,$$

and since τ is continuous around $\widetilde{\xi}$, there exists a small interval I' around $\widetilde{\xi}$ with $|\tau(\xi)| < p$ for $\xi \in I'$. We will see that $I' \subset \Pi^{2^*}(\Lambda) \cup \Pi^3(\Lambda)$.

Given $\xi \in I'$, consider a sequence $\{\xi_k\} \subset \Pi^{2^*}(\Lambda) \cap I'$ with $\xi_k \to \xi$, and for each ξ_k let $\eta_k \neq \varrho_k \in Img(\xi_k)$. There exist subsequences $\{\eta_k^*\}$ and $\{\varrho_k^*\}$ such that $\eta_k^* \to \eta^*$ and $\varrho_k^* \to \varrho^*$ for some $\eta^*, \varrho^* \in [0, 2]$. Since the set Λ is closed and 2-periodic with respect to the second coordinate, we may assume WLOG that $\xi \in \Pi(\Lambda)$ with $\eta^* \neq \varrho^* \in Img(\xi)$. Otherwise,

$$|\tau(\xi)| \longleftarrow |\tau(\xi_k^*)| = \left| e^{(p-1)\pi i \eta_k^*} + e^{(p-2)\pi i \eta_k^*} e^{\pi i \varrho_k^*} + \cdots + e^{(p-1)\pi i \varrho_k^*} \right|$$
$$\longrightarrow \left| e^{(p-1)\pi i \eta^*} + e^{(p-1)\pi i \eta^*} + \cdots + e^{(p-1)\pi i \eta^*} \right| = p,$$

which is a contradiction with the fact that $\xi \in I'$.

So $I' \subset \Pi^2(\Lambda) \cup \Pi^3(\Lambda)$, and since the extended functions τ, Φ are continuous on I', we also have that $\xi \in \Pi^{2^*}(\Lambda)$ for any $\xi \in \Pi^2(\Lambda) \cap I'$. Also, we can conclude that $I' \subset \Pi^{2^*}(\Lambda) \cup \Pi^3(\Lambda)$. $\qquad \square$

3. Proof of the main result

This section is devoted to the proof of Theorem 1. The proof of the sufficiency of condition (2.1) is rather easy. Let μ be an admissible measure. Then there exist functions $f, g, h \in L^1(\mathbb{R})$ such that

$$\widehat{\mu}(\xi, \eta) = \widehat{f}(\xi) + e^{\pi i \eta}\widehat{g}(\xi) + e^{p\pi i \eta}\widehat{h}(\xi), \quad \text{for any} \quad (\xi, \eta) \in \mathbb{R}^2.$$

Since $\mathfrak{F}$ is dense in $\mathbb{R}$ we will be done if we show that $\widehat{f}(\xi) = \widehat{g}(\xi) = \widehat{h}(\xi) = 0$ for any $\xi \in \mathfrak{F} = \Pi^3(\Lambda) \cup (\Pi^2(\Lambda) \setminus \Pi^{2^*}(\Lambda)) \cup (\Pi^1(\Lambda) \setminus \Pi^{1^*}(\Lambda))$.

If $\xi \in \Pi^3(\Lambda)$, let $\eta_0, \eta_1, \eta_2 \in Img(\xi)$ be different. Since $\widehat{\mu}_{|\Lambda} = 0$ and $\frac{e^{p\pi i \eta_1} - e^{p\pi i \eta_0}}{e^{\pi i \eta_1} - e^{\pi i \eta_0}} \neq \frac{e^{p\pi i \eta_2} - e^{p\pi i \eta_0}}{e^{\pi i \eta_2} - e^{\pi i \eta_0}}$, it follows that $\widehat{f}(\xi) = \widehat{g}(\xi) = \widehat{h}(\xi) = 0$.

If $\xi \in \Pi^2(\Lambda)$, let $\eta_0 \neq \eta_1 \in Img(\xi)$. Since $\widehat{\mu}_{|\Lambda} = 0$, then $\widehat{g}(\xi) = -\tau(\xi)\widehat{h}(\xi)$ and $\widehat{f}(\xi) = \Phi(\xi)\widehat{h}(\xi)$. Suppose $\widehat{h}(\xi) \neq 0$. Then by Wiener's lemma and Fubini's theorem, $\tau, \Phi \in \mathcal{A}_{loc}^{\Pi^2(\Lambda), \xi}$, which implies that $\xi \in \Pi^{2^*}(\Lambda)$. So if $\xi \in \Pi^2(\Lambda) \setminus \Pi^{2^*}(\Lambda)$, then $\widehat{f}(\xi) = \widehat{g}(\xi) = \widehat{h}(\xi) = 0$.

Finally, if $\xi \in \Pi^1(\Lambda)$ and $\eta_0 \in Img(\xi)$, since $\widehat{\mu}_{|\Lambda} = 0$, then $\widehat{f}(\xi) + \chi(\xi)\widehat{g}(\xi) + \chi^p(\xi)\widehat{h}(\xi) = 0$, where $\chi(\xi) = e^{\pi i \eta_0}$. Suppose $\widehat{h}(\xi) \neq 0$; then $\chi \in P^{1,p}[\mathcal{A}_{loc}^{\Pi^1(\Lambda), \xi}]$ and $\xi \in \Pi^{1^*}(\Lambda)$. Otherwise, if $\widehat{h}(\xi) = 0$ and $\widehat{g}(\xi) \neq 0$, then by Wiener's lemma and Fubini's theorem, $\chi \in \mathcal{A}_{loc}^{\Pi^1(\Lambda), \xi}$ and also $\chi^p \in \mathcal{A}_{loc}^{\Pi^1(\Lambda), \xi}$, so $\chi \in P^{1,p}[\mathcal{A}_{loc}^{\Pi^1(\Lambda), \xi}]$ and $\xi \in \Pi^{1^*}(\Lambda)$. This means that if $\xi \in \Pi^1(\Lambda) \setminus \Pi^{1^*}(\Lambda)$, then $\widehat{f}(\xi) = \widehat{g}(\xi) = \widehat{h}(\xi) = 0$.

For the proof of the necessity of condition (2.1), suppose that the set $\mathfrak{F}$ is not dense in $\mathbb{R}$ and let us pick an open interval I that has empty intersection with $\mathfrak{F}$, i.e.,

$$\Pi(\Lambda) \cap I = (\Pi^{1^*}(\Lambda) \cup \Pi^{2^*}(\Lambda)) \cap I.$$

We consider three cases:

- There exists a small interval $I_\xi \subset I$ around $\xi \in \Pi^{1^*}(\Lambda)$ such that all the points in $I_\xi \cap \Pi(\Lambda)$ belong to $\Pi^{1^*}(\Lambda)$. Since $\chi \in P^{1,p}[\mathcal{A}_{loc}^{\Pi^1(\Lambda), \xi}]$, there exist an interval $I' \subset I_\xi$ around ξ and functions $\varphi_0, \varphi_1 \in L^1(\mathbb{R})$ such that

$$\chi^p(\xi^*) + \widehat{\varphi_1}(\xi^*)\chi(\xi^*) + \widehat{\varphi_0}(\xi^*) = 0$$

for any $\xi^* \in I' \cap \Pi(\Lambda)$. Let $h \in L^1(\mathbb{R})$ with $\widehat{h}(\xi) \neq 0$ and $supp\,\widehat{h} \Subset I'$, and define $f, g \in L^1(\mathbb{R})$ via $\widehat{f} = \widehat{h}\widehat{\varphi_0}$, and $\widehat{g} = \widehat{h}\widehat{\varphi_1}$. Now,

$$\widehat{\mu}(\xi^*, \eta^*) = \widehat{f}(\xi^*) + \widehat{g}(\xi^*)\chi(\xi^*) + \widehat{h}(\xi^*)\chi^p(\xi^*) = 0$$

for $\xi^* \in I' \cap \Pi^{1^*}(\Lambda)$, $\eta^* \in Img(\xi^*)$. Finally, since $supp\,\widehat{h} \Subset I'$ and $I' \cap \Pi(\Lambda) = I' \cap \Pi^{1^*}(\Lambda)$, we can conclude that $\widehat{\mu}_{|\Lambda} \equiv 0$, and we have that μ is a nontrivial admissible measure. So (Γ, Λ) is not a Heisenberg uniqueness pair.

- There exists a small interval $I_\xi \subset I$ around $\xi \in \Pi^{2^*}(\Lambda)$ such that all the points in $I_\xi \cap \Pi(\Lambda)$ belong to $\Pi^{2^*}(\Lambda)$. Now there exists a small interval $I' \subset I_\xi$ around ξ and functions $\Phi_1, \tau_1 \in L^1(\mathbb{R})$ such that $\widehat{\tau_1} = \tau$ and $\widehat{\Phi_1} = \Phi$ on $I' \cap \Pi(\Lambda)$. Consider a function $h \in L^1(\mathbb{R})$ with $supp\,\widehat{h} \subset I'$ and $\widehat{h}(\xi) \neq 0$, and define $f, g \in L^1(\mathbb{R})$ as

$$g = -h * \tau_1 \quad \text{and} \quad f = h * \Phi_1.$$

Now, given a point $\xi^* \in I' \cap \Pi^{2^*}(\Lambda)$, let $\eta^* \neq \varrho^* \in Img(\xi^*)$. Since $\tau(\xi^*) = \frac{e^{p\pi i\eta^*} - e^{p\pi i\varrho^*}}{e^{\pi i\eta^*} - e^{\pi i\varrho^*}}$ and $\Phi(\xi) = e^{\pi i\eta^*} \frac{e^{p\pi i\eta^*} - e^{p\pi i\varrho^*}}{e^{\pi i\eta^*} - e^{\pi i\varrho^*}} - e^{p\pi i\varrho^*}$, we have that

$$\widehat{\mu}(\xi^*, \eta^*) = \widehat{f}(\xi^*) + \widehat{g}(\xi^*)e^{\pi i\eta^*} + \widehat{h}(\xi^*)e^{p\pi i\eta^*} = 0$$

and also that $\widehat{\mu}(\xi^*, \varrho^*) = 0$. So, the corresponding measure μ is a nontrivial admissible measure and (Γ, Λ) is not a Heisenberg uniqueness pair.

- All the intervals $I_\xi \subset I$ contain points in $\Pi^{1^*}(\Lambda)$ and points in $\Pi^{2^*}(\Lambda)$. That is, the sets $\Pi^{1^*}(\Lambda)$ and $\Pi^{2^*}(\Lambda)$ are dense in $I \cap (\Pi^{1^*}(\Lambda) \cup \Pi^{2^*}(\Lambda)) = I \cap \Pi(\Lambda)$. But this is not possible. In fact, if $\Pi^{2^*}(\Lambda)$ is dense in I, by Lemma 3, there exists a subinterval $I' \subset I$ such that $I' \subset \Pi^{2^*}(\Lambda) \cup \Pi^3(\Lambda)$.

This finishes the proof of the theorem.

4. Examples and further results

Given a point $\xi \in \Pi(\Lambda)$ such that $\sharp\{\eta \in Img(\xi)\} \geq 3$, we will state a criteria to decide whether the point ξ belongs to $\Pi^3(\Lambda)$ or to $\Pi^2(\Lambda)$. But before this we prove the following lemma.

Lemma 4. *Given $C \in \mathbb{C}$, there exist at most p different points $\rho^{(k)} \in [0, 2)$ such that for any $j \neq k$,*

$$(4.1) \qquad \frac{x^p - y^p}{x - y} = C, \quad where \quad x = e^{\pi i\rho^{(k)}}, \; y = e^{\pi i\rho^{(j)}}.$$

Proof. Observe that for fixed C, there exists a constant $C^* \in \mathbb{C}$ such that

$$(4.2) \qquad\qquad xC - x^p = C^*$$

for any $x = e^{\pi i\rho^{(k)}}$ solution of (4.1). Now it is obvious that there are at most p different solutions $\rho^{(k)} \in [0, 2)$ of the equation (4.2). $\square$

Corollary 5. *Given a point $\xi \in \Pi(\Lambda)$, if $\sharp\{\eta \in Img(\xi)\} > p$, then $\xi \in \Pi^3(\Lambda)$.*

In particular, if Γ consists of three parallel equidistant lines in the plane ($p = 2$), we have

$$\Pi^3(\Lambda) = \{\, \xi \in \Lambda \text{ such that } \sharp\{\eta \in Img(\xi)\} \geq 3 \,\},$$
$$\Pi^2(\Lambda) = \{\, \xi \in \Lambda \text{ such that } \sharp\{\eta \in Img(\xi)\} = 2 \,\}.$$

Example 6. The following example shows that Corollary 5 is sharp:

- Let $\Lambda = \mathbb{R} \times \{2k/p\}_{k=0,\cdots,p-1}$. Then for any $\xi \in \mathbb{R}$,

$$\sharp\{\eta \in Img(\xi)\} = p$$

and $\xi \in \Pi^{2^*}(\Lambda)$. Observe that in this case, (Γ, Λ) is not an HUP.

This lemma will be useful for another example.

Lemma 7. *For any $z \in \mathbb{C}$ with $|z| < 1$, there exist $w_1, w_2 \in \mathbb{C}$ unimodular with $z = w_1 + w_2$.*

Proof. Let $z = re^{i\sigma}$ and let $v \in [0, \pi/2]$ with $\cos v = r/2$. Let's take

$$w_1 = e^{i(v+\sigma)}, \quad w_2 = e^{i(-v+\sigma)}.$$

Then,

$$w_1 + w_2 = e^{i(v+\sigma)} + e^{i(-v+\sigma)} = e^{i\sigma} 2\cos(v) = re^{i\sigma} = z,$$

and this finishes the proof. $\square$

Example 8. Let $p = 2$. Let g be a bounded, continuous function with $|g| < 1$ that is nowhere locally the Fourier transform of an L^1 function. There exists a set $\Lambda \subset \mathbb{R} \times [0, 2)$ such that $\Pi(\Lambda) = \Pi^{2^*}(\Lambda)$ is dense in $\mathbb{R}$ and the function $\Phi \equiv g$ on $\Pi^{2^*}(\Lambda)$. So, (Γ, Λ) is not an HUP.

Let's first prove the existence of the function g. Let E be a dense set of measure zero on the circle $\mathbb{T}$. By [6] there exists a continuous function f such that the Fourier series of f fails to converge on any point of E. Now let $g : \mathbb{R} \to \mathbb{C}$ be the 2-periodic function defined as $g(t) = f(e^{\pi i t})$. It is easy to see that this function g is continuous but it is not a Fourier transform of an L^1 function locally at any point. By a standard argument we can think that g is bounded with $|g| < 1$.

Now we will define the set Λ. By Lemma 7, for any $\xi \in \mathbb{R}$ there exist $w_1(\xi) = e^{\pi i \eta_0}$, $w_2(\xi) = e^{\pi i \eta_1}$ with $w_1(\xi) + w_2(\xi) = g(\xi)$. Observe also that there is a dense set Ψ of $\mathbb{R}$ such that $\eta_0 \neq \eta_1$ for any $\xi \in \Psi$. Otherwise the function g is constant on an interval, and we get a contradiction with the fact that g is not locally the Fourier transform of an L^1 function.

We define $\Lambda = \{(\xi, \eta_0) \cup (\xi, \eta_1)\}_{\xi \in \Psi}$. Now $\Pi(\Lambda) = \Pi^2(\Lambda)$ and

$$\Phi(\xi) = e^{\pi i \eta_0} + e^{\pi i \eta_1} = g(\xi), \text{ for any } \xi \in \Psi.$$

Since $\Phi \notin \mathcal{A}_{loc}^{\Pi^2(\Lambda), \xi}$ for any $\xi \in \Pi^2(\Lambda)$, we have that $\Pi(\Lambda) = \Pi^{2^*}(\Lambda)$, and so (Γ, Λ) is not an HUP.

Acknowledgements

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